LINEAR AND NONLINEAR SHELL THEORY

Contents

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Nonlinear strain-displacement relations

We have already been introduced to two shell theories, nonlinear shallow shell theory and nonlinear Donnell-Mushtari-Vlasov theory, and we have solved a few shell problems related to linear behavior, including boundary layers, and nonlinear behavior such as buckling. We have also covered the fundamentals of tensor and surface theory which provide the analytical underpinnings of shell theory. Therefore, we will start by deriving exact nonlinear strain-displacement relations for 2D shell theory and subsequently introduce some of the various approximate versions that are used, including those for exact linear first order theories. For each set of strain-displacement relations, equilibrium equations are derived from the principle of virtual work. In these notes, the entire emphasis will be on linear stress-strain constitutive behavior for shells made of isotropic materials, but extensions to anisotropic elastic materials or to plasticity can also be made.

Let the two surface coordinates be (ξ^1, ξ^2) and denote the positions of a point on the middle surface in the undeformed and deformed state by $\vec{x}(\xi^1, \xi^2)$ and $\vec{\bar{x}}(\xi^1, \xi^2)$, respectively. The middle surface is determined by the metric tensor and curvature tensor in the respective states. In the undeformed state these are denoted by $g_{\alpha\beta}$ and $b_{\alpha\beta}$ while in the deformed state they are denoted by $G_{\alpha\beta}$ and $B_{\alpha\beta}$. The most obvious choice for the membrane strain tensor and the bending strain tenor are

$$E_{\alpha\beta} = \frac{1}{2}\left(G_{\alpha\beta} - g_{\alpha\beta}\right) \text{ and } K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta}$$

although we will find that it will be convenient to modify this choice in the sequel.

The base vectors of the undeformed middle surface will be used as the reference:

$$\vec{e}_\alpha = \frac{\partial \vec{x}}{\partial \xi^\alpha} = \vec{x}_{,\alpha} \text{ and } \vec{N}$$

The displacement vector is $\vec{U}(\xi^1, \xi^2) = \vec{\bar{x}} - \vec{x}$. Introduce components of the displacement with respect to tangent and normal vectors in the undeformed state according to

$$\bar{x}^i = x^i + u^\beta x^i_{,\beta} + wN^i$$

Note that

$$\bar{x}^i_{,\alpha} = x^i_{,\alpha} + u^\beta_{,\alpha} x^i_{,\beta} + u^\beta x^i_{,\beta\alpha} + w_{,\alpha} N^i + wN^i_{,\alpha}$$

which by the Gauss and Weingarten equations become

$$\bar{x}^i_{,\alpha} = x^i_{,\alpha} + \left(u^{\gamma}_{,\alpha} + b^{\gamma}_{\alpha} w\right) x^i_{,\gamma} + \left(w_{,\alpha} - b_{\alpha\beta} u^{\beta}\right) N^i = x^i_{,\alpha} + d^{\gamma}_{\cdot\alpha} x^i_{,\gamma} - \varphi_{\alpha} N^i$$

where

$$d^{\gamma}_{\cdot\alpha} \equiv u^{\gamma}_{,\alpha} + b^{\gamma}_{\alpha} w \quad (\text{or} \quad d_{\gamma\alpha} \equiv u_{\gamma,\alpha} + b_{\gamma\alpha} w) \quad \text{and} \quad \varphi_{\alpha} \equiv -w_{,\alpha} + b_{\alpha\beta} u^{\beta}$$

Now, compute the deformed metric tensor:

$$G_{\alpha\beta} = \bar{\vec{x}}_{,\alpha} \bullet \bar{\vec{x}}_{,\beta} = g_{\alpha\beta} + \left(g_{\gamma\beta} d^{\gamma}_{\cdot\alpha} + g_{\gamma\alpha} d^{\gamma}_{\cdot\beta}\right) + g_{\gamma\mu} d^{\gamma}_{\cdot\alpha} d^{\mu}_{\cdot\beta} + \varphi_{\alpha}\varphi_{\beta}$$

