1 Implications of No Arbitrage, State Prices, and SDF (Notes 4)

1.1 Case 1: Finite State Space

- Prop 1: Complete markets \( \Rightarrow \varphi \) (state prices) unique
- Prop 2: Incomplete markets \( \Rightarrow \exists \) infinitely-many \( \varphi \) (state prices)
- Theo 1: No arbitrage \( \iff \exists \) strictly positive state price vector \( \varphi \in \mathbb{R}_{++}^s \) s.t. \( P = X\varphi \)
- Corr 1: No arbitrage \( \iff \exists \) strictly positive SDF \( m \in \mathbb{R}_{++}^s \) s.t. \( P = E(mX) \)
- Corr 2: No arbitrage in a complete market \( \iff \exists \) unique and strictly positive state price vector \( \varphi \in \mathbb{R}_{++}^s \) (and SDF \( m \in \mathbb{R}_{++}^s \))
- Corr 3: No arbitrage in an incomplete market \( \iff \exists \) infinitely-many \( \varphi \) (state prices) and SDF \( m \in \mathbb{R}_{++}^s \) (Not everyone has to be str +, but at least one is str + under no arbitrage)

1.2 Case 2: General State Space

- Prop 1: LOOP \( \iff \) Linear Price Functional \( [p(ax + by) = ap(x) + bp(y)] \)
- Prop 2: LOOP \( \iff \) basis vector of random payoffs \( \bar{x} \) is nondegenerate (i.e. non-redundant)
- Prop 3: LOOP \( \implies \exists E(\bar{x}\bar{x}^T)^{-1} \) (i.e. \( E(\bar{x}\bar{x}^T) \) invertible)
- Prop 4: No arbitrage \( \Rightarrow \) LOOP (But LOOP \( \not\Rightarrow \) No arbitrage)
- Theo 1: Under frictionless portfolio formation and additional technical assumptions, there is \( \exists \) ARBITRAGE \( \iff \exists \) a strictly positive \( m \in L^2 \) s.t. \( \forall x \in X, p(x) = E(mX) \)
- Theo 2: Under frictionless portfolio formation, LOOP holds \( \iff \exists \) a SDF \( x^* \in X \) and it is unique. (so that \( \forall x \in X, p(x) = E(x^*x) \))
- Theo 3: Under LOOP (so that \( x^* \) exists), \( m \in L^2 \) is a SDF \( \iff \pi(m|X) = x^* \)

2 Mean Variance Frontier and CAPM (Notes 5)

- Prop 1: For a mean portfolio return of \( \mu_p \), the MV portfolio has variance and portfolio weights given by:
  \[
  \sigma^2(\mu_p) = \frac{A\mu^2 - 2B\mu + C}{D} \quad \text{and} \quad \omega(\mu_p)_{n \times 1} = g + h\mu_p
  \]
  where \( g_{n \times 1} = \frac{C\Sigma^{-1}\bar{1} - B\Sigma^{-1}\bar{E}}{D}, \quad h_{n \times 1} = \frac{A\Sigma^{-1}\bar{E} - B\Sigma^{-1}\bar{1}}{D}, \quad A = \bar{1}^T\Sigma^{-1}\bar{1}, \quad B = \bar{1}^T\Sigma^{-1}\bar{E}, \quad C = \bar{E}^T\Sigma^{-1}\bar{E}, \quad D = AC - B^2 \)
- Prop 2: MV frontier form a convex set. (i.e. if \( \omega_1, \omega_2 \) frontier portfolios with \( \mu_1, \mu_2 \). Then, \( \forall \lambda \in \mathbb{R}, \omega^* = \lambda\omega_1 + (1 - \lambda)\omega_2 \) also frontier.)
- Prop 3: Mean, variance, covariance of the MVP (minimum variance portfolio) is given by:
  \[
  \mu_{mvp} = \frac{B}{A}, \quad \sigma^2_{mvp} = \frac{1}{A} \alpha^2(R_i, R_{mvp}) = \frac{1}{A} = \sigma^2_{mvp} \ \forall i \text{ on frontier}
  \]
- Prop 4: If \( p^T \) is the tangency portfolio, then all securities / portfolios can be priced according to the following beta-representation:
  \[
  E(R_i - R_f) = \frac{\text{Cov}(R_i, R_f)}{\text{Var}(R_f)} [E(R_f^T) - R_f] \quad \forall i
  \]
- Prop 5: If \( \exists \) R.F. asset, then ANY portfolio \( p \) on the MV frontier (other than the r.f. asset) can be used to price other assets/portfolios according to the following beta-representation:
  \[
  E(R_i - R_f) = \frac{\text{Cov}(R_i, R_p)}{\text{Var}(R_p)} [E(R_p) - R_f] \quad \forall i
  \]
3 CAPM (The beta-rep model with market portfolio as factor. Or, the model with SDF $m = a + bR_m$) (Notes 6)

