ON SHARP BURKHOLDER–ROSENTHAL-TYPE INEQUALITIES FOR INFINITE-DEGREE U-STATISTICS

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ABSTRACT. – In this paper, we present a method that allows one to obtain a number of sharp inequalities for expectations of functions of infinite-degree U-statistics. Using the approach, we prove, in particular, the following result: Let $D$ be the class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that the function $f(x + z) - f(x)$ is concave in $x \in \mathbb{R}_+$ for all $z \in \mathbb{R}_+$. Then the following estimate holds:

$$Ef\left(\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l})\right) \leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} Ef\left(\sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n \setminus \{j_1, \ldots, j_q\}} E\left(Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l}) \mid X_{j_1}, \ldots, X_{j_q}\right)\right)$$

for all $f \in D$ and all U-statistics $\sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l})$ with nonnegative kernels $Y_{i_1, \ldots, i_l} : \mathbb{R}^l \to \mathbb{R}_+$, $1 \leq i_k \leq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m$, in independent r.v.’s $X_1, \ldots, X_n$. Similar inequality holds for sums of decoupled U-statistics. The class $D$ is quite wide and includes all nonnegative twice differentiable functions $f$ such that the function $f''(x)$ is nonincreasing in $x > 0$, and, in particular, the power functions $f(x) = x^t$, $1 < t \leq 2$; the power functions multiplied by logarithm $f(x) = (x + x_0)^t \ln(x + x_0)$, $1 < t < 2$, $x_0 \geq \max\left(e^{(3t^2 - 6t + 2)/(t(t-1)(2-t)}), 1\right)$; and the entropy-type functions $f(x) = (x + x_0) \ln(x + x_0)$, $x_0 \geq 1$. As an application of the results, we determine the best constants in Burkholder–Rosenthal-type inequalities for sums of U-statistics and prove new decoupling inequalities for U-statistics.

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those objects. The results obtained in the paper are, to our knowledge, the first known results on the best constants in sharp moment estimates for $U$-statistics of a general type.

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1. Introduction

particular, the following Burkholder–Rosenthal-type inequality:

In this paper, we present a method that allows one to obtain sharp inequalities for expectations of sums of multilinear forms in independent nonnegative and symmetric r.v.'s. de la Peña et al. [10] found the best constants in Burkholder–Rosenthal-type inequalities for bilinear forms in the case of the fixed number of random variables (r.v.’s). de la Peña et al. [2] determined the best constants in Burkholder–Rosenthal-type inequalities for sums of multilinear forms in independent nonnegative and symmetric r.v.’s.

In this paper, we present a method that allows one to obtain sharp inequalities for expectations of sums of multilinear forms in independent nonnegative and symmetric r.v.’s.

In particular, for the power functions $f(x)$ by logarithm

In the Burkholder–Rosenthal-type estimates for $U$-statistics the following inequalities

In particular, the following Burkholder–Rosenthal-type inequality:

$$Ef\left(\sum_{l=0}^{m} \sum_{1 \leq i_{1} \prec \cdots \prec i_{l} \leq n} Y_{i_{1}, \ldots, i_{l}}(X_{i_{1}}, \ldots, X_{i_{l}})\right)$$

$$\leq \sum_{q=0}^{m} \sum_{1 \leq j_{1} \prec \cdots \prec j_{q} \leq n} Ef\left(\sum_{l=q}^{m} \sum_{i_{1} \prec \cdots \prec i_{l-q} \in [1, \ldots, n] \setminus \{j_{1}, \ldots, j_{q}\}} E\left(Y_{j_{1}, \ldots, j_{q}, i_{1}, \ldots, i_{l-q}}(X_{j_{1}}, \ldots, X_{j_{q}}, X_{i_{1}}, \ldots, X_{i_{l-q}}) \mid X_{j_{1}}, \ldots, X_{j_{q}}\right)\right)$$

for all $U$-statistics $\sum_{1 \leq i_{1} \prec \cdots \prec i_{l} \leq n} Y_{i_{1}, \ldots, i_{l}}(X_{i_{1}}, \ldots, X_{i_{l}})$ with nonnegative kernels $Y_{i_{1}, \ldots, i_{l}} : \mathbb{R}^{l} \to \mathbb{R}_{+}$, $1 \leq i_{k} \leq n$; $i_{t} \not= i_{s}$, $r \not= s$; $k$, $r$, $s = 1, \ldots, l$; $l = 0, \ldots, m$ $(Y_{1, \ldots, 1} \equiv \text{const} \geq 0$ for $l = 0)$ in independent r.v.'s $X_{1}, \ldots, X_{n}$ and all functions $f : \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that the function $f(x + z) - f(x)$ is concave in $x \in \mathbb{R}_{+}$ for all $z \in \mathbb{R}_{+}$. A similar inequality holds for sums of decoupled $U$-statistics. The above condition is satisfied for all twice differentiable functions $f$ such that the function $f''(x)$ is nonincreasing in $x > 0$, and, in particular, for the power functions $f(x) = x^{t}$, $1 < t \leq 2$; the power functions multiplied by logarithm $f(x) = (x + x_{0})^{t} \ln(x + x_{0})$, $1 < t < 2$, $x_{0} \geq \max(e^{(2t^{-1}d^{-1} - 1/2) / (2-1)} - 1)$; and the entropy-type functions $f(x) = (x + x_{0})\ln(x + x_{0})$, $x_{0} \geq 1$. As an application of the results, we determine the best constants in Burkholder–Rosenthal-type inequalities for sums of regular and decoupled $U$-statistics with nonnegative kernels and prove new decoupling inequalities for sums of $U$-statistics. We show, for instance, that the constant in the following Burkholder–Rosenthal-type inequality is sharp:

$$E\left(\sum_{l=0}^{m} \sum_{1 \leq i_{1} \prec \cdots \prec i_{l} \leq n} Y_{i_{1}, \ldots, i_{l}}(X_{i_{1}}, \ldots, X_{i_{l}})\right)^{t}$$