Next note that

$$g_{\gamma\beta} d^{\gamma}_{\cdot\alpha} + g_{\gamma\alpha} d^{\gamma}_{\cdot\beta} = d_{\beta\alpha} + d_{\alpha\beta} = u_{\beta,\alpha} + u_{\alpha,\beta} + 2b_{\alpha\beta} w \equiv 2e_{\alpha\beta} = 2e_{\beta\alpha}$$

where as noted two lines below $e_{\alpha\beta} \equiv \frac{1}{2}\left(u_{\beta,\alpha} + u_{\alpha,\beta}\right) + b_{\alpha\beta} w$ is the *linearized membrane strain tensor*. Thus,

$$G_{\alpha\beta} = g_{\alpha\beta} + 2e_{\alpha\beta} + d^{\gamma}_{\cdot\alpha} d_{\gamma\beta} + \varphi_{\alpha}\varphi_{\beta},$$

and

$$E_{\alpha\beta} = \frac{1}{2}\left(G_{\alpha\beta} - g_{\alpha\beta}\right) = e_{\alpha\beta} + \frac{1}{2} d^{\gamma}_{\cdot\alpha} d_{\gamma\beta} + \frac{1}{2}\varphi_{\alpha}\varphi_{\beta}$$

Further refine this expression for the membrane strain by defining φ as

$$\varphi \equiv \frac{1}{2}\varepsilon^{\alpha\beta} u_{\beta,\alpha} = \frac{1}{2\sqrt{g}}\left(u_{2,1} - u_{1,2}\right) \quad \text{or} \quad \varepsilon_{\alpha\beta}\varphi = \frac{1}{2}\left(u_{\beta,\alpha} - u_{\alpha,\beta}\right)$$

such that φ is the rotation about the normal in the linear theory. Then, $d_{\alpha\beta} = e_{\alpha\beta} - \varepsilon_{\alpha\beta}\varphi$, and

$$E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2}\left[e^{\gamma}_{\alpha} e_{\gamma\beta} - \left(\varepsilon_{\gamma\alpha} e^{\gamma}_{\beta} + \varepsilon_{\gamma\beta} e^{\gamma}_{\alpha}\right)\varphi + \varphi_{\alpha}\varphi_{\beta} + g_{\alpha\beta}\varphi^2\right]$$

We emphasize that no approximations have been made.

Now we proceed to derive an exact expression for the bending strain. The curvature tensor of the deformed middle surface is given by $B_{\alpha\beta} = \bar{\vec{N}}_{,\alpha} \bullet \bar{\vec{x}}_{,\beta} = -\bar{\vec{N}} \bullet \bar{\vec{x}}_{,\alpha\beta}$.

Noting that $\bar{\vec{N}} = \left(\bar{\vec{x}}_{,1} \times \bar{\vec{x}}_{,2}\right) / \left|\bar{\vec{x}}_{,1} \times \bar{\vec{x}}_{,2}\right|$, a lengthy but straight forward calculation gives

$$\bar{N}^i = \sqrt{\frac{g}{G}}\left[\left(\varphi^{\gamma} + R^{\gamma}\right) x^i_{,\gamma} + \left(1 + d^{\omega}_{\omega} + H\right) N^i\right]$$

where

$$G/g = 1 + 2E^{\omega}_{\omega} + 2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}E_{\alpha\beta}E_{\lambda\mu}$$

$$R_{\gamma} = \varphi_{\gamma}d^{\omega}_{\omega} - \varphi^{\omega}d_{\omega\gamma} = \varphi_{\gamma}e^{\omega}_{\omega} - \varphi^{\omega}e_{\omega\gamma} + \varphi\varphi^{\omega}\varepsilon_{\omega\gamma}$$

$$H = \frac{1}{2}\left(d^{\omega\,2}_{\bullet\omega} - d_{\omega\gamma}d^{\gamma\omega}\right) = e/g + \varphi^{2} \quad \text{with} \quad e = \left|e_{\alpha\beta}\right|$$

With $\overline{x}^{i}_{,\alpha} = x^{i}_{,\alpha} + d^{\gamma}_{\bullet\alpha}x^{i}_{,\gamma} - \varphi_{\alpha}N^{i}$ and using the Gauss and Weingarten equations, one finds

$$\vec{\overline{x}}_{,\alpha\beta} = \vec{x}_{,\alpha\beta} + \left(d^{\gamma}_{\bullet\alpha,\beta} - \varphi_{\alpha}b^{\gamma}_{\beta}\right)\vec{x}_{,\gamma} - \left(b_{\gamma\beta}d^{\gamma}_{\bullet\alpha} - \varphi_{\alpha,\beta}\right)\vec{N}$$

