APMTH 105 Final Project: Musical Sounds

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Abstract

Strings, drums, and instruments alike are able to produce a number of sounds. We are able to model these sounds by calculating the eigenvalues and eigenfunctions of the Laplacian equation, \( \nabla^2 u = -\lambda^2 u \). Through the solutions of this equation, we mathematically modeled the sounds we expected to hear for a string, a square drum, and a circular drum. After examining the resulting sounds and frequencies produced through these calculations, we compared this theoretical data to experimental data obtained from recording similar sounds in the field. We used this information to determine how closely our models resembled the data obtained from real instruments. Finally, we used this data to extrapolate about the sounds produced by a number of other instruments. We were able to then incorporate additional songs into our project by analyzing the presence of the vibrations that we modeled in these environments.

1 Materials and Methods

For the well-defined part of the project, we first used our notes and a few online sources to solve the wave equation on a string, square drum, and circular drum. We then tested the predictions made in the solved equation by producing sounds and comparing them to graphs that were made based on the square of the coefficients (power) and lambdas (frequencies). We used an elastic band (to mimic the sound of a string being plucked), a square wooden table (to mimic the sound of a square drum being tapped), and a circular lid (to mimic the sound of a circular drum being tapped). We recorded the sounds on our phones, and then analyzed them using MATLAB. We charted frequency graphs to compare how the differently-shaped drums sound different.

For the open-ended part, we found video clips online of a clarinet, piano, trumpet, and violin playing the middle C note. We then analyzed the sound files in the same manner as we did in the well defined part: with MATLAB. We analyzed the frequency plots to determine why different instruments playing the same notes sounded different. After analysis of the four instruments, we found a song online that featured these instruments and analyzed that sound file in MATLAB. By analyzing the frequencies in the song, we can determine which of the four instruments was being played.
2 Results and Discussion

2.1 String

2.1.1 Solution of Wave Equation on a String

To treat the problem of sounds produced on a string with fixed ends of length \( L \), we start with the wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]  

(1)

Because we wish to solve an initial-value problem with singular boundary conditions, it is appropriate to use the method of separation of variables in two variables, \( x \) and \( t \), using the substitution:

\[
u(x, t) = X(x)T(t)\]

(2)

We thus obtain two boundary value problems for \( X(x) \) and \( T(t) \).

Applying boundary conditions to \( X(x) \) to account for the stability of the string at each end yields the following spatial Sturm-Liouville problem and corresponding eigenfunctions and eigenvalues:

\[
X'' + \lambda^2 X = 0
\]  

(3)

\[
X(0) = 0
\]  

(4)

\[
X(L) = 0
\]  

(5)

\[
X_n(x) = a_n \sin(\lambda_n x)
\]  

(6)

\[
X_n(x) = a_n \sin(\lambda_n x)
\]  

(7)

where \( \lambda_n = \frac{n\pi}{L} \), \( n = 1, 2, 3, \ldots \)

We need these eigenvalues from the \( X \) function, as they characterize the corresponding ODE for the time-dependent function, \( T(t) \). Then, applying the initial condition to the first time derivative, \( T'(t) = 0 \) at \( t \) due to the string begin released with an initial velocity of zero. This yields the following singular Sturm-Liouville problem and corresponding eigenfunctions and eigenvalues:

\[
T'' + c^2 \lambda^2 T = 0
\]  

(8)

\[
T'(0) = 0
\]  

(9)

\[
T_n(t) = b_n \cos(c\lambda_n t)
\]  

(10)
where \( \lambda_n = \frac{n\pi}{L} \), \( n = 1, 2, 3, \ldots \)

Thus, the complete solution \( u(x, t) \) is given by:

\[
    u(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) \tag{12}
\]

Explicitly,

\[
    u(x, t) = \sum_{n=1}^{\infty} \left( a_n \sin\left( \frac{n\pi x}{L} \right) \cos\left( \frac{n\pi ct}{L} \right) \right)
\]

\[
    = \sum_{n=1}^{\infty} \left( a_n \sin\left( \frac{\omega_n x}{c} \right) \cos(\omega_n t) \right)
\]

where \( \omega_n = \lambda_n c \), \( \lambda_n = \frac{n\pi}{L} \), \( n = 1, 2, 3, \ldots \), and \( c \) is the speed of light.