$$\leq \sum_{q=0}^{m} \sum_{1 \leq j_{1} \prec \cdots \prec j_{q} \leq n} E\left(\sum_{l=q}^{m} \sum_{i_{1} \prec \cdots \prec i_{l-q} \in [1, \ldots, n] \setminus \{j_{1}, \ldots, j_{q}\}} E\left(Y_{j_{1}, \ldots, j_{q}, i_{1}, \ldots, i_{l-q}}(X_{j_{1}}, \ldots, X_{j_{q}}, X_{i_{1}}, \ldots, X_{i_{l-q}}) \mid X_{j_{1}}, \ldots, X_{j_{q}}\right)\right)^{t}$$

for all $U$-statistics $\sum_{1 \leq i_{1} \prec \cdots \prec i_{l} \leq n} Y_{i_{1}, \ldots, i_{l}}(X_{i_{1}}, \ldots, X_{i_{l}})$ with nonnegative kernels $Y_{i_{1}, \ldots, i_{l}} : \mathbb{R}^{l} \to \mathbb{R}_{+}$, $1 \leq i_{k} \leq n$; $i_{t} \not= i_{s}$, $r \not= s$; $k$, $r$, $s = 1, \ldots, l$; $l = 0, \ldots, m$, in
independent r.v.'s \(X_1, \ldots, X_n\). A similar result holds for sums of decoupled \(U\)-statistics. To our knowledge, the results obtained in the paper are the first known results on the best constants in sharp two-sided moment estimates for \(U\)-statistics of a general type.

2. Sharp estimates for expectations of functions of sums of \(U\)-statistics

Let \(R_+ = [0, \infty)\), \(1 \leq m \leq n\), \(X_1, \ldots, X_n, X_{p1}, \ldots, X_{pn}\), \(p = 1, \ldots, m\), be independent r.v.'s and let \(Y_{i_1, \ldots, i_l} : R^l \to R_+\), \(1 \leq i_k \leq n\); \(i_r \neq i_s\); \(k, r, s = 1, \ldots, l\); \(l = 0, \ldots, m\), be functions having the property that \(Y_{i_1, \ldots, i_l}(x_1, \ldots, x_l) = Y_{s(i_1), \ldots, s(i_l)}(x_{\pi(1)}, \ldots, x_{\pi(l)})\), \(x_k \in R\), \(k = 1, \ldots, l\), \(1 \leq i_1 \leq \cdots \leq i_l \leq n\), for all permutations \(\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}\), \(l = 2, \ldots, m\) (we assume that \(Y_{i_1, \ldots, i_l} \equiv \text{const} \geq 0\) for \(l = 0\)). Consider the sums of regular \(U\)-statistics (symmetric statistics)

\[
\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l})
\]

and decoupled \(U\)-statistics (symmetric statistics)

\[
\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l}).
\]

In what follows, write

\[
Y^{\text{reg}}(i_1, \ldots, i_l) = Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l}),
\]

\[
Y^{\text{dec}}(i_1, \ldots, i_l) = Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l}).
\]

Denote by \(D\) the class of functions \(f : R_+ \to R_+\) such that the function \(f(x + z) - f(x)\) is concave in \(x \in R_+\) for all \(z \in R_+\). The class \(D\) is quite wide and includes all nonnegative twice differentiable functions \(f\) such that the function \(f''(x)\) is nonincreasing in \(x > 0\) and, in particular, the power functions \(f_1(x) = x^t\), \(1 < t \leq 2\); the power functions multiplied by logarithm \(f_2(x) = (x + x_0)^t \ln(x + x_0)\), \(1 < t < 2\), \(x_0 \geq \max(e^{(t^2 - 6t + 2)/(t(t-1)(2-t))}, 1)\), and the entropy-type functions \(f_3(x) = (x + x_0) \ln(x + x_0)\), \(x_0 \geq 1\). Indeed, if the function \(f''(x)\) is nonincreasing in \(x > 0\), then we have \(f''(x + z) \leq f''(x)\) for all \(x > 0\), \(z \geq 0\), and, therefore, \(f(x + z) - f(x)\) is concave in \(x \in R_+\) for all \(z \in R_+\). It is obvious that \(f''_1(x)\) is nonincreasing in \(x > 0\) and, therefore, \(f_1 \in D\). In addition to that, \(f_{2}''(x) = (x + x_0)^{t-3}((t(t-1)(t-2)) \ln(x + x_0) + 3t^2 - 6t + 2) \leq 0\), \(x > 0\), and, therefore, \(f_2(x)\) is nonincreasing in \(x > 0\), and \(f_2 \in D\). Since \(f_3''(x) = 1/(x + x_0)\) is nonincreasing in \(x > 0\), we have \(f_3 \in D\).

In the inequalities throughout the paper, the extremal cases of the estimates such as \(+\infty \leq +\infty\) are considered to be valid inequalities; we, therefore, do not include assumptions on finiteness of moments of the summand r.v.'s that ensure finiteness of moments of sums of \(U\)-statistics into formulations of the results.

The following theorems give sharp Burkholder–Rosenthal-type inequalities for sums of \(U\)-statistics. In what follows, \(E(\cdot \mid X_{j_1}, \ldots, X_{j_q}) = E(\cdot \mid X_{j_1}, i_{j_1}, \ldots, X_{j_q}, i_{j_q}) = E(\cdot)\), the unconditional expectation operator, for \(q = 0\).
THEOREM 1. — For \( f \in D \),
\[
Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{reg}}(i_1, \ldots, i_l) \right)
\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} Ef \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_l \notin \{j_1, \ldots, j_q\}} E\left( Y^{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right).
\] (1)

THEOREM 2. — For \( f \in D \),
\[
Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{dec}}(i_1, \ldots, i_l) \right)
\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} Ef \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_l \notin \{j_1, \ldots, j_q\}} E\left( Y^{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1}, \ldots, X_{j_q}, X_{i_1}, \ldots, X_{i_{l-q}} \right) \right).
\] (2)

COROLLARY 1. — For a twice differentiable function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that the function \( f''(x) \) is nonincreasing on \( x > 0 \), inequalities (1) and (2) hold.

