Conserved quantities and adaptation to the edge of chaos

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Certain dynamical systems, such as the shift map and the logistic map, have an edge of chaos in their parameter spaces. On one side of this edge, the dynamics are chaotic for many parameter values, on the other side of the edge they are periodic. We find that discrete-time dynamical systems with wavelet filtered feedback from the dynamical variable to the parameters are attracted to a narrow parameter range near the edge of chaos, the periodic boundary regime. We show that the migration from the chaotic regime to the periodic boundary regime can be attributed to a conserved quantity, and find that such adaptation to the edge of chaos is accompanied by a depopulation of the chaotic regime. We use this conserved quantity to determine the location of the periodic boundary regime and show that its size is proportional to the size of the feedback. Further, we compute the dynamics of the probability density for the parameter for a specific example.

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I. INTRODUCTION

The concept “adaptation to the edge of chaos” refers to the idea that many complex adaptive systems, including those found in biology, seem to naturally evolve toward a narrow regime near the boundary between order and chaos [1]. The suppression of chaos is not due to a sophisticated external control [2–9] but induced by some simple self-adjustment of the system. Packard [10] first showed that adaptation to the edge of chaos occurs for a population of cellular automata rules evolving with a genetic algorithm, though the conclusions drawn from this work have come under some dispute [11]. Self-organized critically (SOC) [12] in avalanche and earthquake models is believed to be a related phenomenon. However the connection between SOC and edge of chaos is not completely obvious. This is mainly due to the fact that in the SOC models (e.g., the sand pile) some randomness is involved; therefore there is not a unique way to define the Lyapunov exponent and this can induce some confusion [13–15]. Models of coupled neurons with self-adjusting coupling strength have been found to exhibit robust synchronization and suppression of chaos [16]. The edge of chaos occupies a prominent position because it has been found to be not only the optimal setting for control of a system [17], but also an optimal setting under which a physical system can support primitive functions for computation [18], though once again this claim has been disputed [19]. Zaslavsky and others [20,21] noted that the irregular dynamics near the edge of chaos has unique properties due to very long transients and they call that motion pseudochaos.

Possibly the simplest models for adaptation to the edge of chaos are self-adjusting map dynamics [22]. The numerical findings have been confirmed experimentally [23] with Chua’s circuit [24]. However, the theoretical analysis does not predict the location of the narrow parameter regime near the boundary to which the system evolves. Furthermore the distribution function for the limiting parameter values differ from numerical findings. This is believed to be due to the fact that the dynamics of the parameters are approximated by a diffusion process with a large diffusion constant in the chaotic regime. In Melby’s system, the feedback from the dynamical variable to the map parameter is computed with a windowed Fourier band filter. This is a rather complicated algorithm whereas wavelet filters [25,26] have a similar effect, but are much simpler in both implementation [27–29] and analysis. Wavelet filters have been successfully used for the compression of experimental data [30–37] as well as images in JPEG format [38]. Finitely supported wavelet filters can be good models for the dynamics of slow variables in naturally occurring processes [39,40], though not all wavelets filters have these properties [41].

In this paper, we study the evolution of self-adjusting maps toward the edge of chaos. Maps can be good models nonlinear and chaotic motion, and are mathematically more tractable than nonlinear differential equations. In contrast to earlier work, we assume that the feedback from the dynamical variable is low-pass wavelet filtered. Wavelet filters can mimic the filter function of damped oscillators which occur in many natural systems. We use discrete wavelet filters that have a finite support and have zero mean. Further we consider the impact of correlations in the parameter dynamics and determine the location of the narrow regime near the edge of chaos to which the dynamics evolve. Finally we determine the dynamics of the probability density of the parameter for a specific example.