- Theo 1: (CAPM with RF) Suppose $\exists$ a R.F. asset, and suppose CAPM assumptions hold (agents optimize, have full information, and no trading frictions) $\Rightarrow$ in equilibrium the market portfolio is the tangency portfolio.

So, $E(R_i) - R_f = \frac{Cov(R_i, R_m)}{Var(R_m)} [E(R_m) - R_f] \ \forall i$

- Theo 2: (Zero-Beta CAPM) If no R.F. asset, under CAPM assumptions $\Rightarrow$ in equilibrium the market portfolio is on the MV frontier and

$E(R_i) - R_{zc} = \frac{Cov(R_i, R_m)}{Var(R_m)} [E(R_m) - R_{zc}] \ \forall i$

4 SDF, Beta Representation, and MV Frontier (Notes 7)

4.1 Beta Representation of Linear Factor Pricing Models

$(E(R_i) = \gamma + \beta^T\lambda_k$ where $\beta : R_i = a_i + \beta^T f_k + \varepsilon_i$)

- Prop 1: If a factor $f_k$ is a gross return on a portfolio (e.g. $R_m$ in CAPM) and $\exists$ R.F. asset $\Rightarrow$ $\lambda_k$ should always be a risk-premium on that portfolio (i.e. $\lambda_k = E(R_m) - R_f$)

- Prop 2: If a factor $f_k$ is an excess return (e.g. $R_H - R_L$ in FF3) $\Rightarrow$ $\lambda_k$ is always the expected excess return (i.e. $\lambda_k = E(f_k)$)

4.2 SDF and Beta Representation ($m = a + b^T f \iff$ Beta Representation with $f$ as factor)

- Theo 1: $\exists a \in \mathbb{R} \& b \in \mathbb{R}^k$ s.t. $m = a + b^T f$ is a SDF $\iff \exists \gamma \in \mathbb{R} \& \lambda \in \mathbb{R}^k$ s.t. $\forall$ securities/portfolios $i$, $E(R_i) = \gamma + \beta_i^T \lambda \forall i$

where $\beta_i \equiv \Sigma_f^{-1} Cov(\hat{f}, R_i)$ is the reg coeff of $R_i$ on $f$ and a constant

- Prop 3: $m = a + b^T f \Rightarrow$ in beta representation, $\lambda = E(\hat{f}) - \gamma E(m\hat{f})$ (where $\gamma = \frac{1}{E(m)}$)

4.3 Single-Factor Special Case: CAPM ($CAPM \iff m = a + bR_m$)

- Prop 4: CAPM $\iff m = a_m + b_m R_m$ for some $a_m, b_m \in \mathbb{R}$

Moreover,

if $\exists$ R.F. asset, $a_m = \frac{1}{R_f} + \frac{1}{R_f} \frac{E(R_m)(E R_m - R_f)}{Var(R_m)}$ and $b_m = -\frac{1}{R_f} \frac{E R_m - R_f}{Var(R_m)}$

if no R.F. asset, $a_m = \frac{1}{R_{zc}} + \frac{1}{R_{zc}} \frac{E(R_m)(E R_m - R_{zc})}{Var(R_m)}$ and $b_m = -\frac{1}{R_{zc}} \frac{E R_m - R_{zc}}{Var(R_m)}$

- Corr 1: (Complete Mkt + CAPM don’t mix!) Suppose market is complete. If CAPM holds and market portfolio return distribution is normal (or has large enough support) $\Rightarrow \exists$ arbitrage opportunities.
4.4 Single-Factor Special Case: SDF as Factor

- Prop 5: Any valid SDF $m$ yields a beta representation with $m$ as factor.
  
  \[ i.e. \quad E(R_i) = \frac{1}{E(m)} \times \frac{\text{Var}(m)}{\lambda} - \frac{\text{Cov}(m, R_i)}{\beta} \]