THEOREM 3. — The constants in the following inequalities are sharp:
\[
E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{reg}}(i_1, \ldots, i_l) \right)^t
\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E\left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_l \notin \{j_1, \ldots, j_q\}} E\left( Y^{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t,
\] (3)

\[
E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{dec}}(i_1, \ldots, i_l) \right)^t
\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E\left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_l \notin \{j_1, \ldots, j_q\}} E\left( Y^{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1}, \ldots, X_{j_q}, X_{i_1}, \ldots, X_{i_{l-q}} \right) \right)^t,
\] (4)

Remark 1. — It is not difficult to see that moment inequalities (2) and (4) for sums of decoupled \( U \)-statistics follow from their counter-parts (1) and (3) for sums of
regular $U$-statistics, using the fact that any decoupled $U$-statistic can be represented as an undecoupled $U$-statistic with many zero kernels (it suffices to consider new r.v.’s $\tilde{X}_{(p-1)n+i} = X_{p+1}$ for $1 \leq i \leq n$, and new kernels $\tilde{Y}_{i_1,n+i_2,...,(l-1)n+i_l} = Y_{i_1,i_2,...,i_l}$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, ..., m$; $\tilde{Y}_{j_1,j_2,...,j_l} = 0$, $1 \leq j_1 < j_2 < \cdots < j_l \leq mn$, $(j_1, j_2, ..., j_l) \neq (i_1, n + i_2, ..., (l-1)n + i_l)$ for $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 1, ..., m$). In the case of, let us say, sums of multilinear forms the terms in the bounds depend only on the moments of individual terms is significant, as it was shown in [6], see also [5]). In the case of, let us say, sums in terms of expressions that do not contain moments of r.v.’s as an undecoupled $U$-statistic contain only directly computable expressions. For example, in the case of regular $U$-statistics of order $m$ in identically distributed r.v.’s the bounds consist of terms equivalent to $n^{(m-k)r+k}E(Y_{ij}(X_1, ..., X_n) | X_1, ..., X_k)^t$, $k = 0, 1, ..., m$ (and each of the terms is significant, as it was shown in [6], see also [5]). In the case of, let us say, sums of multilinear forms the terms in the bounds depend only on the moments of individual variables (see also [2]).

**Remark 2.** The essence of the Burkholder–Rosenthal-type bounds for sums of $U$-statistics given by Theorems 1–3 is that they give (sharp) estimates for moments of the sums in terms of expressions that do not contain moments of *sums* of r.v.’s. The bounds contain only directly computable expressions. For example, in the case of regular $U$-statistics of order $m$ in identically distributed r.v.’s the bounds consist of terms equivalent to $n^{(m-k)r+k}E(Y_{ij}(X_1, ..., X_n) | X_1, ..., X_k)^t$, $k = 0, 1, ..., m$ (and each of the terms is significant, as it was shown in [6], see also [5]). In the case of, let us say, sums of multilinear forms the terms in the bounds depend only on the moments of individual variables (see also [2]).

**Remark 3.** From the results obtained in [5–9,11] (see also [3]) it follows that the following non-sharp (in the sense of constants) Burkholder–Rosenthal-type inequality holds for regular $U$-statistics of second order with nonnegative kernels (below, $C_i(t)$, $C_i^{\text{dec}}(t)$ and $C_i^{\text{reg}}(t)$ are constants depending on $t$ only):

$$E \left( \sum_{1 \leq i < j \leq n} Y_{ij}^{\text{reg}}(X_i, X_j) \right)^t \leq C_1(t) \sum_{1 \leq i < j \leq n} E \left( Y_{ij}^{\text{reg}}(X_i, X_j) \right)^t$$

$$+ C_2(t) \sum_{i=1}^{n-1} E \left( \sum_{j=i+1}^{n} E \left( Y_{ij}^{\text{reg}}(X_i, X_j) | X_i \right) \right)^t$$

$$+ C_3(t) \sum_{j=2}^{n} E \left( \sum_{i=1}^{j-1} E \left( Y_{ij}^{\text{reg}}(X_i, X_j) | X_j \right) \right)^t$$

$$+ C_4(t) \left( \sum_{1 \leq i < j \leq n} EY_{ij}^{\text{reg}}(X_i, X_j) \right)^t, \quad t > 1.$$
Moreover, the best constants in the inequality are given by $C_i^{\text{reg}}(t) = 1$, $i = 1, 2, 3$, for $1 < t \leq 2$. Similarly, from (4) it follows that a “natural” form of Burkholder–Rosenthal-type inequality for decoupled $U$-statistics of second order with nonnegative kernels contains four terms similar to those in [8], namely,

$$E\left( \sum_{1 \leq i < j \leq n} Y_{ij}^{\text{dec}}(X_{1i}, X_{2j}) \right)^t \leq C_1^{\text{dec}}(t) \sum_{1 \leq i < j \leq n} E\left( Y_{ij}^{\text{dec}}(X_{1i}, X_{2j}) \right)^t,$$

$$+ C_2^{\text{dec}}(t) \sum_{i=1}^{n-1} E \left( \sum_{j=i+1}^{n} E\left( Y_{ij}^{\text{dec}}(X_{1i}, X_{2j}) \mid X_{1i} \right) \right)^t,$$

$$+ C_3^{\text{dec}}(t) \sum_{j=2}^{n} E \left( \sum_{i=1}^{j-1} E\left( Y_{ij}^{\text{dec}}(X_{1i}, X_{2j}) \mid X_{2j} \right) \right)^t,$$

$$+ C_4^{\text{dec}}(t) \left( \sum_{1 \leq i < j \leq n} E Y_{ij}^{\text{reg}}(X_{1i}, X_{2j}) \right)^t,$$  

and, moreover, the best constants in the above inequality are given by $C_i^{\text{dec}}(t) = 1$, $i = 1, 2, 3, 4$, for $1 < t \leq 2$.