Thus, it follows that

$$\begin{aligned} B_{\alpha\beta} &= -\vec{\overline{N}}\bullet\vec{\overline{x}}_{,\alpha\beta} \\ &= \sqrt{g/G}\left[\left(1 + e^{\omega}_{\omega} + H\right)\left(b_{\alpha\beta} + \varphi_{\alpha,\beta} + b^{\gamma}_{\beta}d_{\gamma\alpha}\right) - \left(\varphi^{\gamma} + R^{\gamma}\right)\left(d_{\gamma\alpha,\beta} - b_{\gamma\beta}\varphi_{\alpha}\right)\right] \end{aligned}$$

While it is not obvious that $B_{\alpha\beta} = B_{\beta\alpha}$, this symmetry must hold because the result is exact. We now have one possible exact bending strain measure, $K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta}$. Because the above stretching strain and bending strain measures are exact, they must vanish for arbitrary large rigid body displacements (translations and rotations).

Various Approximate strain-displacement relations

Sanders (1963) gives an excellent discussion of most of the commonly used approximations of the strain-displacement relations. The following treatment largely follows those in his article. In each case, equilibrium equations follow from the principle of virtual work, but we will defer deriving the equilibrium equations until after we have discussed the various approximate strain-displacement relations. We have already had some experience in considering various approximations when we considered nonlinear curved beam theory and when we derived shallow shell and DMV theories from nonlinear plate theory.

Exact linear membrane and bending measures for first order shell theories

Linearize the expressions for $E_{\alpha\beta}$ and $B_{\alpha\beta}$ to obtain

$$E_{\alpha\beta} = e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) + b_{\alpha\beta}w$$

$$B_{\alpha\beta} = b_{\alpha\beta} + \varphi_{\alpha,\beta} + b_\beta^\gamma d_{\gamma\alpha} = b_{\alpha\beta} + b_{\alpha\gamma} u^\gamma{}_{,\beta} + b_{\alpha\gamma} b_\alpha^\gamma w + \varphi_{\beta,\alpha}$$
$$= b_{\alpha\beta} + \frac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) + \frac{1}{2}\left(b_\beta^\gamma u_{\gamma,\alpha} + b_\alpha^\gamma u_{\gamma,\beta}\right) + b_\beta^\gamma b_{\alpha\gamma} w$$

where the Codazzi equations have been used in obtaining the second expression. Note that it reveals the indicial symmetry explicitly. The *linearized bending strain tensor* is

$$K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} = \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) + \tfrac{1}{2}\left(b_\beta^\gamma u_{\gamma,\alpha} + b_\alpha^\gamma u_{\gamma,\beta}\right) + b_\beta^\gamma b_{\alpha\gamma} w$$
$$= \tfrac{1}{2}\left(b_\alpha^\gamma e_{\gamma\beta} + b_\beta^\gamma e_{\gamma\alpha}\right) + \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) - \tfrac{1}{2}\left(b_\beta^\gamma \varepsilon_{\gamma\alpha} + b_\alpha^\gamma \varepsilon_{\gamma\beta}\right)\varphi$$
$$= -w_{,\alpha\beta} + b_{\alpha\gamma} u^\gamma{}_{,\beta} + b_{\beta\gamma} u^\gamma{}_{,\alpha} + \tfrac{1}{2}\left(b_{\alpha\gamma,\beta} + b_{\beta\gamma,\alpha}\right) u^\gamma + b_\beta^\gamma b_{\alpha\gamma} w$$

These results will be used to generate equilibrium equations later in the notes. Because they are exact linear measures, $E_{\alpha\beta}$ and $K_{\alpha\beta}$, vanish for all rigid body translations and the vanish to lowest order for rigid body rotations, i.e. for any rigid body rotation θ, the measures are of order θ^2. It is possible to modify these measures using the fact that contributions to the curvature changes like $b_\alpha^\gamma e_{\gamma\beta}$ can be neglected if one so desires, for reasons which we discussed when we formulated curved beam theory. For example, in a linear first order shell theory, one could use

$$K_{\alpha\beta} = \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) - \tfrac{1}{2}\left(b_\beta^\gamma \varepsilon_{\gamma\alpha} + b_\alpha^\gamma \varepsilon_{\gamma\beta}\right)\varphi$$

involving only the rotation components. Note that this modification does not alter the fact that the new measures vanish for rigid body translations and are of order θ^2 for rigid body rotations.