- Prop 6: Given any valid SDF $m$ and under LOOP (so that $x^*$ exists). We can use $m$-mimicking payoff $x^* \equiv \pi(m \mid X)$ as factor to yield a beta representation.

  \[ i.e. \quad E(R_i) = \frac{1}{E(x^*)} + \frac{-\text{Var}(x^*) \cdot \text{Cov}(x^*, R_i)}{E(x^*) \cdot \text{Var}(x^*)} \]

- Prop 7: Given SDF $m$ and under LOOP (so that $x^*$ exists). Then, \( \exists \) beta representation with \( R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^*)} \) as factor.

  \[ i.e. \quad E(R_i) = \frac{E(R^* \mid x^*)}{E(R^*)} - \frac{-\text{Var}(R^*) \cdot \text{Cov}(R^*, R_i)}{E(R^*) \cdot \text{Var}(R^*)} \]

4.5 MV and an Orthogonal Characterization of the MV Frontier (Under LOOP so that $x^*$ exists)

- Prop 8: (R* on MV frontier) \( R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^*)} \) is on the MV frontier (bc \( E(R_i) - \gamma = \beta_i, R^* = [ER^* - \gamma] \))

- Prop 9: (Any frontier portf can be a factor) \( \exists a, b \in \mathbb{R} \) and a return \( R_p \) s.t. \( m = a + bR_p \) is a SDF \( \iff R_p \) is on the MV frontier

- Prop 10: (Representation of $R^* \equiv \pi(1\mid R^*)$)
  
  a) If \( \exists R_f \), then \( R^* = 1 - \frac{R^*}{R_f} \)
  
  b) If no \( R_f \), then \( R^* \equiv \pi(1\mid X) - \frac{E(R^*)}{E(R^*)} R^* \) where \( \pi(1\mid X) \equiv E(\bar{x}^T) E(\bar{x}\bar{x}^T)^{-1} \bar{x} \)

- Prop 11: Every $R_i$ can be expressed as
  \[ R_i = R^* + \omega_i R^* + \eta_i \]
  
  where \( \omega \in \mathbb{R}, \eta_i \) an excess return with \( E(\eta_i) = 0 \), and \( E(R^* R^*) = E(R^* \eta_i) = E(R^* \eta_i^2) = 0 \)

- Corr 2: $R^{mv}$ is on the MV frontier \( \iff R^{mv} = R^* + \omega R^* \) for some $\omega$

- Prop 12: Under LOOP (so that $x^*$ exists), $R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^*)}$ is the MV frontier portfolio with the smallest second moment.

5 Arbitrage Pricing Theory (APT) (Notes 8)

- Prop 1 (APT): Suppose all $N$ security returns are described by the following statistical factor decomposition

  \[ R_i = a_i + \bar{f}^T \beta_i + \epsilon_i \quad \forall i \quad (1) \]

  where \( E(\epsilon_i) = E(\epsilon_i \bar{f}_k) = \text{Cov}(\epsilon_i, \bar{f}_k) = 0 \quad \forall i, \forall k \quad (2) \)

  and \( E(\epsilon_i \epsilon_j) = \sigma^2_i \leq \sigma^2 \) if $i=j$ and 0 o.w. \( (3) \)

  or in a system form... \( R_{N\times1} = A_{N\times1} + B_{N\timesK} f_{K\times1} + E_{N\times1} \quad (1') \)

  \[ E(E) = E(\bar{E} f) = 0 \quad (2') \]

  \[ \sum_{N\times N} \equiv E(EE^T) \text{ is diagonal with } + \text{ and bdd entries } (3') \]
Then, "arbitrage considerations" alone ⇒ expected returns of securities "approximately" satisfy beta representation with factors $\tilde{f}$

\[ i.e. E(R_i) \equiv \gamma + \lambda^T \beta_i \quad (4) \]

Moreover, if $\exists$ R.F. asset and $\tilde{f}$ are returns, then

\[ E(R_i) \equiv R_f + \beta_i^T [E(\tilde{f}) - R_f] \quad \forall i \quad (5) \]

- Ex 1: (Exact pricing with $\epsilon_i = 0$) When $\epsilon_i = 0$, under APT / no arbitrage, $E(R_i) = R_f = \beta_i^T (E(f) - R_f \tilde{f})$