**Remark 4.** Similarly to Remark 3, from moment inequalities for sums of multilinear forms obtained by Peña et al. [2] and Theorems 1–3 it follows that a “natural” form of Burkholder–Rosenthal-type inequalities for expectations of functions of sums of regular $U$-statistics of order not greater than $m$ with nonnegative kernels contains $m + 1$ terms and a “natural” form of Burkholder–Rosenthal-type inequalities for expectations of functions of sums of decoupled $U$-statistics of order not greater than $m$ with nonnegative kernels contains $2^m$ terms. Moreover, those theorems imply the following inequalities:

$$E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{reg}}(i_1, \ldots, i_l) \right)^t \leq (m + 1) \max_{q=0, \ldots, m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_{l-q} \leq [1, \ldots, n] \setminus \{j_1, \ldots, j_q\}} E\left( Y^{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t,$$

$$E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{dec}}(i_1, \ldots, i_l) \right)^t \leq 2^m \max_{q=0, \ldots, m} \max_{1 \leq j_1 < \cdots < j_q \leq m} \sum_{1 \leq i_1 < \cdots < i_{l-q} \leq n} \sum_{l=q}^{m} \sum_{p, p_1 < p_2, p_1 < j_1, p_2 < j_2} E \left( Y^{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1, i_1}, \ldots, X_{j_q, i_{l-q}} \right)^t,$$

for $1 < t \leq 2$. 

$$E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{reg}}(i_1, \ldots, i_l) \right)^t \leq (m + 1) \max_{q=0, \ldots, m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_{l-q} \leq [1, \ldots, n] \setminus \{j_1, \ldots, j_q\}} E\left( Y^{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t,$$

$$E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{dec}}(i_1, \ldots, i_l) \right)^t \leq 2^m \max_{q=0, \ldots, m} \max_{1 \leq j_1 < \cdots < j_q \leq m} \sum_{1 \leq i_1 < \cdots < i_{l-q} \leq n} \sum_{l=q}^{m} \sum_{p, p_1 < p_2, p_1 < j_1, p_2 < j_2} E \left( Y^{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1, i_1}, \ldots, X_{j_q, i_{l-q}} \right)^t,$$

for $1 < t \leq 2$. 

$$E\left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y^{\text{reg}}(i_1, \ldots, i_l) \right)^t \leq (m + 1) \max_{q=0, \ldots, m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_{l-q} \leq [1, \ldots, n] \setminus \{j_1, \ldots, j_q\}} E\left( Y^{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t,$$
From the estimate
\[ \sum_{k=1}^{N} z_k^t \leq \left( \sum_{k=1}^{N} z_k \right)^t, \quad z_1, \ldots, z_N \geq 0, \quad t > 1, \quad (5) \]
and Jensen’s inequality it follows that
\[ E \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{reg}}(i_1, \ldots, i_l) \right)^t \]
\[ \geq \max_{q=0, \ldots, m} \sum_{1 \leq j_1 < \cdots < j_q \leq m} E \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} E \left( Y_{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t, \quad (6) \]
\[ E \left( \sum_{l=0}^{m} \sum_{1 \leq i_j \leq n} Y_{\text{dec}}(i_1, \ldots, i_l) \right)^t \]
\[ \geq \max_{q=0, \ldots, m} \sum_{1 \leq j_1 < \cdots < j_q \leq m} E \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} E \left( Y_{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1}, \ldots, X_{j_q} \right) \right)^t, \quad (7) \]
1 < t \leq 2. Assume that \( X_{p1}', \ldots, X_{pn}', \ p = 1, \ldots, m, \) are independent copies of the r.v.’s \( X_1, \ldots, X_n \) (the primes are used to remind us about the independence between the sequences). From estimate (5), the inequality \( (\sum_{k=1}^{N} z_k)^t \leq N^{-1} \sum_{k=1}^{N} z_k, \quad z_1, \ldots, z_N \geq 0, \quad t > 1, \) and estimates (3), (4), (6) and (7) it follows that the following theorem holds
\( (C_m^k) \leq 2^m. \) As far as we know, the constants in the estimates in Theorem 4 are the best available so far, and it is likely that they are the sharp ones.

**Theorem 4.** – The following decoupling inequalities hold:
\[
(m+1)^{-1} E \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l} (X_{i_1,j_1}', \ldots, X_{i_l,j_l}') \right)^t \]
\[
\leq E \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l} (X_{i_1}, \ldots, X_{i_l}) \right)^t \]
\[
\leq \left( \sum_{k=0}^{m} (C_m^k) \right) E \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l} (X_{i_1,j_1}', \ldots, X_{i_l,j_l}') \right)^t, \quad 1 < t \leq 2. \]
Similarly, the estimate
\[ \sum_{k=1}^{N} f(z_k) \leq f \left( \sum_{k=1}^{N} z_k \right), \quad z_1, \ldots, z_N \geq 0 \]  
(8)
for all convex functions \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( f(0) = 0 \) and Jensen’s inequality imply that
\[
E_f \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{reg}}(i_1, \ldots, i_l) \right) \\
\geq \max_{q=0, \ldots, m} \max_{1 \leq j_1 < \cdots < j_q \leq n} E_f \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q} \right),
\]
(9)
\[
E_f \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{dec}}(i_1, \ldots, i_l) \right) \\
\geq \max_{q=0, \ldots, m} \max_{1 \leq j_1 < \cdots < j_q \leq m} \max_{p, p_1, p_2=1, \ldots, l, p \neq j_1, \ldots, j_q} E_f \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{dec}}(i_1, \ldots, i_l) \mid X_{j_1}, \ldots, X_{j_q}, X_{i_1}, \ldots, X_{i_l} \right),
\]
(10)
for all convex functions \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( f(0) = 0 \). From (8), the inequality
\[ f(\sum_{k=1}^{N} z_k) \leq \frac{1}{N-1} \sum_{k=1}^{N} f(Nz_k), \quad z_1, \ldots, z_N \geq 0, \]
for all convex functions \( f \in D \) with \( f(0) = 0 \).

