Problem Set 3

EC2450A

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Problem 1

There are two types of individuals, \( i = 1, 2 \) with different abilities \( w_i \). Let \( c_i \) be type \( i \)'s consumption, \( l_i \) be his hours worked and income \( y_i = w_i l_i \). Utility is increasing in consumption and decreasing in hours worked.

\[ u_i = u_i (c_i, l_i) \]

Suppose there is a tax \( T(y_i) \) on income.

1) Write the budget constraint of type \( i \) under this tax system.

The budget constraint is:

\[ c_i \leq y_i - T(y_i) \]

2) Write the maximization problem of the individual under this tax system and derive the first-order conditions.

\[ \max_{c_i, l_i} u_i (c_i, l_i) \]

\[ c_i \leq y_i - T(y_i) \]

The FOC is:

\[ \frac{\partial u_i}{\partial l_i} = -w_i (1 - T'(y_i)) \]

3) Express the marginal tax rate as a function of the primitives (marginal utilities and ability levels). This gives you a characterization of the marginal tax rate as a function of the allocations.

We can write the marginal tax rate as a function of the allocations as:

\[ T'(y_i) = 1 + \frac{1}{w_i} \frac{\partial u_i}{\partial c_i} \]
4) Define \( V_i(c_i, y_i; w_i) = u_i\left( c_i, \frac{y_i}{w_i}\right) \) to be the utility expressed as a function of observable \( y_i \). Express the derivatives: \( \frac{\partial V_i}{\partial c_i}, \frac{\partial V_i}{\partial y_i}, \frac{\partial V_i}{\partial w_i} \) as functions of the derivatives of \( u_i \). Express the marginal tax rate \( 1 - T \)' as a function of the derivatives of \( V_i \).

\[
\frac{\partial V_i}{\partial c_i} = \frac{\partial u_i}{\partial c_i} ; \quad \frac{\partial V_i}{\partial y_i} = \frac{\partial u_i}{\partial l_i} \frac{y_i}{w_i} ; \quad \frac{\partial V_i}{\partial w_i} = -\frac{\partial u_i}{\partial l_i} \frac{y_i}{w_i}^2 = -\frac{\partial V_i}{\partial \frac{y_i}{w_i}} \frac{y_i}{w_i} \\
\frac{\partial V_i}{\partial y_i} / \frac{\partial y_i}{\partial c_i} = 1 - T'(y_i)
\]

5) What first-order condition would characterize the first-best allocation (in the first best allocation, there are lump-sum, type-specific taxes available since there is no private information).

In the first-best there is lump-sum taxation only:

\[
\frac{\partial V_i}{\partial y_i} / \frac{\partial y_i}{\partial c_i} = 1
\]

6) Suppose that \( w_2 > w_1 \). Draw one indifference curve for each type in the \((y, c)\) space. Which one is flatter? Explain intuitively which type experiences a higher reduction in utility from the same increase in income \( dy \).

Type 2 experiences a smaller utility change for the same increase \( dy \). The high type is willing to trade a smaller amount of consumption for a reduction in income since she needs to forgo a lower amount of effort to reach the same income.

7) Types \( w_i \) are unobserved. Explain why choosing optimal taxes in this setting is equivalent to choosing a menu of \((c_i, y_i)\) pairs. Is there a more general mechanism that we could come up with that would do better than setting this menu (please justify your answer).

We know we can design a mechanism that chooses allocations for every type because of the revelation principle (see also PS1).

8) Suppose that there are \( n_i \) individuals of type \( i \) and that there is a revenue requirement \( \bar{R} \). Write the revenue constraint in terms of allocations \((y_i, c_i)\).

The revenue constraint is:

\[
R = (y_1 - c_1)n_1 + (y_2 - c_2)n_2 \geq \bar{R}
\]

9) Set up the Pareto problem: the Pareto problem maximizes the utility of type 2 subject to type 1 reaching some target utility \( \bar{u}_1 \) and subject to constraints. Think carefully what the constraints are (don’t forget that types are unobservable).

The Pareto problem is:

\[
\max_{c_1, c_2, y_1, y_2} V^2(c_2, y_2)
\]
s.t.

\[ V^1(c_1, y_1) \geq U^1 \]

\[ V^2(c_2, y_2) \geq V^2(c_1, y_1) \]

\[ V^1(c_1, y_1) \geq V^1(c_2, y_2) \]

\[ R = (y_1 - c_1)n_1 + (y_2 - c_2)n_2 \geq \bar{R} \]

10) Write the Lagrangian for this constrained maximization and provide the first-order conditions. Show that there are (only) three possible regimes, depending on which constraints bind.