### 2.1.2 Theoretical Data

To determine how the sound of a string changes, we can analyze the calculated eigenfunctions after applying different initial conditions. The string has fixed ends and we assume a length of 1. If we pluck the string to a constant height of 1 and change the location along the length of the string from which it is being plucked, we can model this through the following initial conditions:

\[
    u(x, 0) = f(x) = \begin{cases} 
        \frac{x}{M} & \text{if } 0 < x < M, \\
        \frac{1-x}{1-M} & \text{if } M < x < 1. 
    \end{cases} \tag{13}
\]

If we plug in these initial conditions to our eigenfunctions derived in Section 1.1, we derive the following system of equations:

\[
    u(x, t) = \sum_{n=1}^{\infty} \left( a_n \sin\left( \frac{n\pi x}{L} \right) \cos\left( \frac{n\pi ct}{L} \right) \right)
\]

At \( t = 0 \) and \( L = 1 \), we get

\[
    a_n = 2 \int_{0}^{1} f(x) \sin(n\pi x) dx
\]

\[
    a_n = 2 \int_{0}^{M} \frac{x}{M} \sin(n\pi x) dx + 2 \int_{M}^{1} \frac{1-x}{1-M} \sin(n\pi x) dx
\]
Thus, the complete solution $u(r,t)$ with the $M$-dependent solution is given by:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \cos(n\pi ct) \tag{14}$$

where $\lambda_n = \frac{n\pi}{M}$, $n = 1, 2, 3, \ldots$

Plugging in the first 10 values of $n$ for $\lambda$, one obtains both the frequencies $\lambda_n$ and the amplitudes $a_n$, and we plot the power (the square of the amplitudes) vs. the corresponding frequency modes in Figures 1 and 2.
Figure 1: Bar plot of power contribution vs. frequency at the ten frequencies modeling the plucking of a string from the center. Bars are centered about these harmonic frequencies. Energy contribution at a frequency $\lambda_n$ obtained as the square of the coefficients of the eigenfunction product $u(x,t) = XT$ corresponding to that $\lambda_n$, the total sum over which is equal to the time-domain solution to the wave equation with initial conditions given in Equation 13.
Figure 2: Bar plot of power contribution vs. frequency at the ten frequencies modeling the plucking of a string from two different points. The left plot presents the values calculated when plucking the string at 0.75 units when the string is of length 1. The right plot presents the values calculated when plucking the string at 0.9 units when the string is of length 1. Similar to Figure 1, bars are centered about these harmonic frequencies and energy contribution is obtained as the square of the coefficients of the eigenfunction.

We can observe from these plots that the rates at which the amplitude decreases changes according the the value of $M$ at which you pluck the string. In the first plot, the value of $M$ is set to 0.5, so the string is being plucked from the center. In this case, the first coefficient, $a_1$ is much larger relative to the latter coefficients. As the value of $M$ reaches closer to the ends of the string, towards values of 0 or 1, the power attenuates more gradually. While the shape and oscillatory pattern remains quite similar throughout each scenario, the rate at which the coefficients decrease with time changes significantly for different values of $M$.