Theorem 5. – The following decoupling inequalities hold:
\[
(m+1)^{-1} E_f \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1,\ldots,i_l}(X'_{1,i_1}, \ldots, X'_{1,i_l}) \right) \\
\leq E_f \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1,\ldots,i_l}(X_{i_1}, \ldots, X_{i_l}) \right) \\
\leq \sum_{k=0}^{m} E_f \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1,\ldots,i_l}(X'_{1,i_1}, \ldots, X'_{1,i_l}) \right)
\]
for all convex functions \( f \in D \) with \( f(0) = 0 \).

Remark 5. – It is easy to see, using the derivations at the beginning of the section, that the class of convex functions \( f \in D \) with \( f(0) = 0 \) includes the functions \( f(x) = x^t, 1 < t \leq 2; f(x) = (x + x_0)^t \ln(x + x_0) - x_0^t \ln x_0, 1 < t < 2, x_0 \geq \max(e^{3r^2-6r+2}/(t(t-1)(2-t)), 1); \) and \( f(x) = (x + x_0) \ln(x + x_0) - x_0 \ln x_0, x_0 \geq 1 \).
Remark 6. From Khintchine–Marcinkiewicz–Zygmund inequalities for $U$-statistics (e.g., [1,5–9]) it follows that analogues of inequalities (3) and (4) with appropriately adjusted constants hold for sums of $U$-statistics with degenerate kernels. Moreover, by Hoeffding’s expansion, this implies corresponding inequalities for sums of $U$-statistics with not necessarily degenerate kernels.

3. Proof of the theorems

Let us prove Theorem 1. Let us use induction on the number of r.v.’s $X_1, \ldots, X_n$. Let us first demonstrate the argument in the case $m = 2$. Suppose that $f \in D$, $c_0 \geq 0$, and $Y_i : \mathbb{R} \to \mathbb{R}_+$, $Y_{ij} : \mathbb{R}^2 \to \mathbb{R}_+$, $1 \leq i, j \leq n$, $i \neq j$, are functions such that $Y_{ij}(x_i, x_j) = Y_{ji}(x_j, x_i)$, $x_i, x_j \in \mathbb{R}$, $1 \leq i < j \leq n$. Let $Y_{\text{reg}}^i(x_i) = Y_i(X_i)$, $Y_{\text{reg}}^{i,j}(x_i, x_j) = Y_{ij}(X_i, X_j)$, $E_j(\cdot) = E(\cdot | X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$, $1 \leq i, j \leq n$, $i \neq j$, and let $E(\cdot)$ be the unconditional expectation operator. Let us show that

$$Ef\left(c_0 + \sum_{i=1}^{n} Y_{\text{reg}}^i + \sum_{1 \leq i < j \leq n} Y_{\text{reg}}^{i,j}\right) \leq \sum_{1 \leq i < j \leq n} Ef\left(Y_{\text{reg}}^{i,j}\right) + \sum_{i=1}^{n} Ef(\sum_{j=1, j \neq i}^{n} E_j Y_{\text{reg}}^{i,j}) + f\left(c_0 + \sum_{i=1}^{n} EY_{\text{reg}}^i + \sum_{1 \leq i < j \leq n} EY_{\text{reg}}^{i,j}\right). \quad (11)$$

Suppose that it is already known that estimate (11) holds in the case of $n - 1$ r.v.’s $X_1, \ldots, X_{n-1}$. Let us prove that this implies that the inequality is valid in the case of $n$ r.v.’s $X_1, \ldots, X_n$. From the inequality

$$Ef(X + z) - Ef(X) \leq f(EX + z) - f(EX) \quad (12)$$

for $f \in D$ and for an arbitrary nonnegative r.v. $X$ and all $z \in \mathbb{R}_+$ (implied by Jensen’s inequality) we have, letting $X = Y_{\text{reg}}^n + \sum_{i=1}^{n-1} Y_{\text{reg}}^i$ and $z = c_0 + \sum_{i=1}^{n-1} Y_{\text{reg}}^i + \sum_{1 \leq i < j \leq n-1} Y_{\text{reg}}^{i,j}$,

$$Ef\left(c_0 + \sum_{i=1}^{n} Y_{\text{reg}}^i + \sum_{1 \leq i < j \leq n} Y_{\text{reg}}^{i,j}\right) = Ef\left(Y_{\text{reg}}^n + \sum_{i=1}^{n-1} Y_{\text{reg}}^i + \left(c_0 + \sum_{i=1}^{n-1} Y_{\text{reg}}^i + \sum_{1 \leq i < j \leq n-1} Y_{\text{reg}}^{i,j}\right)\right) \leq Ef\left(Y_{\text{reg}}^n + \sum_{i=1}^{n-1} Y_{\text{reg}}^i\right) + Ef\left(EY_{\text{reg}}^n + c_0 + \sum_{i=1}^{n-1} \left(Y_{\text{reg}}^i + E_n Y_{\text{reg}}^i\right) + \sum_{1 \leq i < j \leq n-1} Y_{\text{reg}}^{i,j}\right).$$
Conditioning on $X_n$ and using the induction hypothesis, we obtain
\[
Ef\left(Y_{\text{reg}}(n) + \sum_{i=1}^{n-1} Y_{\text{reg}}(i, n)\right) \\
\leq \sum_{i=1}^{n-1} Ef\left(Y_{\text{reg}}(i, n)\right) + Ef\left(Y_{\text{reg}}(n) + \sum_{i=1}^{n-1} E_i Y_{\text{reg}}(i, n)\right).
\]

In addition to that (also by the induction hypothesis),
\[
Ef\left(EY_{\text{reg}}(n) + c_0 + \sum_{i=1}^{n-1} (Y_{\text{reg}}(i) + E_n Y_{\text{reg}}(i, n)) + \sum_{1 \leq i < j \leq n-1} Y_{\text{reg}}(i, j)\right) \\
\leq \sum_{1 \leq i < j \leq n-1} Ef\left(Y_{\text{reg}}(i, j)\right) + \sum_{i=1}^{n-1} Ef\left(Y_{\text{reg}}(i) + \sum_{j=1, j \neq i}^{n} E_j Y_{\text{reg}}(i, j)\right) \\
+ f\left(EY_{\text{reg}}(n) + c_0 + \sum_{i=1}^{n-1} (EY_{\text{reg}}(i) + EY_{\text{reg}}(i, n)) + \sum_{1 \leq i < j \leq n-1} EY_{\text{reg}}(i, j)\right).
\]