- Ex 2: (Approx pricing with $\epsilon_i \neq 0$) When $\epsilon_i \neq 0$, under APT / no arbitrage, $E(R_i) - R_f = \beta_i^T (E(f) - R_f \tilde{f}) - R_f Cov(m, \epsilon_i) \equiv \beta_i^T (E(f) - R_f \tilde{f})$

- Prop 2: (Exact beta-rep at the limit) For a fixed SDF $m$ that prices the factors $\tilde{f}$, as $Var(\epsilon_i) \to 0$ for a security/portfolio $i \Rightarrow E(R_i) \to R_f + \beta_i^T (E(f) - R_f \tilde{f})$

- Corr 1: (Exact beta-rep at the limit for well-div portfolio) For a fixed SDF $m$ that prices the factors $\tilde{f}$, consider a portfolio $p$ with equilibrium weights in $n$ securities. As $n \to \infty$, $E(R_p) \to R_f + \beta_i^T (E(f) - R_f \tilde{f})$

- Prop 3: (Approx APT Pricing for all but finite securities) For a fixed SDF $m$ that prices the factors $\tilde{f}$, $\forall \delta > 0$ small, $\exists$ only finitely many securities s.t.

\[ |E(R_i) - R_f - \beta_i^T [E(\tilde{f}) - R_f]| > \delta \]

6 Option Pricing (Notes 11)

6.1 13 Important Properties of Options

We can derive the following with the following 2 no-arbitrage principles: 1) Portfolios with non-neg payoff has non-neg cost. 2) If one portfolio’s payoff dominates another then so must its cost.

1. $c, C, p, P \geq 0$ for all parameter values
2. $C(S, t; E, T) \geq S(t) - E$ and $P(S, t; E, T) \geq E - S(t)$
3. For $T_2 > T_1$, $C(S, t; E, T_2) \geq C(S, t; E, T_1)$ and $P(S, t; E, T_2) \geq P(S, t; E, T_1)$
4. $C(\cdot) \geq c(\cdot)$ and $P(\cdot) \geq p(\cdot)$ given same input values
5. For $E_1 > E_2$, all else equal,

\[ C(\cdot; E_1) \leq C(\cdot; E_2) \]
\[ c(\cdot; E_1) \leq c(\cdot; E_2) \]
\[ P(\cdot; E_1) \geq P(\cdot; E_2) \]
\[ p(\cdot; E_1) \geq p(\cdot; E_2) \]

6. $S(t) \equiv (S, t; 0, \infty) \geq C(S, t; E, T) \geq c(S, t; E, T)$
7. $C(0, t; E, T) = c(0, t; 0, \infty) = 0$
8. If the stock pays no dividends between $t$ and $T$, then $c(S, t; E, T) \geq S - E \cdot B(t, T)$ where $B(t, T)$ is the time $t$ PV of receiving $\$1$ at $T$.
9. European call options are convex functions of $E$: i.e. $\alpha c(\cdot; E_1) + (1 - \alpha) c(\cdot; E_2) \geq c(\cdot; \alpha E_1 + (1 - \alpha) E_2)$
10. "A portfolio of options is more valuable than an option on a portfolio" (i.e. call option on a portfolio of securities w/ an exercise price of $X$ is less valuable than a portfolio of call options on the underlying with equal exercise)
11. Put-Call Parity (0 dividend stock): $p(S, t; E, T) = c(S, t; E, T) - S(t) + E \cdot B(t, T)$
12. European put option price is convex in $E$: i.e. $\alpha p(\cdot; E_1) + (1 - \alpha) p(\cdot; E_2) \geq p(\cdot; \alpha E_1 + (1 - \alpha) E_2)$
13. For a stock paying no dividends over $(t, T)$, an American call will never be exercised early. So, $C = c$ when $S$ pays no dividends.
7 Binomial Pricing

- We can replicate the derivatives by holding a portfolio of the underlying stock and cash bond \((N, B)\).

\[
N = \frac{f_u - f_d}{S_u - S_d} \quad \text{and} \quad B = f_{\text{now}} - N \cdot S_{\text{now}}
\]

- We can also use risk-neutral pricing

\[
f_{\text{now}} = E_Q \left[ \frac{1}{R} f_{\text{tomor}} \right] = \frac{1}{R} \{ qf_u + (1 - q) f_d \}
\]

where \( q = \frac{RS_{\text{now}} - S_d}{S_u - S_d} \)