2.1.3 Experimental Data

We present sound file data for an actual string sound, which we produced by plucking a thick elastic band. In Figure 3 and 4, we depict the signal amplitude vs. time for multiple string plucks and for a single string pluck, respectively. In Figure 5, we depict the corresponding frequency distribution of power.
Figure 3: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained by plucking an elastic band (i.e. one-dimensional string) multiple times and recorded on a smartphone.
Figure 4: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 3) for one pluck of an elastic band. Audio file obtained by plucking an elastic band (i.e. one-dimensional string) multiple times and recorded on a smartphone.
Figure 5: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 4.
2.2 Square Drum

2.2.1 Solution of Wave Equation on a Square Drum

For a square drum, we have the following wave equation:

\[
\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{15}
\]

To be able to solve this equation as an initial value problem with singular boundary conditions, we will use the method of separation of variables in three variables, \(x\), \(y\), and \(t\) using the ansatz:

\[
u(x, y, t) = X(x)Y(y)T(t) \tag{16}
\]

where \(0 < x, y < L\) and \(t > 0\). Since the ends of the drums are bounded, the equation equals 0 at all bounds. Thus, we get two boundary value problems for \(X(x)\) and \(Y(y)\).

Plugging the ansatz back into the wave equation, we get:

\[
\frac{\partial^2 T}{Tc^2} = \frac{\partial^2 X}{X} + \frac{\partial^2 Y}{Y} \tag{17}
\]

let

\[
\frac{\partial^2 X}{X} = -p^2 \tag{18}
\]

\[
\frac{\partial^2 Y}{Y} = -k^2 \tag{19}
\]

\[
\frac{\partial^2 T}{Tc^2} = -c^2(p^2 + k^2) \tag{20}
\]

\[
\frac{p^2 + k^2}{\lambda^2}, c \text{ is the speed of sound, and frequency } \omega = c\lambda \text{ to get the following equations for } X \text{ and } Y.
\]

\[
X = A \cos(px) + B \sin(px) \tag{22}
\]

\[
Y = C \cos(ky) + D \sin(ky) \tag{23}
\]

Applying the boundary conditions \(X(0) = X(L) = 0\) and \(Y(0) = Y(L) = 0\), we get

\[
X = B \sin(n\pi x/L) \tag{25}
\]

\[
Y = D \sin(m\pi y/L) \tag{26}
\]

where \(p = n\pi/L, k = m\pi L, \lambda = \sqrt{(n^2m^2\pi^2)/L^2}\), and \(n, m = 0, 1, 2, \ldots\).
These two boundary value problems for $X(x)$ and $Y(y)$ define the corresponding spectra of $T(t)$. The ODE and the eigenfunctions and eigenvalues are given by:

$$T'' + \omega^2 T = 0 \quad (28)$$

$$T_{n,m}(t) = c_{n,m} \cos(\omega_{n,m} t) + d_{n,m} \sin(\omega_{n,m} t) \quad (30)$$

$$\omega_{n,m} = c \sqrt{l_{n,m}} \quad (31)$$

where $n = 0, 1, 2, \ldots$, and $l_{n,m}$ are given as before.

Thus, the complete solution $u(x, y, t)$ is given by:

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} X_n(x) Y_{n,m}(y) T_{n,m}(t) \quad (33)$$

Explicitly,

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (\sin(nx\pi/L)) \sin(my\pi/L) (e_{n,m} \cos(\omega_{n,m} t) + f_{n,m} \sin(\omega_{n,m} t)) \quad (34)$$

### 2.2.2 Theoretical Data

To determine how the sound of a square drum changes, we can analyze the calculated eigenfunctions after applying different initial conditions. The drum has fixed ends and we assume a length of 1 on each side. If we tap on the drum to a constant height of 1 and change the location over the area of the drum from which it is being tapped, we can model this through the following initial conditions: $u(x, y, 0) = 1$ and $\frac{\partial u}{\partial t} = 0$, to describe a drum membrane that has been stretched, but is not moving at that time.

We now apply the following initial conditions to the solution to the equation.