From the latter relations it follows that
\[
Ef\left(c_0 + \sum_{i=1}^{n} Y_{\text{reg}}(i) + \sum_{1 \leq i < j \leq n} Y_{\text{reg}}(i, j)\right) \\
\leq \sum_{i=1}^{n} Ef\left(Y_{\text{reg}}(i, n)\right) + Ef\left(Y_{\text{reg}}(n) + \sum_{i=1}^{n-1} E_i Y_{\text{reg}}(i, n)\right) \\
+ \sum_{1 \leq i < j \leq n-1} Ef\left(Y_{\text{reg}}(i, j)\right) + \sum_{i=1}^{n-1} Ef\left(Y_{\text{reg}}(i) + \sum_{j=1, j \neq i}^{n} E_j Y_{\text{reg}}(i, j)\right) \\
+ f\left(EY_{\text{reg}}(n) + c_0 + \sum_{i=1}^{n-1} (EY_{\text{reg}}(i) + EY_{\text{reg}}(i, n)) + \sum_{1 \leq i < j \leq n-1} EY_{\text{reg}}(i, j)\right) \\
= \sum_{1 \leq i < j \leq n} Ef\left(Y_{\text{reg}}(i, j)\right) + \sum_{i=1}^{n} Ef\left(Y_{\text{reg}}(i) + \sum_{j=1, j \neq i}^{n} E_j Y_{\text{reg}}(i, j)\right) \\
+ f\left(c_0 + \sum_{i=1}^{n} EY_{\text{reg}}(i) + \sum_{1 \leq i < j \leq n} EY_{\text{reg}}(i, j)\right).
\]

The fact that by (12)
\[
Ef\left(c_0 + Y_1(X_1)\right) \leq Ef\left(Y_1(X_1)\right) + f\left(c_0 + EY_1(X_1)\right)
\]
for all $f \in D$ and $c_0 \geq 0$, that is, (11) is valid in the case $n = 1$, completes the proof. Let us follow the same approach in the case of arbitrary $m$. Suppose that $f \in D$, and $Y_{i_1, \ldots, i_l} : \mathbf{R} \to \mathbf{R}_+$, $1 \leq i_k \leq n; i, i_i \neq i_j; k, r, s = 1, \ldots, l; l = 0, \ldots, m$, are functions such that $Y_{i_1, \ldots, i_l}(x_1, \ldots, x_l) = Y_{i_k(1), \ldots, i_k(l)}(x_{\pi(1)}, \ldots, x_{\pi(l)})$, $x_k \in \mathbf{R}$, $k = 1, \ldots, l$, $1 \leq i_1 < \cdots < i_l \leq n$, for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$, $l = 2, \ldots, m$. 
Let $Y_{\text{reg}}(i_1, \ldots, i_l) = Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l})$, $E_{i_1, \ldots, i_l}() = E(\cdot | X_k, k = 1, \ldots, n; k \neq i_1, \ldots, i_l)$, $1 \leq i_k \leq n$; $i_r \neq i_l$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m$, and let $E(\cdot)$ be the unconditional expectation operator. Suppose that we already have the bound

$$Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} Y_{\text{reg}}^* (i_1, \ldots, i_l) \right)$$

$$\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} \sum_{l=q}^{m} i_{l} Y_{\text{reg}}^* (j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \right)$$

for all $f \in D$. From inequality (12) we obtain, letting $X = \sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} Y_{\text{reg}}(i_1, \ldots, i_l, n)$ and $\tilde{z} = \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} Y_{\text{reg}}(i_1, \ldots, i_l)$,

$$Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{\text{reg}}(i_1, \ldots, i_l) \right)$$

$$\leq Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} E_n Y_{\text{reg}}(i_1, \ldots, i_l) \right)$$

$$+ Ef \left( \sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} Y_{\text{reg}}(i_1, \ldots, i_l, n) \right). \quad (14)$$

From the induction hypothesis we get (we assume $Y_{\text{reg}}(i_1, \ldots, i_m, n) = 0$ for all $1 \leq i_k \leq n-1$; $i_r \neq i_l$, $r \neq s$; $k, r, s = 1, \ldots, m$)

$$Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} E_n Y_{\text{reg}}(i_1, \ldots, i_l) \right)$$

$$= Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} \left( Y_{\text{reg}}(i_1, \ldots, i_l) + E_n Y_{\text{reg}}(i_1, \ldots, i_l, n) \right) \right)$$

$$\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} \sum_{l=q}^{m} i_{l} Y_{\text{reg}}^* (j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \right)$$

$$+ Ef_{i_1, \ldots, i_{l-q}} Y_{\text{reg}}^* (j_1, \ldots, j_q, i_1, \ldots, i_{l-q}, n)$$