II. SELF-ADJUSTING MAP DYNAMICS

We consider a self-adjusting map dynamics with the dynamical variable \( x_n \) on the interval \([0,1]\) and a self-adjusting parameter \( a_n \)

\[
x_{n+1} = f(x_n, a_n),
\]

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FIG. 1. The value of the dynamical variable \( x_n \) versus time step \( n \) (a) and the value of the parameter \( a_n \) versus time step \( n \) (b) for a shift map, where \( s=0.4, \, N=150 \) and a Haar wavelet filter. \( a_n \) is constant if the self-adjustment is off. If the self-adjustment is on and \( a_n > 1 \), the parameter value has an irregular time dependence, but stays within a small range. In this simulation the parameter value \( a_n \) never reaches the boundaries of the parameter range \( a_{\min}=0 \) and \( a_{\max}=2 \).

\[
a_{n+1} = a_n + s_n \Delta F_n, \tag{1}
\]

where the wavelet filter \( \Delta F_n = \sum_{j=0}^{M-1} g_j x_{n+1-j} \) has finite support \( M \) and zero mean. For Daubechies wavelets of order one with support \( M=4 \) the wavelet coefficients are \( g_0 = (1 - \sqrt{3})/\sqrt{2} \), \( g_1 = -(3 - \sqrt{3})/\sqrt{2} \), \( g_2 = (3 + \sqrt{3})/\sqrt{2} \), and \( g_3 = -(1 + \sqrt{3})/\sqrt{2} \) [26]. For a Haar wavelet the coefficients are \( g_0 = -g_1 = 1 \). During the adaptation periods the size of the feedback is small and constant, i.e., \( 0 < s_n = s < 1 \) at the time step \( n=\text{int}(n) \), \( \text{int}(n), \text{int}(n+1), \ldots, (\text{int}(n)+1)N-1 \), \( n=0,2, \ldots, M-1 \). During the relaxation periods, \( n=\text{int}(n) \), \( \text{int}(n)+1, \ldots, (\text{int}(n)+1)N-1 \), \( n=0,2, \ldots, M-1 \) there is no feedback, i.e., \( s_n = 0 \). For systems with a bounded parameter range, \( a_{\min} \leq a_n \leq a_{\max} \), the parameter is set equal to the boundary value if the new value would be outside the parameter range, i.e., \( a_{n+1} = a_{\max} \) if \( a_n + s_n \Delta F_n > a_{\max} \) and \( a_{n+1} = a_{\min} \) if \( a_n + s_n \Delta F_n < a_{\min} \). Further we assume that the adaptation and relaxation periods are long compared to the support of the filter, i.e., \( N \gg M \), and long compared to the relaxation time of the dynamical system \( x_{n-j} \), \( j=1,2, \ldots \), so that it can reach the vicinity of an attractor before adaptation is turned on or off again. The initial state \( x_0 \) is assumed to be random and equally distributed.

Figure 1 shows typical numerical results generated from Eq. (1) for the shift map \( f = \text{mod}(a_n x_n + r_n) \), where \(-5 \times 10^{-6} < r_n < 5 \times 10^{-6} \) are random and equally distributed and \( 0 \leq a_n \leq 2 \). The modulo function is defined as \( \text{mod}(x) = x - \text{int}(x) \), where \( \text{int}(x) \) returns the integer portion of \( x \). If the initial parameter value is in the chaotic regime, i.e., \( a_n > 1 \) the parameter value is changing within a certain range during the adaptation periods. Even though \( a_n \) stays within a small range during each adaptation period, these ranges are different at each adaptation period and eventually the parameter value reaches the period regime \( a_n < 1 \) and stays there.

The total wavelet filter is defined as

\[
F_n = \sum_{j=0}^{M-1} w_j x_{n-j}, \tag{2}
\]

where \( w_j = \sum_{k=0}^{\infty} g_j \). \( j=0,1, \ldots, M-1 \) are the coefficients of the integrated wavelet. \( w_{M-1} = \sum_{k=0}^{\infty} g_k = 0 \), since we assume that the wavelet has zero mean. Hence \( F_{n+1} - F_n = \Delta F_n \) and \( F_n = F_{n_0} + \sum_{n=0}^{n-1} \Delta F_i \) if \( n_0 \equiv n \). In contrast to the parameter \( a_n \), the quantity \( b_n \) defined as

\[
b_n = a_n - s_n F_n, \tag{3}
\]

is conserved, i.e.,

\[
b_{n+1} = b_n \quad \text{if} \quad s_{n+1} = s_n, \tag{4}
\]

