1. The general second order homogeneous linear differential equation with constant coefficients looks like

\[ Ay'' + By' + Cy = 0, \]

where \( y \) is an unknown function of the variable \( x \), and \( A, B, \) and \( C \) are constants. If \( A = 0 \) this becomes a first order linear equation, which we already know how to solve. So we will consider the case \( A \neq 0 \). We can divide through by \( A \) and obtain the equivalent equation

\[ y'' + by' + cy = 0 \]

where \( b = B/A \) and \( c = C/A \).

“Linear with constant coefficients” means that each term in the equation is a constant times \( y \) or a derivative of \( y \). “Homogeneous” excludes equations like \( y'' + by' + cy = f(x) \) which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

\[ y = e^{\lambda x}, \]

where \( \lambda \) is an unknown constant. Why? Because it works!

We substitute \( y = e^{\lambda x} \) in our equation. This gives

\[ \lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0. \]

Since \( e^{\lambda x} \) is never zero, we can divide through and get the equation

\[ \lambda^2 + b\lambda + c = 0. \]

Whenever \( \lambda \) is a solution of this equation, \( y = e^{\lambda x} \) will automatically be a solution of our original differential equation, and if \( \lambda \) is not a solution, then \( y = e^{\lambda x} \) cannot solve the differential equation. So the substitution \( y = e^{\lambda x} \) transforms the differential equation into an algebraic equation!
Example 1. Consider the differential equation

\[ y'' - y = 0. \]

Plugging in \( y = e^{\lambda x} \) give us the associated equation

\[ \lambda^2 - 1 = 0, \]

which factors as

\[ (\lambda + 1)(\lambda - 1) = 0; \]

this equation has \( \lambda = 1 \) and \( \lambda = -1 \) as solutions. Both \( y = e^x \) and \( y = e^{-x} \) are solutions to the differential equation \( y'' - y = 0 \). (You should check this for yourself!)

Example 2. For the differential equation

\[ y'' + y' - 2y = 0, \]

we look for the roots of the associated algebraic equation

\[ \lambda^2 + \lambda - 2 = 0. \]

Since this factors as \( (\lambda - 1)(\lambda + 2) = 0 \), we get both \( y = e^x \) and \( y = e^{-2x} \) as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form

\[ y'' + by' + cy = 0, \]

we need to find the roots of \( \lambda^2 + b\lambda + c = 0 \), which we can do using the quadratic formula to get

\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]

If the discriminant \( b^2 - 4c \) is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if \( y_1 \) and \( y_2 \) are solutions of the homogeneous differential equation \( y'' + by' + cy = 0 \), then so is the linear combination \( py_1 + qy_2 \) for any numbers \( p \) and \( q \). This fact is easy to check (just plug \( py_1 + qy_2 \) into the equation and regroup terms; note that the coefficients \( b \) and \( c \) do not need to be constant for this to work. This means that for the differential equation in Example 1 \( (y'' - y = 0) \), any function of the form

\[ pe^x + qe^{-x}, \quad \text{where } p \text{ and } q \text{ are any constants}. \]
is a solution. Indeed, while we can’t justify it here, all solutions are of this form. Similarly, in Example 2, the general solution of

\[ y'' + y' - 2y = 0 \]

is

\[ y = pe^x + qe^{-2x}, \quad \text{where } p \text{ and } q \text{ are constants.} \]

4. If the discriminant \( b^2 - 4c \) is negative, then the equation \( \lambda^2 + b\lambda + c = 0 \) has no solutions, unless we enlarge the number field to include \( i = \sqrt{-1} \), i.e. unless we work with complex numbers. If \( b^2 - 4c < 0 \), then since we can write any positive number as a square \( k^2 \), we let \( k^2 = -(b^2 - 4c) \). Then \( ik \) will be a square root of \( b^2 - 4c \), since \( (ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2 - 4c \). The solutions of the associated algebraic equation are then

\[ \lambda_1 = \frac{-b + ik}{2}, \quad \lambda_2 = \frac{-b - ik}{2}. \]

**Example 3.** If we start with the differential equation \( y'' + y = 0 \) (so \( b = 0 \) and \( c = 1 \)) the discriminant is \( b^2 - 4c = -4 \), so \( 2i \) is a square root of the discriminant and the solutions of the associated algebraic equation are \( \lambda_1 = i \) and \( \lambda_2 = -i \).

**Example 4.** If the differential equation is \( y'' + 2y' + 2y = 0 \) (so \( b = 2 \) and \( c = 2 \) and \( b^2 - 4c = 4 - 8 = -4 \)). In this case the solutions of the associated algebraic equation are \( \lambda = (-2 \pm 2i)/2 \), i.e. \( \lambda_1 = -1 + i \) and \( \lambda_2 = -1 - i \).

5. Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting \( e^{\lambda x} \) as a function of \( x \) when \( \lambda \) is a complex number. Suppose \( \lambda \) has real part \( a \) and imaginary part \( ib \), so that \( \lambda = a + ib \) with \( a \) and \( b \) real numbers. Then

\[ e^{\lambda x} = e^{(a+ib)x} = e^{ax}e^{ibx} \]

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor \( e^{ax} \) does not cause a problem, but what is \( e^{ibx} \)? Everything will work out if we take

\[ e^{ibx} = \cos(bx) + i \sin(bx), \]

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

6. Let us try this formula with our examples.

**Example 3.** For \( y'' + y = 0 \) we found \( \lambda_1 = i \) and \( \lambda_2 = -i \), so the solutions are \( y_1 = e^{ix} \) and \( y_2 = e^{-ix} \). The formula gives us \( y_1 = \cos x + i \sin x \) and \( y_2 = \cos x - i \sin x \).