With the second initial condition,

$$\frac{\partial u}{\partial t} = 0 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (\sin(nx\pi/L)) \sin(my\pi/L) (-e_{n,m} \omega_{n,m} \sin(\omega_{n,m}0) + \omega_{n,m} f_{n,m} \cos(\omega_{n,m}t)) \quad (35)$$

and $f_{n,m} = 0$, leaving:

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (\sin(nx\pi/L)) \sin(my\pi/L) (e_{n,m} \cos(\omega_{n,m}t)) \quad (36)$$

Applying $u(x, y, 0) = 1$, we get
\[ u(x, y, 0) = 1 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (\sin(nx\pi/L)) \sin(my\pi/L)(e_{n,m}) \] 

(37)

Multiply each side by \( \sin(Mx\pi/L) \) and \( \sin(Nx\pi/L) \) (where M and N are different from m and n), and then integrate twice to get:

\[ \int \int_A \sin(Mx\pi/L) \sin(Ny\pi/L) \, dx \, dy = e_{n,m} \frac{L}{2} \frac{L}{2} \] 

(38)

Rearranging and solving for \( e_{n,m} \), we get

\[ e_{n,m} = 4L^2((\sin(M\pi/2))^2(\sin(N\pi/2))^2)/(MN\pi^2) \] 

(39)

where \( M = n, N = m, p = n\pi/L, k = m\pi L, \lambda_{n,m} = \sqrt{(n^2m^2\pi^2)/L^2} \), and \( n, m = 0, 1, 2, \ldots \). For simplicity, we allow \( L = 1 \).

Plugging in the first 10 values of \( n \) and \( m \) into \( \sqrt{(n^2 + m^2) \pi} \), one obtains both the frequencies \( \lambda_{n,m} \) and the amplitudes \( e_{n,m} \), and we plot the power (the square of the amplitudes) vs. the corresponding frequency modes in Figure 6.
Figure 6: Bar plot of power contribution vs. frequency at the ten frequencies modeling the tapping of a square drum. Bars are centered about these harmonic frequencies. Energy contribution at a frequency $\omega_{n,m}$ obtained as the square of the coefficients of the eigenfunction product $u(x, y, t) = XYT$ corresponding to that $\omega_{n,m}$, the total sum over which is equal to the time-domain solution to the wave equation with initial conditions given in equation.
Figure 7: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained by tapping a square wooden table (i.e. square drum) multiple times and recorded on a smartphone.

We can observe from this plot that when either $m$ or $n$ is even, the frequency is 0 (because of the coefficient $e_{n,m}$. The amplitude dramatically decreases as frequency increases.

2.2.3 Experimental Data

We present sound file data for an actual square drum sound, which we produced by tapping on a square wooden table. In the first two figures, we depict the signal amplitude vs. time for multiple drum beats and for a single drum beat, respectively. In the last figure, we depict the corresponding frequency distribution of power.
Figure 8: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 7) for one beat of a square drum. Audio file obtained by tapping a square wooden table (i.e. square drum) multiple times and recorded on a smartphone.
Figure 9: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 8.
2.3 Circular Drum

2.3.1 Solution of Wave Equation for Circular Drum

To treat the problem of sounds produced on a circular drum, we start with the wave equation in polar coordinates:

\[
\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad (40)
\]

Because we wish to solve an initial-value problem with singular boundary conditions, it is appropriate to use the method of separation of variables in three variables, \(r, \theta, t\), using the substitution:

\[
u(r, \theta, t) = R(r)\Theta(\theta)T(t) \quad (41)\]

We thus obtain two boundary value problems for \(\Theta(\theta)\) and \(R(r)\).

Applying periodic boundary conditions to \(\Theta(\theta)\) yields the following periodic Sturm-Liouville problem and the corresponding eigenfunctions and eigenvalues:

\[
\Theta'' + A\Theta = 0 \quad (42)
\]
\[
\Theta(0) = \Theta(2\pi) \quad (43)
\]
\[
\Theta'(0) = \Theta'(2\pi) \quad (44)
\]

\[
\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \quad (45)
\]

\[
A_n = n^2 \quad (46)
\]

\[
\sqrt{l_n,m} = 1 / R_0 z_n,m \quad (52)
\]

\[
z_{n,m} = \text{mth positive zero of } J_n(r) \quad (53)
\]

where \(n = 0, 1, 2, \ldots\).