since \( b_{n+1} = a_{n+1} - s_{n+1} F_{n+1} = (a_n + s_n \Delta F_n - s_{n+1} F_n) - s_{n+1} F_{n+1} = (a_n - s_n F_n) - (s_n - s_{n+1}) F_{n+1} = a_n - s_n F_n = b_n \) except when \( s_{n+1} \neq s_n \) at the time steps when the adaptation is switched on or off. If \( a_n \) reaches the boundary of the parameter range during the adaptation period, \( b_n \) is not constant. Then \( b_n = b_N + \sum_{n=1}^{n-1} (a_{n+1} - a_n - s_n \Delta F_n) \), where \( s_m \) is constant. In the following we consider trajectories where \( a_n \) does not reach the boundary. Figure 2 shows the time dependence of \( b_n \) for the dynamics in Fig. 1.

We can use the conserved quantity to eliminate the dynamical variable \( a_n \) from Eq. (1)

\[
x_{n+1} = g(x_n, \ldots, x_{n-M+2}, s_n, b_{N}), \tag{5}
\]

where \( g(x_n, \ldots, x_{n-M+2}, s_n, b_{N}) = f(x_n, b_{N} + s F_n) \) and \( F_n = F(x_n, \ldots, x_{n-M+2}) \). While the conserved quantity is constant during the adaptation periods and the relaxation periods, it may change whenever the adaptation is switched on or off.

From Eqs. (1) and (4) we conclude that the dynamics of \( b_n \), as illustrated in Fig. 2, is governed by the mapping function
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FIG. 3. The edge of chaos \( a_c \) as detected by a wavelet filter as a function of length of the support \( M \), for the shift map \( x_n+1 = \text{mod}(a_x_n+r_n) \) (a) and a logistic map \( x_n+1 = a_x x_n(1-x_n) \) (b). The wavelet coefficients are \( g_0 = 1 \), \( g_M = -1 \), and \( g_i = 0 \) for \( i = 1, 2, \ldots, M - 2 \). The threshold is \( \Delta F = 0.002 \). The theoretical values are \( a_c = 3.57 \) for the logistic map and \( a_c = 1 \) for the shift map.

\[ b_n = \begin{cases} b_{n-1} - (-1)^i F(i, N) & \text{if } n = iN, \ i \in \mathbb{N} \\ b_{n-1} & \text{else} \end{cases} \]

where \( F(i, N) \) is the value of the filter function at the beginning of an adaptation period, for \( i = 0, 2, \ldots \) and otherwise a value of filter function at the beginning of a relaxation period. Since we assume that the adaption and relaxation periods are long enough for the system to reach its attractor, the range of the values of the filter function \( F(i, N) \) at the end of each periods depends only on the value of the conserved quantity \( b \) and the size of the feedback \( s \), if there is only one attractor which covers a finite region of the state space. Hence for \( i = 0, 2, \ldots \) we find \( F(i, N) \in [F_{\min}(b_{iN}, s), F_{\max}(b_{iN}, s)] \) and otherwise \( F(i, N) \in [F_{\min}(b_{iN}, 0), F_{\max}(b_{iN}, 0)] \). Whenever adaptation is switched on or off, the conserved quantities changes by a certain amount which is proportional to \( s \)

\[ b_{iN} + s F_{\min}(b_{iN}, s) \leq b_{(i+1)N} \leq b_{iN} + s F_{\max}(b_{iN}, s) \]

if \( i = 0, 2, \ldots \),

\[ b_{iN} - s F_{\max}(b_{iN}, 0) \leq b_{(i+1)N} \leq b_{iN} - s F_{\min}(b_{iN}, 0) \quad \text{else}. \]

Since \( b_n \) is conserved during adaptation periods [see Eq. (6)] and \( M \) is constant, \( a_c \) stays within a small range of order \( s \)

\[ b_{iN} + s F_{\min}(b_{iN}, s) < a_c < b_{iN} + s F_{\max}(b_{iN}, s), \]

where \( iN \leq n < (i+1)N \) and \( i = 1, 3, \ldots \) (see Fig. 1).