Finite strain membrane theory

Budiansky in the article *Notes on nonlinear shell theory* discusses aspects of nonlinear membrane theory which uses the exact stretching strain measure, $E_{\alpha\beta}$, and neglects any effect of bending. Note that $E_{\alpha\beta}$ is a combination of linear and quadratic contributions of the displacements (although there are additional "hidden" dependencies on g. The reader is referred to Budiansky's article for further details and discussion.

Small strain, large rotation theory

Some simplification of the bending strain measures are obtained under the assumption that the strains are small compared to unity. Specifically, one assumes that $\left| E_{\alpha\beta} t^{\alpha} t^{\beta} \right| << 1$ for all directions tangent to the shell $\vec{t}$. Thus, for example, it follows that one can make the approximation $G/g \cong 1$ and certain other simplifications. We will not pursue this class of theories here.

Small strain, moderate rotation theory

This is analogous to the assumptions made in deriving nonlinear plate theory except that now we do not assume that the rotation about the normal is small. Recall that in deriving the nonlinear plate theory we assumed that only the out of plane rotations were moderate—the rotation about the normal was assumed to be of order of the strains. To proceed, assume that the strains are of order ρ^2 and the rotations are of order ρ with $\rho << 1$, and neglect terms of relative order ρ. Note that for moderate rotations, the out-of-plane rotations are φ_α while φ is the rotation about the normal. Thus, these quantities are assumed to be of order ρ. For small strains and small rotations, the strains are $e_{\alpha\beta}$ and these are assumed to be of order ρ^2. Noting that $d_{\alpha\beta} = e_{\alpha\beta} - \varepsilon_{\alpha\beta}\varphi$, $d_{\alpha\beta} \cong -\varepsilon_{\alpha\beta}\varphi$ in this approximation. The *stretching strain tensor* becomes

$$E_{\alpha\beta} = e_{\alpha\beta} + \tfrac{1}{2}\varphi_\alpha \varphi_\beta + \tfrac{1}{2} g_{\alpha\beta}\varphi^2$$

where terms of order ρ^3 have been neglected. Note that the stretching strains are of order ρ^2. The *bending strain tensor* becomes the linearized bending strain tensor:

$$\begin{aligned} K_{\alpha\beta} &= +\tfrac{1}{2}\left(b^{\gamma}_{\alpha} e_{\gamma\beta} + b^{\gamma}_{\beta} e_{\gamma\alpha}\right) + \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) - \tfrac{1}{2}\left(b^{\gamma}_{\beta}\varepsilon_{\gamma\alpha} + b^{\gamma}_{\alpha}\varepsilon_{\gamma\beta}\right)\varphi \\ &= -w_{,\alpha\beta} + b_{\alpha\gamma} u^{\gamma}_{,\beta} + b_{\beta\gamma} u^{\gamma}_{,\alpha} + \tfrac{1}{2}\left(b_{\alpha\gamma,\beta} + b_{\beta\gamma,\alpha}\right) u^{\gamma} + b^{\gamma}_{\beta} b_{\alpha\gamma} w \end{aligned}$$

It is of order ρ; terms of order ρ^2 have been neglected. It can be simplified further, if desired by noting that the terms involving the linearized strain tensor are of order ρ^2. With this approximation,

$$K_{\alpha\beta} = \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) - \tfrac{1}{2}\left(b^{\gamma}_{\beta}\varepsilon_{\gamma\alpha} + b^{\gamma}_{\alpha}\varepsilon_{\gamma\beta}\right)\varphi$$