We need these eigenvalues for the \(\Theta\) function, as they characterize the corresponding ODE for \(R(r)\). Then, applying the singular boundary condition to \(R(r)\) at \(r = 0\) and the homogeneous boundary condition at \(r = R_0\) yields the following singular Sturm-Liouville problem and the corresponding eigenfunctions and eigenvalues:

\[
(rR')' + (lr - n^2)R = 0 \quad (48)
\]
\[
|R(0)| < \infty \quad (49)
\]
\[
R(R_0) = 0 \quad (50)
\]

\[
R_{n,m}(r) = J_n(\sqrt{l_{n,m}}r) \quad (51)
\]
where \( n = 0, 1, 2, \ldots \), and \( J_n(r) \) is the \( n \)th Bessel function of the first kind.

These two boundary value problems for \( \Theta(\theta) \) and \( R(r) \) define the corresponding spectra of \( T(t) \) (for which there are no boundary conditions, hence there is no Sturm-Liouville problem). The ODE and the eigenfunctions and eigenvalues are given by:

\[
T'' + \omega^2 T = 0 \quad (54)
\]

\[
T_{n,m}(t) = c_{n,m} \cos(\omega_{n,m} t) + d_{n,m} \sin(\omega_{n,m} t) \quad (55)
\]

\[
\omega_{n,m} = c\sqrt{l_{n,m}} \quad (56)
\]

where \( n = 0, 1, 2, \ldots \), and \( l_{n,m} \) are given as before.

Thus, the complete solution \( u(r,\theta,t) \) is given by:

\[
u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Theta_n(\theta) R_{n,m}(r) T_{n,m}(t) \quad (57)
\]

Explicitly,

\[
u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) J_n(\sqrt{l_{n,m}} r)(c_{n,m} \cos(\omega_{n,m} t) + d_{n,m} \sin(\omega_{n,m} t))
\]

\[
(58)
\]

### 2.3.2 Theoretical Data

We present theoretical data for the distribution of energy over the first ten (circularly symmetric) modes for a circular drum subject to the circularly symmetric initial condition of the form:

\[
u(r,\theta,0) = f(r) \quad (59)
\]

\[
\partial_t u(r,\theta,0) = 0 \quad (60)
\]

where

\[
f(r) = M(1 - r/R_0) \quad (61)
\]

In our theoretical data calculations below (Figure 10), for simplicity, we take \( R_0 = 1 \) for the radius of the drum and \( c = 1 \) for the speed of propagation. Since this merely scales the angular frequency modes \( \omega_{n,m} \) by a constant multiplicative factor, the relative values of the frequency modes are preserved. The specific initial condition we analyze in the data below is for \( M = 1 \).

Recall that
\[ u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))J_n(\sqrt{l_{n,m}} r)(c_{n,m} \cos(\omega_{n,m} t) + d_{n,m} \sin(\omega_{n,m} t)) \]  

(62)

The condition that \( \partial_t u(r, \theta, 0) = 0 \) therefore implies that \( d_{n,m} = 0 \). And the circularly symmetric condition implies that there is no \( \theta \) dependence, so only \( n = 0 \) terms survive. Thus we can more simply write the eigenfunction decomposition as

\[ u(r, \theta, t) = \sum_{m=1}^{\infty} c_m J_0(\sqrt{l_{0,m}} r) \cos(\omega_{0,m} t) \]  

(63)

or, more explicitly in terms of the zeros of the 0th Bessel function of the 1st kind,

\[ u(r, \theta, t) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{r}{R_0} z_{0,m}\right) \cos \left(\frac{c t}{R_0} z_{0,m}\right) \]  

(64)

To calculate the distribution of energy over the frequency modes, we therefore use the formulas

\[ c_m = \frac{\int_0^{R_0} f(r) J_0\left(\frac{r}{R_0} z_{0,m}\right) r dr}{\int_0^{R_0} f(r) \left( J_0\left(\frac{r}{R_0} z_{0,m}\right) \right)^2 r dr} \]  

(65)

which follow from the orthogonality of the Bessel functions with respect to the \( r dr \) measure, given that they satisfy the Sturm-Liouville problem Eqn. 48.