In the following, we assume that for \( s = 0 \) the parameter \( a \) of the map dynamics has an edge of chaos at \( a_c \), i.e., there exists a band of width \( \epsilon > 0 \) about \( a_c \) such that when \( a_c - \epsilon \)

\[ \Delta F_n = \begin{cases} 0 & \text{if } a_{iN} < a_c, \ a_n < a_c \text{, where } a_c = 1 \end{cases} \]

and if \( i \in \{0, 2, \ldots \} \). This means that the periodic boundary region is an attractor for the parameter dynamics.

To investigate the statistical properties of the system, we study a large ensemble where the initial parameter values are homogeneously distributed in the interval \( I = [a_{\min}, a_{\max}] \), where \( a_{\min} \) and \( a_{\max} \) are far away from the edge of chaos, so that the boundary has no impact on the parameter dynamics near the edge of chaos. \( P_n(b) \) is probability density of the \( b_n \) values at time \( n \), hence \( P_0 = 1/(a_{\max} - a_{\min}) \). Since \( a_n = b_n \) during the relaxation periods, the probability density of the \( a_n \) values equals the probability density of the \( b_n \) values, i.e., the probability density of the \( a_n \) values is \( P_n(a) \). Next we discuss the change of the \( b_n \) values from relaxation period to relax-

FIG. 4. Typical \( \Delta F_n \)-values versus the value for the parameter \( a \) for a shift map with a Haar wavelet filter. This plot illustrates that \( \Delta F_n = 0 \) if \( a_n < a_c \), where \( a_c = 1 \).

\[ < a_n < a_c \]
The squares indicate numerical values, the continuous lines defined the probability density $P_n$ of the parameter $b$ below the edge of chaos it is constant even during adaptation periods [see Eq. (9)]. Hence, in the periodic regime the probability density $P_n$ of the parameter $b$ stays same or increases at the expense of the probability density in the chaotic regime for each $b$ value

$$P_{(i+2)N} \geq P_{iN} \quad \text{if } b < a_c$$

(10)

for $i \in \{0, 2, \ldots \}$. In the following we show that growth of the probability density occurs mostly in the boundary of the periodic regime. Since the $b_n$ changes only by a small amount given by Eq. (7) whenever adaptation is switched on or off, and otherwise $b_n$ is constant, only parameter values in the vicinity of the edge of chaos can reach the periodic regime during one adaptation period. Systems with parameters further away from the edge of chaos can thus reach the periodic regime only after adaptation has been repeatedly switched on and off.

Systems with $b_n > a_c$ during the relaxation period may have a parameter value below $a_c$ during the adaptation period. Figure 5 shows typical $b$ values during the adaptation period values as a function of the $a$ value during the relaxation period and how this affects $P_n$. If the parameter value is below the edge of chaos it is constant even during adaptation periods [see Eq. (9)]. Hence, in the periodic regime the probability density $P_n$ of the parameter $b$ stays same or increases at the expense of the probability density in the chaotic regime for each $b$ value

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FIG. 5. Typical $b$ values during the adaptation period versus the value for the parameter $a$ during the preceding relaxation period for a shift map with a Haar wavelet filter and $s=0.3$. This plot illustrates that $b_n=a_n$ if $a_n<a_c$, where $a_c=1$.

FIG. 6. The lower bound of the periodic boundary regime $a_{b_n}$. The squares indicate numerical values, the continuous lines defined by Eq. (12) for a logistic map $x_{n+1}=a_n x_n (1-x_n)$ with a Haar wavelet filter, where $a_c=3.449$. The probability density increases in the interval $a_b<a<a_c$.

The minimum $b$ value during the next adaptation period is the lowest $b$ value in the chaotic regime minus the maximum change of the $b$ values when the adaptation starts, i.e., $b_{min} = a_c - F_{max}(a_c,0)$. This assumes that $b$ values which are initially inside the chaotic regime do not have a significantly larger change. Hence the $b$ values which the system can reach from the edge of chaos is lower than the $b$ values that it can reach from inside the chaotic regime, $b_{min} = -F_{max}(a_c,0) < b - F_{max}(b,0)$ for $b > a_c$. This is the case if $F_{max}(b,0)$ does not increase rapidly at the edge of chaos, i.e., $(F_{max}(b,0)-F_{max}(a_c,0))/(b-a_c) \leq 1/s$.