$$= \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} \sum_{l=q}^{m} i_{l} Y_{\text{reg}}^* (i_1, \ldots, i_l, n) \right). \quad (15)$$
Conditioning on the variable $X_n$ we also get by the induction assumptions
\[
Ef\left(\sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} Y_{reg}^{i_1, \ldots, i_l, n}\right)
\leq \sum_{q=0}^{m-1} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef\left(\sum_{l=q}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq \{1, \ldots, n\}\backslash \{j_1, \ldots, j_q\}} Y_{reg}^{i_1, \ldots, i_l, n}\right)
E_{i_1, \ldots, i_{l-q}} Y_{reg}^{j_1, \ldots, j_q, i_1, \ldots, i_{l-q}, n}.
\]
From (14)–(16) it follows that
\[
Ef\left(\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{reg}^{i_1, \ldots, i_l}\right)
\leq \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef\left(\sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq \{1, \ldots, n\}\backslash \{j_1, \ldots, j_q\}} Y_{reg}^{i_1, \ldots, i_l, n}\right)
E_{i_1, \ldots, i_{l-q}} Y_{reg}^{j_1, \ldots, j_q, i_1, \ldots, i_{l-q}, n}
+ \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} Ef\left(\sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq \{1, \ldots, n\}\backslash \{j_1, \ldots, j_q\}} Y_{reg}^{i_1, \ldots, i_l, n}\right)
E_{i_1, \ldots, i_{l-q}} Y_{reg}^{j_1, \ldots, j_q, i_1, \ldots, i_{l-q}, n}
= \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} Ef\left(\sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq \{1, \ldots, n\}\backslash \{j_1, \ldots, j_q\}} Y_{reg}^{i_1, \ldots, i_l, n}\right)
E_{i_1, \ldots, i_{l-q}} Y_{reg}^{j_1, \ldots, j_q, i_1, \ldots, i_{l-q}, n}.
\]
The fact that by (13) inequality (1) holds in the case of one r.v. $X_1$ completes the proof of Theorem 1. Theorem 2 might be proven in a similar way (or deduced from Theorem 1, see Remark 1). Corollary 1 is an immediate consequence of Theorems 1 and 2. Applying Theorems 1 and 2 for $f(x) = x^t$, we obtain inequalities (3) and (4). Let $1 < t \leq 2$, $a_k, b_k > 0$, $a'_k < b_k$, and let $c_{i_1, \ldots, i_l} \geq 0$, $1 \leq i_k \leq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m$; $c_{i_1, \ldots, i_l} = c_{i_1(1), \ldots, i_l(0)}$, $1 \leq i_1 < \cdots < i_l \leq n$, for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$, $l = 2, \ldots, m$ (we assume that $c_{i_1, \ldots, i_l} = c_0 \geq 0$ for $l = 0$). Let us set $Y_{i_1, \ldots, i_l}(x_1, \ldots, x_l) = c_{i_1, \ldots, i_l} x_1 \ldots x_l$, $1 \leq i_k \leq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m$ ($Y_{i_1, \ldots, i_l}(x_1, \ldots, x_l) = c_0$ for $l = 0$). Consider, similarly to [12], independent nonnegative r.v.’s $X_{1}^{(n_1)}, \ldots, X_{n}^{(n_n)}$, $s_k = 1, 2, \ldots, k = 1, 2, \ldots, n$, with the following distributions: $P(X_{k}^{(n_k)} = a_k) = 1 - 1/s_k$, $P(X_{k}^{(n_k)} = b_k^{(n_k)}) = a_k/(s_k b_k^{(n_k)})$, $P(X_{k}^{(n_k)} = 0) = 1/s_k - a_k/(s_k b_k^{(n_k)})$, where $b_k^{(n_k)} = (\frac{a_k}{s_k} - \frac{a_k}{s_k})^{1/(t-1)}$. It is not difficult
to see that \( b_k(s_k) \geq a_k \), \( 0 \leq a_k/(s_k b_k(s_k)) \leq 1/s_k \), \( b_k(s_k) \rightarrow \infty \), \( (b_k(s_k))^{t-1} a_k/s_k = b_k - a_k (1 - 1/s_k) \rightarrow b_k - a_k \) as \( s_k \rightarrow \infty \). We have that for all nonnegative r.v.'s \( Z_1 \) and \( Z_2 \) with finite \( r \)th moment independent of \( X_k(s_k) \),

\[
E(Z_1 X_k(s_k) + Z_2) = E(Z_1 a_k + Z_2)(1 - 1/s_k) + EZ_k(a_k/(s_k b_k(s_k))) + (E(Z_1 b_k + Z_2) - EZ_k(b_k) a_k/(s_k b_k(s_k))) + EZ_k(b_k(s_k))^{t-1} a_k/s_k.
\]

(17)

It is not difficult to see that \((1 + x)^t - 1 \leq t(x + x^t)\) for all \( t \in (1, 2] \) and all \( x \geq 0 \). Consequently,

\[
0 \leq E(Z_1 + Z_2/b_k(s_k))^t - EZ_1 \leq t(EZ_1^{-1} Z_2/b_k(s_k) + EZ_2/(b_k(s_k))^t).
\]

Therefore,

\[
(E(Z_1 b_k + Z_2)^t - EZ_k(b_k(s_k)^t) a_k/(s_k b_k(s_k))) = (E(Z_1 + Z_2/b_k(s_k))^t - EZ_k(b_k(s_k)^t) a_k/s_k \rightarrow 0
\]
as \( s_k \rightarrow \infty \), and from (17) we obtain

\[
E(Z_1 X_k(s_k) + Z_2)^t = E(Z_1 b_k - a_k)^t + E(Z_1 a_k + Z_2)^t
\]

(18)
as \( s_k \rightarrow \infty \), for all r.v.'s \( Z_1 \) and \( Z_2 \) defined above. Let us show that

\[
E\left(\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} Y_{i_1, \ldots, i_l} X_{i_1}^{(s_1)} \cdots X_{i_l}^{(s_l)}\right)^t
\]

\[
\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1, \ldots, i_l} X_{i_1}^{(s_1)} \cdots X_{i_l}^{(s_l)}
\]

\[
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} \prod_{r=1}^{q} (b_{j_r} - a_{j_r})
\]

\[
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} \prod_{r=1}^{q} (b_{j_r} - a_{j_r})
\]

as \( s_1 \rightarrow \infty, \ldots, s_n \rightarrow \infty \). Let us use induction on the number of the r.v.'s \( X_1^{(s_1)}, \ldots, X_n^{(s_n)} \).