However, we actually want to calculate the coefficients of the normalized eigenfunctions, since only those coefficients would correspond to the actual amplitudes of the corresponding frequency modes. We can ignore the normalization of the cosines since they all have the same normalization. Putting a normalization factor into the Bessel functions, one is left with the following coefficients \( \hat{c}_m \) as the relative amplitudes of the corresponding frequency modes:

\[ \hat{c}_m = \frac{\int_0^{R_0} f(r) J_0\left(\frac{r}{R_0} z_{0,m}\right) r dr}{\sqrt{\int_0^{R_0} f(r) \left( J_0\left(\frac{r}{R_0} z_{0,m}\right) \right)^2 r dr}} \]  

(66)

Plugging in \( R_0 = 1 \), and the Bessel zeros \( z_{0,m} \), one obtains both the frequencies \( \omega_{0,m} \) and the amplitudes \( c_m \), and we plot the power (the square of the amplitudes) vs. the corresponding frequency modes in Figure 10.

2.3.3 Experimental Data

We present sound file data for an actual circular drum sound, which we produced by tapping a circular lid. In Figure 11 and 12, we depict the signal amplitude vs. time for multiple circular drum beats and for a single circular drum beat, respectively. In Figure 13, we depict the corresponding frequency distribution of power.
Figure 10: Bar plot of energy contribution vs. frequency at the ten frequencies of the circular symmetric modes of the circular drum. Bars are centered about these harmonic frequencies. Energy contribution at a frequency $\omega_{n,m}$ obtained as the square of the coefficients of the normalized eigenfunction product $\Theta RT$ corresponding to $\omega_{n,m}$, the total sum over which is equal to the time-domain solution to the wave equation with initial conditions given in equations 59 and 61.
Figure 11: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained by tapping a circular lid (i.e. two-dimensional drum) multiple times and recorded on a smartphone.
Figure 12: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 11) for one tap of a circular lid (audio file obtained by tapping a circular lid (i.e. two-dimensional drum) multiple times and recorded on a smartphone).
Figure 13: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 12.
2.4 Discussion

We now compare our theoretical calculation of the energy distribution over frequency (Figure 1, 10) vs. our experimental data for the energy distribution over frequency (Figure 5, 13).

We first note that because we did not measure either velocity of wave propagation in the drum or the drum radius for the experimental data, a direct comparison of the spectral absolute values is impossible. However, a comparison of the relative distribution will suffice to determine accuracy of our 2-D wave equation model and corresponding initial conditions in describing the sound of a circular drum beat.

A direct graphical comparison illustrates several features:

1. Both sets of data have a dominant peak.

2. The theoretical spectral distribution dies off rapidly away from the dominant peak. Meanwhile, the experimental spectral distribution exhibits several peaks similar in size to the dominant peak.

3. The theoretical spectral distribution has no contributions below the dominant peak. Meanwhile, there is still significant power contribution to the experimental spectral distribution below the dominant peak.

4. The experimental spectral distribution has a much more varied landscape.

There are several hypotheses that may explain the disparities in the spectral distributions:

1. Our model is wrong, or at least, we have applied our frequency analysis across a time interval over which the wave equation loses validity. We may need to have selected smaller time intervals over which the circular drum beat had more uniform amplitude.

2. There is no time interval over which the wave equation is valid for the circular drum, i.e. one needs to add in dissipative terms which yield the attenuation exhibited in our sound file.