We use the same kind of reasoning to estimate the minimum $b$ value during the following relaxation period. The minimum $b$ value during the following relaxation period is $a_{b} = b_{min} + s F_{min}(b_{min},s)$. This assumes that $b$ values which are initially above $b_{min}$ do not have a significantly smaller change. Hence the $b$ values which the system can reach from $b_{min}$ is lower than the $b$ values that it can reach from $b_{max}$ values that are above $b_{min}$, i.e., $b_{min} + s F_{min}(b_{min},s) \leq b + s F_{min}(b,s)$ for $b > b_{min}$. This is the case if $F_{min}(b,s)$ does not decrease rapidly at the $b_{min}$, i.e., $(F_{min}(b,s)-F_{min}(b_{min},s))/(b-b_{min}) \leq 1/s$.

Next we have to check if $a_b < a_c$. If this is the case then $b$ values that are initially above $a_c$ can end up in the interval $[a_b,a_c]$ within one adaptation/relaxation cycle. Once they are in this interval they are trapped since this interval is in the

FIG. 7. $R_{a,b}(x)=P_{a,b}(x) \Delta x$, the probability of the $x_n$ values in a bin of size $\Delta x=0.04$ of a self-adjusting shift map with a Haar wavelet filter and $s=0.3$. The squares are a histogram of the numerical values of relative class frequencies of the $x$-values determined numerically where the sample size is $N=10^6$.

FIG. 8. Shows the conditional transition probability $T(b|a)/K$ of the conserved quantity at the beginning of the adaptation period and the corresponding numerical class frequencies (squares) for $a=1.1$, sample size $N=10^6$, $s=0.3$, and $K=200$ classes.
periodic regime and their $b$ value does not change from then on. Typically $a_b = a_c - s F_{\text{max}}(a_c,0) + s F_{\min}(b_{\text{min}},s)$ is less than $a_c$ since $F_{\min}(b_{\text{min}},0) = F_{\min}(a_c,0) < F_{\max}(a_c,0)$.

If $a_b < a_c$, then we conclude from Eq. (7) that the probability density of the $b$ values increases in the periodic boundary region and remains constant in the remainder of the periodic regime

$$P_{(i+2)n} = \begin{cases} P_n & \text{if } b = a_c, \\ P_n & \text{if } b < a_c, \\ a_b & \text{if } b < a_b, \\ a_c & \text{if } b > a_c, \\ 0 & \text{if } b > a_c. \\ \end{cases}$$

for $i \in \{0, 2, \ldots\}$, where $a_b = a_c - s F_{\max}(a_c,0) + s F_{\min}(b_{\text{min}},0)$. The periodic boundary region, $a_b \leq b \leq a_c$, is just below $a_c$. If we approximate $F_{\min}(b_{\text{min}},s)$, then $F_{\min}(b_{\text{min}},0) = s F_{\min}(b_{\text{min}},0) + O(s^2)$, we obtain for the lower bound of the periodic boundary region

$$a_b = a_c - s F_{\max}(a_c,0) - F_{\min}(b_{\text{min}},0) - s^2 + O(s^3) \frac{ds}{ds} F_{\min}(b_{\text{min}},0).$$

Figure 6 illustrates the periodic boundary regime for a self-adjusting logistic map dynamics and the parameter range where the class frequencies increase. For the self-adjusting logistic map we find numerically $F_{\max}(a_c,0) = F_{\min}(b_{\text{min}},0)$ and $\frac{ds}{ds} F_{\min}(b_{\text{min}},0) = 0.036$ for $\lambda = 2$. Since the integral of the probability density is equal to one, the increase of the probability density in the periodic boundary region is at the expense of the probability density in the chaotic regime. The migration of the population from the chaotic regime to the periodic boundary region is a concrete model for adaptation to the edge of chaos. The population in the chaotic regime evolves toward a narrow regime near the boundary between order and chaos.