Small strains, small rotation about the normal & moderate out-of-plane rotations

The same assumptions hold as above except now it is also assumed that φ is of order ρ^2. The resulting strain-displacement relations are

$$E_{\alpha\beta} = e_{\alpha\beta} + \tfrac{1}{2}\varphi_\alpha\varphi_\beta$$

$$K_{\alpha\beta} = \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right)$$

Donnell-Mushtari-Vlasov Approximation (DMV theory)

As mentioned in the previous chapter of the notes on shallow shells and DMV theory, these two theories are probably the most widely used shell theory for carrying out analytical analyses of shell buckling. The justification underlying the DMV strain-displacement relations is the same as those enumerated above (i.e. small strains and rotation about the normal and moderate out-of-plane rotations) *plus* short wavelength deformations. Specifically, we assume that

$$\varphi_\alpha = b_\alpha^\gamma u_\gamma - w_{,\alpha} \quad \text{can be approximated by} \quad \varphi_\alpha = -w_{,\alpha}$$

With ℓ as the characteristic length characterizing the deformation and R as a characteristic radius of curvature of the shell, this approximation is tantamount to assuming $|u/R| << |w/\ell|$. With $\varphi_\alpha = -w_{,\alpha}$,

$$E_{\alpha\beta} = \tfrac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) + b_{\alpha\beta}w + \tfrac{1}{2}w_{,\alpha}w_{,\beta}$$

$$K_{\alpha\beta} = -w_{,\alpha\beta}$$

These expressions will be recognized from the previous chapter of notes on DMV theory. The systematic derivation given in the present section of the notes provides more revealing insights into the approximations underlying the theory. The tensoral nature of the quantities involved in the strain-displacement relations is also clarified. The equilibrium equations and the $w-F$ formulation have been fully detailed in the previous section of the notes. Recall that these were presented in general tensoral form.

Linear shell theory—equilibrium, stress-strain and boundary conditions

We proceed to derive equilibrium equations, boundary conditions and to postulate the constitutive relation for linear shell theory following the same procedures we employed when we address plate theory and shallow shell theory. Namely, we start with the strain-displacement relations, postulate the principle of virtual work and derive the equilibrium equations and consistent boundary conditions.

The linear strain displacement relations we will use are

$$E_{\alpha\beta} = e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) + b_{\alpha\beta} w$$

$$K_{\alpha\beta} = \tfrac{1}{2}\left(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}\right) - \tfrac{1}{2}\left(b^{\gamma}_{\beta}\varepsilon_{\gamma\alpha} + b^{\gamma}_{\alpha}\varepsilon_{\gamma\beta}\right)\varphi$$

with

$$\varphi_{\alpha} \equiv -w_{,\alpha} + b_{\alpha\beta}u^{\beta} \quad \text{and} \quad \varphi \equiv \frac{1}{2}\varepsilon^{\alpha\beta}u_{\beta,\alpha} \quad \left(\varepsilon_{\alpha\beta}\varphi = \frac{1}{2}\left(u_{\beta,\alpha} - u_{\alpha,\beta}\right)\right)$$

Note that we have deleted the terms $\tfrac{1}{2}\left(b^{\gamma}_{\alpha}e_{\gamma\beta} + b^{\gamma}_{\beta}e_{\gamma\alpha}\right)$ in the exact linearized bending strain measure. The bending strain above is a linear combination of the exact bending measure and $E_{\alpha\beta}$, thus one set can be expressed in terms of the other. As discussed in connection with the derivation of curved beam theory, Koiter argued that bending strain contributions on the order of e/R can be neglected, if desired, since errors inherent to the constitutive law for a first order shell theory are of this order. We take the bending strain measure above because of its greater simplicity. Budiansky and Sanders (19??) argue that this is the “best” choice for the bending strain measure for further reasons detailed in their paper.