3. The initial condition was poorly chosen, but the model is right.

Of these hypotheses, it is most likely that the first one, at least, is correct. In our comparison, we essentially compared a frequency decomposition of an attenuated sound (experimental) to that of a periodic sound (theoretical). This seems to be a flawed comparison to start with, although nevertheless informative.

2.4.1 Comparison of Notes for String, Square Drum, and Circular Drum

After scaling the lowest 10 frequency modes for the string, square drum, and circular drum, such that the lowest note is 440 Hz, we obtain the table depicted in Figure 14. From the table, we see that the notes become more closely spaced as one goes from the string, to the square drum, and then to the circular drum.
3 Various Instruments and Songs

We then analyzed the sound produced by 4 instruments consisting of the trumpet, violin, clarinet, and piano. Similar to the previous processes, we calculated the coefficients of their respective Fourier series through frequency analysis. We analyzed the frequency of notes for each instrument when playing the middle C for consistency.

3.1 Instrument Analysis

3.1.1 Trumpet

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>String</th>
<th>Square Drum</th>
<th>Circular Drum</th>
</tr>
</thead>
<tbody>
<tr>
<td>440</td>
<td>440.</td>
<td>440.</td>
<td></td>
</tr>
<tr>
<td>880</td>
<td>622.254</td>
<td>589.73</td>
<td></td>
</tr>
<tr>
<td>1320</td>
<td>880.</td>
<td>732.643</td>
<td></td>
</tr>
<tr>
<td>1760</td>
<td>983.87</td>
<td>805.609</td>
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<td>1244.51</td>
<td>871.38</td>
<td></td>
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<td>2640</td>
<td>1320.</td>
<td>966.564</td>
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<td>3080</td>
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<td>1120.87</td>
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<td>1140.98</td>
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</tr>
<tr>
<td>4400</td>
<td>1814.17</td>
<td>1168.23</td>
<td></td>
</tr>
</tbody>
</table>

Figure 14: Lowest 10 notes (in Hz) for string, square drum, and circular drum, respectively.
Figure 15: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained from a clip on Youtube.com.
Figure 16: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 15) for one note (middle C). Audio file obtained from a clip on Youtube.com.
Figure 17: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 16
Figure 18: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained from a clip on Youtube.com.

3.1.2 Violin
Figure 19: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 18) for one note (middle C). Audio file obtained from a clip on Youtube.com.
Figure 20: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 19
Figure 21: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained from a clip on Youtube.com.

3.1.3 Clarinet
Figure 22: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 21) for one note (middle C). Audio file obtained from a clip on Youtube.com.
Figure 23: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 22.
Figure 24: Plot of the amplitude of incoming sound vs. the sample number of the audio file. Audio file obtained from a clip on Youtube.com.

3.1.4 Piano
Figure 25: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 24) for one note (middle C). Audio file obtained from a clip on Youtube.com.
Figure 26: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 25.
3.1.5 Discussion

When analyzing the component frequencies of a single middle C note being played by all of the instruments, we can see that the violin has the most power at a frequency of around $0.05 \times 10^4 \text{Hz}$; the trumpet at around $0.1 \times 10^4 \text{Hz}$; the piano at approximately $47 \text{Hz}$ and the clarinet at around $120 \text{Hz}$. This means that (as one can hear when each instrument plays its notes) even though they play the same notes, the instruments do not sound the same. Why is this? The biggest reason is that the thickness/depth of the surface over which the sound wave travels in each of these instruments is different. This results in vibrations occur at different frequencies, which produces different tones for the same note being played.

3.2 Analysis of Last Home by Typesun

Using the frequencies we plotted for the four instruments above, we analyzed the presence of the instruments in the song Last Home. This is an electronic song with the presence of the trumpet intermittently throughout the recording. We wanted to discover whether we could distinguish the trumpet notes from other clips of the song. We analyzed an eight second clip with both electronically produced sounds as well as trumpet interludes, as shown in Figure 27.