In the following we compute dynamics of the probability density of the parameter values $p_n$ and the dynamics of the probability density of the $x$-values $x_n$ for a specific example, a self-adjusting shift map $x_{n+1} = \text{mod}(a_n x_n + r_n)$ with a Haar wavelet filter, where $a_{\text{min}} = 0$, $a_{\text{max}} = 2$ and $r_n$ is small band limited white noise, $-10^{-7} < r_n < 10^{-7}$. For the self-adjusting shift map, the edge of chaos is $a_c = 1$ and $F_{\max}(a_c,0) = 1$ and $F_{\min}(b_{\text{min}},0) = 0$. Therefore the periodic boundary region is $1 - s < b < 1$.

If $0 \leq b_{\text{in}} < 1$, the dynamics have a fixed point at $x_n = 0$, and the limiting probability density of the $x_n$ values for $s = 0$ is $\rho_{\text{in}} = \delta(x)$, where $\delta$ is the Kronecker’s $\delta$ function. For $b_{\text{in}} > 1$, the dynamics are chaotic. For small $s$, i.e., $0 \leq s \leq 0.3$ and $b_{\text{in}}$ values close to unity, i.e., $1 < b_{\text{in}} < 1.5$, the limiting probability density of the $x$ values can be approximated by

$$\rho_{\text{in}}(x) = \begin{cases} \delta(x) & \text{if } a_c, \\ a_{\text{in}} & \text{if } 0 \leq x < b_{\text{in}} - a_c, \\ d(x) & \text{if } b_{\text{in}} - a_c \leq x \leq 1, \\ 0 & \text{else}, \end{cases}$$

where $d = 1/[1 - \ln(b - a_c)] = a + \beta b + O^2(a_c + 0.25 - b)$, for $a_c = 1$ where $\alpha = (2 \ln 2 - 4)/(1 + 2 \ln 2)$, and $\beta = 4/(1 + 2 \ln 2)^2$. Figure 7 shows a comparison between a histogram of the numerical class frequencies and analytical results in Eq. (13).

Since $b_{i+1} = b_{i} - s F_{i+1} = b_{i} - s x_{i+1}$, the conditional probability $T(b|a)$ that the conserved quantity has the value $b$ during an adaptation period, given that it is the value $a$ during the preceding relaxation period, is $T(b|a) = \rho_{i+1}(x) = (a_i - b)/s$ for $i = 0, 2, \ldots$. Hence we replace $x$ by $(a - b)/s$ in Eq. (13) and obtain

$$T(b|a) = \begin{cases} \delta(a - b) & \text{if } a \leq a_c, \\ \frac{d}{s(a - a_c)} & \text{if } 0 \leq a - b \leq a - a_c, \\ \frac{d}{a - b} & \text{if } a - a_c \leq a - b \leq 1, \\ 0 & \text{else.} \end{cases}$$

Figure 8 shows the conditional probability $T(b|a)$ and the corresponding class frequencies for $a = 1.1$. The probability
of the conserved quantity \( p_{(i+1)N}(b) = \int_{a_{\text{min}}}^{a_{\text{max}}} T(b|a) p_{N}(a) da \). For \( b < a_c \), this integral simplifies to \( p_{(i+1)N}(b) = 1 + \int_{a_{\text{min}}}^{b} T(b|a) p_{N}(a) da \). For the first few adaptation/relaxation cycles we can assume that the probability density in the chaotic regime near the edge of chaos is roughly constant \( p(a) = 1 \) for \( a < a_c + s \). Then \( p_{(i+1)N}(b) = 1 + \int_{a_{\text{min}}}^{b} T(b|a) da \). We use the approximation for \( T \) given in Eq. (14) and evaluate these integrals. We find that for the initial adaptation/relaxation cycles \( i = 0, 2, \ldots, 2(a_{\text{max}} - a_c)/s \), the probability of the conserved quantity is approximately

\[
p_{(i+1)N}(b) = \begin{cases} 
1 & \text{if } a_{\text{min}} \leq b \leq a_c - s \\
\beta(b + s - b_c) + (\beta b + \alpha) \ln \frac{s}{a_c - b} + 1 & \text{if } a_c - s < b < a_c \\
1 - C/s & \text{if } a_c \leq b \leq a_{\text{max}} - is/2 \\
1 & \text{if } a_{\text{max}} - is/2 \leq b \leq a_{\text{max}}, 
\end{cases}
\]