Our earlier observation that if \( y_1 \) and \( y_2 \) are solutions of the linear differential equation, then so is the combination \( py_1 + qy_2 \) for any numbers \( p \) and \( q \) holds even if \( p \) and \( q \) are complex constants.
Using this fact with the solutions from our example, we notice that \( \frac{1}{2}(y_1 + y_2) = \cos x \) and \( \frac{1}{2}(y_1 - y_2) = \sin x \) are both solutions. When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions. Using the fact about linear combinations again, we can say that \( y = p \cos x + q \sin x \) is a solution for any \( p \) and \( q \). This is the general solution. (It is also correct to call \( y = pe^{ix} + qe^{-ix} \) the general solution; which one you use depends on the context.)

Example 4. \( y'' + 2y' + 2y = 0 \). We found \( \lambda_1 = -1 + i \) and \( \lambda_2 = -1 - i \). Using the formula we have

\[
y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i \sin x),
\]

\[
y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i \sin x).
\]

Exactly as before we can take \( \frac{1}{2}(y_1 + y_2) \) and \( \frac{1}{2}(y_1 - y_2) \) to get the real solutions \( e^{-x}\cos x \) and \( e^{-x}\sin x \). (Check that these functions both satisfy the differential equation!) The general solution will be \( y = pe^{-x}\cos x + qe^{-x}\sin x \).

7. Repeated roots. Suppose the discriminant is zero: \( b^2 - 4c = 0 \). Then the “characteristic equation” \( \lambda^2 + b\lambda + c = 0 \) has one root. In this case both \( e^{\lambda x} \) and \( xe^{\lambda x} \) are solutions of the differential equation.

Example 5. Consider the equation \( y'' + 4y' + 4y = 0 \). Here \( b = c = 4 \). The discriminant is \( b^2 - 4c = 4^2 - 4 \times 4 = 0 \). The only root is \( \lambda = -2 \). Check that both \( e^{-2x} \) and \( xe^{-2x} \) are solutions. The general solution is then \( y = pe^{-2x} + qxe^{-2x} \).

8. Initial Conditions. For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition \( y(0) = y_0 \); in the same way the \( p \) and the \( q \) in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some “initial” value of \( x \).

Example 5. Suppose that for the differential equation of Example 2, \( y'' + y' - 2y = 0 \), we want a solution with \( y(0) = 1 \) and \( y'(0) = -1 \). The general solution is \( y = pe^x + qe^{-2x} \), since the two roots of the characteristic equation are 1 and \( -2 \). The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for \( p \) and \( q \). In this case we have

\[
1 = y(0) = pe^0 + qe^{-2 \times 0} = p + q
\]

\[
-1 = y'(0) = pe^0 - 2qe^{-2 \times 0} = p - 2q.
\]

This leads to the set of linear equations \( p + q = 1, p - 2q = -1 \) with solution \( q = 2/3, p = 1/3 \). You should check that the solution

\[
y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}
\]
satisfies the initial conditions.

*Example 6.* For the differential equation of Example 4, $y'' + 2y' + 2y = 0$, we found the general solution $y = pe^{-x} \cos x + qe^{-x} \sin x$. To find a solution satisfying the initial conditions $y(0) = -2$ and $y'(0) = 1$ we proceed as in the last example:

$$-2 = y(0) = pe^{-0} \cos 0 + qe^{-0} \sin 0 = p$$

$$1 = y'(0) = -pe^{-0} \cos 0 - pe^{-0} \sin 0 - qe^{-0} \sin 0 + qe^{-0} \cos 0 = -p + q.$$  

So $p = -2$ and $q = -1$. Again check that the solution

$$y = -2e^{-x} \cos x - e^{-x} \sin x$$

satisfies the initial conditions.
Problems cribbed from Salas-Hille-Etgen, page 1133

In exercises 1-10, find the general solution. Give the real form.

1. \(y'' - 13y' + 42y = 0\).
2. \(y'' + 7y' + 3y = 0\).
3. \(y'' - 3y' + 8y = 0\).
4. \(y'' - 12y = 0\).
5. \(y'' + 12y = 0\).
6. \(y'' - 3y' + \frac{9}{4}y = 0\).
7. \(2y'' + 3y' = 0\).
8. \(y'' - y' - 30y = 0\).
9. \(y'' - 4y' + 4y = 0\).
10. \(5y'' - 2y' + y = 0\).

In exercises 11-16, solve the given initial-value problem.

11. \(y'' - 5y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 1\)
12. \(y'' + 2y' + y = 0, \quad y(2) = 1, \quad y'(2) = 2\)
13. \(y'' + \frac{1}{4}y = 0, \quad y(\pi) = 1, \quad y'(\pi) = -1\)
14. \(y'' - 2y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -1\)
15. \(y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1\)
16. \(y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2\)