We then analyzed two clips where high peaks of incoming sound persisted. The first clip, Clip A, is one where electronic music could be detected. The second clip, Clip B, is one where a trumpet can be heard. In hopes of confirming these sound types, we will compare our frequency analysis of each clip to the analysis done on the individual instruments in the previous section. Figures 28 and 30 plot the amplitudes of each sound clip, A and B respectively. Figures 29 and 31 plot the power against the sound’s frequency for these song clips.
Figure 27: Plot of the amplitude of incoming sound vs. the sample number of the song *Last Home* by the artist Typesun. Audio file obtained from a clip on Youtube.com.
Figure 28: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 27) for the electronic music clip. Audio file obtained from a clip on Youtube.com.
Figure 29: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 28.
Figure 30: Plot of the amplitude of incoming sound vs. time. A time slice of the audio file (see Figure 27) for the trumpet note. Audio file obtained from a clip on Youtube.com.
Figure 31: Plot of energy per unit frequency versus frequency. Energy per unit frequency obtained by Fourier transform of time-domain sound signal displayed in Figure 30.
From these frequency plots, we see that although Clip A spans a slightly shorter time scale, there are more peaks occurring at a variety of frequency levels, ranging from 0 Hz to 500 Hz. Clip B shows only one major peak at a frequency of approximately 165 Hz. This does confirm our expectations that Clip A is electronic and Clip B is produced by a trumpet because an instrument, as we have seen, is unlikely to produce sounds at multiple frequencies simultaneously. Each of our frequency plots for the individual instruments showed only one dominant peak at a certain frequency. Electronic music can be fabricated to replicate many different sounds and frequencies, thus we see many peaks over a short time interval. We recognize that the frequency of the trumpet note in Figure 31 does not match that of the frequency determined in our initial analysis, shown in Figure 17. This difference can be accounted for by the fact that the note in the song, *Last Home*, is not the same note as the middle C that we analyzed first. Therefore, the frequencies will be slightly different even though the instrument producing these sounds is the same.

4 Conclusions

Through our calculations of the Laplace equation, we were able to determine the eigenvalues and eigenfunctions to model the sound of three different instruments: the string, the square drum, and the circular drum. After calculating the first ten coefficients of the Fourier Series produced by each of the eigenfunctions, we were able to detect particular frequencies and notes that resulted from the instruments. We found that each of the instruments showed similar attenuation of the coefficients as time increased, but the specific values of these coefficients varied across a significant range. We then created our own recordings of each instrument to compare and contrast the theoretical values to the experimental data. Similarly, we found that patterns could be resembled in both the theoretical and experimental data, but there were rather apparent discrepancies between these frequency plots across the board. In the Discussion Section, we attempted to detect where these discrepancies could have arisen. As previously stated, it’s likely that the model loses accuracy as the time span with which we apply it grows. Next, we uploaded sound data for four additional instruments: the trumpet, violin, clarinet, and piano. We determined the frequencies during which the power output was greatest for each of these instruments. We found that, for the most part, they each had one frequency that was most common for the given note, and this frequency varied amongst the different instruments. We were able to use these findings to analyze a song and determine where an instrument, the trumpet in our case, was being played versus other fabricated sounds. This analysis of musical vibrations has allowed us to form a more detailed understanding of the sounds we hear.

4.1 Acknowledgments

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4.2 Author Contributions

R.L, K.P., and K.S. designed research, carried out experiments, analyzed data, and wrote the paper. Specifically, K.P. carried out experiments on and analyzed the string, K.S. carried out experiments on and analyzed the square drum, and R.L. carried out experiments on and analyzed the circular drum. R.L. and K.P. developed MATLAB scripts; R.L., K.P. and K.S. used MATLAB scripts to perform calculations and data analysis. R.L., K.P., and K.S. designed research for comparison of instrument timbres; K.P. and K.S. obtained sound data and carried out analysis of instrument timbres.

References