(15)

where \( C = \int_{a_{\text{min}}}^{a_c - s} (\beta(b + s - b_c) + (\beta b + \alpha) \ln \frac{s}{a_c - b}) db = (\beta + \alpha)s + \frac{\alpha^2}{2}i^2 \). During the relaxation periods the probabilities for classes in the periodic regime is the same as during the first adaptation period, \( P_{b_{\text{max}}a_c} = P_{b_{\text{max}}a_c} \) if \( a_c < a_c \). Figure 9 shows the class frequencies for the \( b \) values and the probabilities computed with Eq. (15).

Next we consider the limiting population after many cycles. Equation (15) indicates that the population in the periodic boundary region is increasing at each adaptation/relaxation cycle, until the chaotic regime is depopulated. Therefore we set the probability density in the chaotic regime equal to zero. In addition we normalize \( i \) in the expression for the probability density in the periodic boundary region, by a factor which makes the integral of the probability density equal to one, i.e., \( \int_{i}^{b} p_{(i+1)N}(b) db = 1 \). With these two steps we obtain from Eq. (15)

\[
p(b) = \begin{cases} 
1 & \text{if } a_{\text{min}} \leq b \leq a_c - s \\
\beta(b + s - a_c) + (\beta b + \alpha) \ln \frac{s}{a_c - b} + 1 & \text{if } b_c - s < b < b_c \\
0 & \text{if } a_c \leq b \leq a_{\text{max}}.
\end{cases}
\]

(16)

Equation (16) models the limiting distribution of the population after many adaptation/relaxation cycles. Figure 10 shows the limiting class frequencies for the \( b \) values and the limiting probabilities computed with Eq. (16).

This example illustrates that after many adaptation/relaxation cycles even systems which are initially far away from the edge can migrate into the periodic boundary regime [see Eq. (16) and Fig. 10]. Then the periodic boundary regime is a global attractor for the parameter dynamics. In contrast, for a single adaptation/relaxation cycle only systems very close to the edge of chaos can migrate into the periodic boundary regime due to the conserved quantity [see Eq. (7)]. During each cycle, adaptation to the edge of chaos is a very local phenomenon. The periodic boundary regime is an attractor for the parameter dynamics, but the basin of attraction is small.

### III. CONCLUSION

Many dynamical systems have an edge of chaos in their parameter spaces. On one side of this edge, the dynamics is chaotic for many parameter values, on the other side of the edge it is periodic for all parameter values. Our work shows that discrete-time dynamical systems with wavelet filtered feedback from the dynamical variable to the parameters can be attracted to a narrow parameter range near the edge of chaos, the periodic boundary regime.

We have shown that all self-adjusting map dynamics with wavelet filtered feedback [see Eq. (1)] has a conserved quantity [see Eq. (4)], if the wavelet filter has a finite support and a zero mean. This conserved quantity can be used to eliminate the parameter dynamics from Eq. (1) and to obtain a much simpler mapping function [Eq. (5)]. We used this simpler mapping function to show that the parameter dynamics is bounded during an adaptation/relaxation cycle [see Eqs. (7) and (8)]. Then we showed numerically that subband filters, including wavelets, are typically good chaos detectors, i.e., the filter value is zero or very small for periodic motion, and rather large for chaotic motion. We describe the wavelet’s ability to differentiate between periodicity and chaos with the function \( F_{\text{max}} \) and \( F_{\text{max}} \) and showed that adaptation to a narrow parameter regime near the edge of Chaos occurs [see Eq. (11)], if the wavelet filter is a sufficiently good chaos detector [see Eq. (12)]. Finally we studied one system in detail: adaptation to edge of chaos of shift map with a Haar wavelet filtered feedback. We showed that this theory predicts the dynamics of the probability density [see Eqs. (13)–(16)].
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