Suppose we have already proven relation (19) for all sums of multilinear forms of order not greater than \( m \), \( 1 \leq m \leq n - 1 \), in the case of \( n - 1 \) r.v.'s \( X_1^{(s_1)}, \ldots, X_{n-1}^{(s_{n-1})} \), that is suppose that the relation

\[
E\left(\sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1, \ldots, i_l} X_{i_1}^{(s_1)} \cdots X_{i_l}^{(s_l)}\right)^t
\]

\[
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r})
\]

\[
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r})
\]
as $s_1 \to \infty$, $\ldots$, $s_{n-1} \to \infty$, is valid. Letting $k = n$,

\begin{align*}
&Z_1 = \sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l, n} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})}, \\
&Z_2 = \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})},
\end{align*}

from (18) we get

\begin{align*}
&\mathbb{E} \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1, \ldots, i_l, n} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right)^{t} \\
&\quad \to (b_n - a_{n}^t) \mathbb{E} \left( \sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l, n} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right)^{t} \\
&\quad + \mathbb{E} \left( \mathbb{E} \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1, \ldots, i_l, n} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right| X_n^{(s_n)} = a_n \right)^{t}, \quad (20)
\end{align*}

as $s_n \to \infty$. From the induction hypothesis it follows that

\begin{align*}
&\mathbb{E} \left( \sum_{l=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right)^{t} \\
&\quad \to \sum_{q=0}^{m-1} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \\
&\quad \times \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l, n} a_{i_1} \cdots a_{i_l} \right)^{t}, \quad (21)
\end{align*}

as $s_1 \to \infty$, $\ldots$, $s_{n-1} \to \infty$. Moreover (we assume $c_{i_1, \ldots, i_m, n} = 0$ for all $1 \leq i_k \leq n - 1$; $i_r \neq i_s, r \neq s; k, r, s = 1, \ldots, m$)

\begin{align*}
&\mathbb{E} \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1, \ldots, i_l} X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right)^{t} \\
&\quad = \mathbb{E} \left( \sum_{l=0}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} (c_{i_1, \ldots, i_l} + c_{i_1, \ldots, i_l, n} a_n) X_{i_1}^{(s_{i_1})} \cdots X_{i_l}^{(s_{i_l})} \right)^{t} \\
&\quad \to \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \left( \sum_{l=q}^{m} \sum_{1 \leq i_1 < \cdots < i_l \leq n-1} c_{i_1, \ldots, i_l, n} a_{i_1} \cdots a_{i_l} \right)^{t}
\end{align*}
\[
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a'_{j_r})
\times \left( \sum_{l=q}^{m} \sum_{i_1 < \cdots < i_l \in \{1, \ldots, n\}\setminus\{j_1, \ldots, j_q\}} c_{j_1, \ldots, j_q, i_1, \ldots, i_l} a_{i_1} \cdots a_{i_l} \right)^t
\] (22)

as \( s_1 \to \infty, \ldots, s_n \to \infty \). From (20)–(22) it follows that

\[
E \left( \sum_{l=0}^{m} \sum_{1 \leq j_1 < \cdots < j_l \leq n} c_{i_1, \ldots, i_l, 0} x_{i_1}^{(b_{j_1})} \cdots x_{i_l}^{(b_{j_l})} \right)^t
\to (b_n - a'_n) \sum_{q=0}^{m-1} \sum_{1 \leq j_1 < \cdots < j_q \leq n-1} \prod_{r=1}^{q} (b_{j_r} - a'_{j_r})
\times \left( \sum_{l=q}^{m} \sum_{i_1 < \cdots < i_l \in \{1, \ldots, n\}\setminus\{j_1, \ldots, j_q\}} c_{j_1, \ldots, j_q, i_1, \ldots, i_l} a_{i_1} \cdots a_{i_l} \right)^t
\]

\[
= \sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} \prod_{r=1}^{q} (b_{j_r} - a'_{j_r})
\times \left( \sum_{l=q}^{m} \sum_{i_1 < \cdots < i_l \in \{1, \ldots, n\}\setminus\{j_1, \ldots, j_q\}} c_{j_1, \ldots, j_q, i_1, \ldots, i_l} a_{i_1} \cdots a_{i_l} \right)^t.
\] (23)

Therefore, (19) is valid. The following constants satisfy the conditions stated before (17): \( b_k = a_k = 1/n, k = 1, \ldots, n; c_{i_1, \ldots, i_l} = 0, 1 \leq l \leq n; i_r \neq i_s, r \neq s; k, r, s = 1, \ldots, m; l = 0, \ldots, m - 1; c_{i_1, \ldots, i_l} = (\sum q=0^m 1/(m-q))^{-1/t}; 1 \leq i_k \leq n; i_r \neq i_s, r \neq s; k, r, s = 1, \ldots, m \). For these parameters, we get

\[
\sum_{q=0}^{m} \prod_{r=1}^{q} (b_{j_r} - a'_{j_r})
\times \left( \sum_{l=q}^{m} \sum_{i_1 < \cdots < i_l \in \{1, \ldots, n\}\setminus\{j_1, \ldots, j_q\}} c_{j_1, \ldots, j_q, i_1, \ldots, i_l} a_{i_1} \cdots a_{i_l} \right)^t
\]

\[
= \sum_{q=0}^{m} C_n^q \left(n^{-1} - n^{-t}\right)^q C_{n-q}^{m-q} n^{-(m-q)} c_{1, \ldots, m}^t
\]

\[
\to \sum_{q=0}^{m} \frac{1}{q!} \left(\frac{1}{(m-q)!}\right)^t c_{1, \ldots, m}^t = 1,
\] (24)
as $n \to \infty$. Moreover, since $E X_k^{(s_k)} = a_k$, $E (X_k^{(s_k)})' = b_k$, $s_k = 1, 2, \ldots, k = 1, \ldots, n$, we obtain

$$
\sum_{q=0}^{m} \sum_{1 \leq j_1 < \cdots < j_q \leq n} E \left( \sum_{l=q}^{m} \sum_{i_1 < \cdots < i_q \leq n} \prod_{(j) \leq i} E X_j^{(s_j)} \right) \to 1,
$$

as $n \to \infty$. Relations (19), (24) and (25) imply sharpness of the constants in inequality (3). Sharpness of the constants in inequality (4) might be proven in a similar way.

The decoupling inequalities in Theorems 4 and 5 follow from inequalities (1)–(10), as explained before the theorems. The proof is complete.

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REFERENCES