Principle of virtual work, equilibrium equations and boundary conditions

Let the bending moments, $M^{\alpha\beta} = M^{\beta\alpha}$, and resultant stresses, $N^{\alpha\beta} = N^{\beta\alpha}$, be work conjugate to the bending strains, $K_{\alpha\beta}$, and stretching strains, $E_{\alpha\beta}$. Postulate the principle of virtual work (PVW):

$$IVW \equiv \int_S \left[M^{\alpha\beta}\delta K_{\alpha\beta} + N^{\alpha\beta}\delta E_{\alpha\beta}\right] dS =$$

$$EVW \equiv \int_S \left[f^{\alpha}\delta u_{\alpha} + p\delta w\right] dS + \int_C \left[T^{\alpha}\delta u_{\alpha} + Q\delta w - M_n \delta w_{,n}\right] ds$$

where

$$\delta E_{\alpha\beta} = \frac{1}{2}\left(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha}\right) + b_{\alpha\beta}\delta w$$

$$\delta K_{\alpha\beta} = -\delta w_{,\alpha\beta} + b_{\alpha\gamma,\beta}\delta u^{\gamma} + \frac{3}{4}\left(b_{\alpha\gamma}\delta u^{\gamma}_{,\beta} + b_{\beta\gamma}\delta u^{\gamma}_{,\alpha}\right) - \frac{1}{4}\left(b^{\gamma}_{\beta}\delta u_{\alpha.\gamma} + b^{\gamma}_{\alpha}\delta u_{\beta.\gamma}\right)$$

Following now familiar procedures,

$$IVW = \int_S \left\{ M^{\alpha\beta} \left[-\delta w_{,\alpha\beta} + b_{\alpha\gamma,\beta}\delta u^{\gamma} + \tfrac{3}{2} b_{\alpha\gamma}\delta u^{\gamma}{}_{,\beta} - \tfrac{1}{2} b^{\gamma}_{\beta}\delta u_{\alpha,\gamma} \right] + N^{\alpha\beta} \left[\delta u_{\alpha,\beta} + b_{\alpha\beta}\delta w \right] \right\} dS$$

$$= \int_S \left\{ \left[-M^{\alpha\beta}_{\ ,\alpha\beta} + N^{\alpha\beta} b_{\alpha\beta} \right] \delta w - \left[N^{\gamma\beta}_{\ ,\beta} + M^{\alpha\beta}_{\ ,\beta} b^{\gamma}_{\alpha} + \frac{1}{2}\left(M^{\alpha\beta} b^{\gamma}_{\alpha} - M^{\gamma\alpha} b^{\beta}_{\alpha} \right)_{,\beta} \right] \delta u_{\gamma} \right\} dS +$$

$$+ \int_C \left\{ -M^{\alpha\beta} n_{\beta}\delta w_{,\alpha} + M^{\alpha\beta}_{\ ,\beta} n_{\alpha}\delta w + \left(N^{\gamma\beta} n_{\beta} + \tfrac{3}{2} M^{\alpha\beta} n_{\beta} b^{\gamma}_{\alpha} - \tfrac{1}{2} M^{\gamma\beta} b^{\alpha}_{\beta} n_{\alpha} \right) \delta u_{\gamma} \right\} ds$$

We have made use of the divergence theorem for curved surface integration with n_β as the components the of unit tangent vector to the surface that is normal to C. As in the case of plate theory, the term $-M^{\alpha\beta} n_\beta \delta w_{,\alpha}$ in the boundary integral must be additionally manipulated using $\delta w_{,\alpha} = n_\alpha \delta w_{,n} + t_\alpha \delta w_{,t}$ involving derivatives normal and tangent to C ($\vec{t}$ is the unit vector in the surface tangent to C)

$$-\int_C M^{\alpha\beta} n_\beta \delta w_{,\alpha} ds = -\int_C \left(M^{\alpha\beta} n_\beta n_\alpha \delta w_{,n} + M^{\alpha\beta} n_\beta t_\alpha \delta w_{,t} \right) ds =$$

$$= -\int_C \left(M^{\alpha\beta} n_\beta n_\alpha \delta w_{,n} - (M^{\alpha\beta} n_\beta t_\alpha)_{,t}\, \delta w \right) ds - M^{\alpha\beta} n_\beta t_\alpha \, \delta w \big|_{corners}$$

By enforcing the PVW, we are now in a position to read off the equilibrium equations and boundary conditions:

Equilibrium Equations:

$$M^{\alpha\beta}_{\ ,\alpha\beta} - N^{\alpha\beta} b_{\alpha\beta} = -p$$

$$N^{\gamma\beta}_{\ ,\beta} + M^{\alpha\beta}_{\ ,\beta} b^{\gamma}_{\alpha} + \frac{1}{2}\left(M^{\alpha\beta} b^{\gamma}_{\alpha} - M^{\gamma\alpha} b^{\beta}_{\alpha} \right)_{,\beta} = -f^{\gamma}$$

Boundary conditions:

Specify $M^{\alpha\beta}_{\ ,\beta} n_\alpha + (M^{\alpha\beta} n_\beta t_\alpha)_{,t} = Q$ or w

Specify $M^{\alpha\beta} n_\beta n_\alpha = M_n$ or $w_{,n}$

Specify $N^{\gamma\beta} n_\beta + \tfrac{3}{2} M^{\alpha\beta} n_\beta b^{\gamma}_{\alpha} - \tfrac{1}{2} M^{\gamma\beta} b^{\alpha}_{\beta} n_\alpha = T^{\gamma}$ or u_γ

with conditions relating $-M^{\alpha\beta} n_\beta t_\alpha \, \delta w \big|_{corners}$ to the virtual work of concentrated loads at any corners.

Constitutive relation for thin linear isotropic elastic shells

We again make use of Koiter's arguments concerning the inherent errors in the constitutive relation in first order shell theory for thin shells, i.e. that stretching strain contributions to the bending strain can be neglected. This permits to take the same uncoupled relations that we used for linear isotropic elastic plates but listed here in sa manner consistent with the curvilinear tensor framework

$$E_{\alpha\beta} = \frac{1+\nu}{Eh} N_{\alpha\beta} - \frac{\nu}{Eh} N^{\gamma}_{\gamma} g_{\alpha\beta} \ , \qquad N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\left[(1-\nu)E_{\alpha\beta} + \nu E^{\gamma}_{\gamma} g_{\alpha\beta}\right]$$

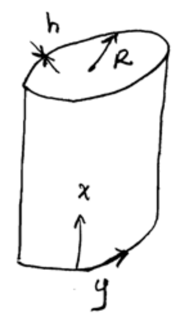
and

$$K_{\alpha\beta} = \frac{12}{Eh^3}\left[(1+\nu)M_{\alpha\beta} - \nu M^{\gamma}_{\gamma} g_{\alpha\beta}\right], \qquad M_{\alpha\beta} = D\left[(1-\nu)K_{\alpha\beta} + \nu K^{\gamma}_{\gamma} g_{\alpha\beta}\right]$$

where $D = Eh^3[12(1-\nu^2)]$ and $M^{\mu\lambda} = g^{\alpha\mu} g^{\beta\lambda} M_{\alpha\beta}$ and $N^{\mu\lambda} = g^{\alpha\mu} g^{\beta\lambda} N_{\alpha\beta}$.

This completes the formulation for linear shell theory. Additional aspects of the formulation such as stress functions and compatibility equations have also been derived in the literature (see, for example Koiter (1966)). As mentioned above, this particular version of shell theory is dubbed "best theory" by Budiansky and Sanders (1963) because, among other reasons, it has an attractive duality known as the static-geometric analogy.

Small strain, moderate rotation theory for circular cylindrical shells



As noted in the figure, we can either use surface coordinates (x,θ) or (x,y) where $y = R\theta$ and R is the cylinder radius. We will use the latter, i.e. we take $(\xi^1, \xi^2) = (x, y)$. For a complete cylinder, $0 \le y \le 2\pi R$ Show that

$$g_{\alpha\beta} = \delta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \Gamma^{\gamma}_{\alpha\beta} = 0, \quad \vec{e}_1 = \vec{x}_{,1} = \vec{i}_3 \ \& \ \vec{e}_2 = \vec{x}_{,2} = \vec{i}_\theta, \ \ b_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & R^{-1} \end{pmatrix}$$

Moreover, the components of the displacement vector (u_1, u_2) are the physical components, as are the components of the strain tensors and the stress and bending moment tensors.

Note the following:

$$u_{1,1} = \partial u_1 / \partial x, \ u_{2,1} = \partial u_2 / \partial x, \ u_{1,2} = \partial u_1 / \partial y, \ \ u_{2,2} = \partial u_2 / \partial y$$