\[ L = V^2(c_2, y_2) + \mu V^1(c_1, y_1) + \lambda_2 (V^2(c_2, y_2) - V^2(c_1, y_1)) + \lambda_1 (V^1(c_1, y_1) - V^1(c_2, y_2)) + \gamma \left[ (y_1 - c_1)n_1 + (y_2 - c_2)n_2 - \bar{R} \right] \]

The FOCs are:

\[ \frac{\partial L}{\partial c_1} = \mu \frac{\partial V^1}{\partial c_1} - \lambda_2 \frac{\partial V^2}{\partial c_1} + \lambda_1 \frac{\partial V^1}{\partial c_1} - \gamma n_1 = 0 \]

\[ \frac{\partial L}{\partial y_1} = \mu \frac{\partial V^1}{\partial y_1} - \lambda_2 \frac{\partial V^2}{\partial y_1} + \lambda_1 \frac{\partial V^1}{\partial y_1} - \gamma n_1 = 0 \]

\[ \frac{\partial L}{\partial c_2} = \mu \frac{\partial V^2}{\partial c_2} + \lambda_2 \frac{\partial V^2}{\partial c_2} - \lambda_1 \frac{\partial V^1}{\partial c_2} - \gamma n_2 = 0 \]

\[ \frac{\partial L}{\partial y_2} = \mu \frac{\partial V^2}{\partial y_2} + \lambda_2 \frac{\partial V^2}{\partial y_2} - \lambda_1 \frac{\partial V^1}{\partial y_2} - \gamma n_2 = 0 \]

Three regimes arise:

i) \( \lambda_1 = 0, \lambda_2 = 0 \)

ii) \( \lambda_1 = 0, \lambda_2 > 0 \)

iii) \( \lambda_1 > 0, \lambda_2 = 0 \)
11) Is there a case in which the first-best solution would apply? Characterize it in terms of the values of the multipliers and the allocation.

When \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \) the equilibrium is fully revealing and the first-best solution applies. Every agent strictly prefers her bundle and there is no risk of imitation.

12) Suppose that the incentive constraint on type 2 is binding. Using your answer to 3) and 10), show that the marginal tax rate faced by type 2 is zero and explain this intuitively. Also show that the marginal tax rate faced by type 1 is positive and explain why. When is this regime likely to occur (as a function of the primitives of the problem)?

Using the FOCs wrt \( c_2 \) and \( y_2 \) we derive:

\[
-\frac{\partial V^2}{\partial y_2} \frac{\partial V^2}{\partial c_2} = -\frac{\partial u^2}{\partial y_2} \frac{1}{\partial u^2/\partial c_2} w_2 = 1
\]

This equation is equal to the FOC for type 2. Therefore, at the optimum agent 2 faces a zero marginal tax rate. From the FOCs for type 1 we derive:

\[
-\frac{\partial V^1}{\partial y_1} \frac{\partial V^1}{\partial c_1} = \frac{1 - \lambda_2 (\partial V^2/\partial y_1) / n_1 \gamma}{1 + \lambda_2 (\partial V^2/\partial c_1) / n_1 \gamma} < 1
\]

To see this define:

\[
\alpha_i = \frac{\partial V^i}{\partial y_1} \frac{\partial V^i}{\partial c_1}
\]

\[
v = \frac{\lambda_1 \partial V^2/\partial c_1}{N_1 \gamma}
\]

Then we can rewrite the expression above as:

\[
\alpha^1 = \frac{1 + v \alpha^2}{1 + v} = \alpha^2 + \frac{1 - \alpha^2}{1 + v}
\]

Since by assumption \( \alpha^1 > \alpha^2 \), it follows that:

\[
\alpha^2 < \alpha^1 < 1
\]

Problem 2

Consider an overlapping generations model. Suppose that there are two types of consumers, 1 and 2 in each generation. In period \( t \) there are \( N^1_t = N^1(1+n)^t \) young of type 1 whose productivity is \( v < 1 \) and \( N^2_t = N^2(1+n)^t \) young of type 2 with productivity 1. Everyone earns a wage equal to their productivity since the labor market is perfectly competitive.

There is an aggregate production function \( F \) such that:
\[ Q_t = F(K_t, E_t) \text{ with: } E_t = vN^1_1 l^1_t + N^2_1 l^2_t \]

\( E_t \) are efficient units of labor. Denote by \( w_t \) the wage per unit of efficient labor. Type \( i \)'s utility is \( u_i(c^i_t, l^i_t) \) where \( c^i_t = (c^i_{yt}, c^i_{ot,t-1}) \) are his consumptions when he is young and when he is old.

1) Suppose that the government can freely choose \( K_t \) and \( K_{t+1} \) (effective saving every period) and the interest rate is \( r = 0 \) (so a unit saved just gives you one unit of consumption later). The government collects taxes to finance revenue requirements \( R_t \). Write down the government budget constraint at time \( t \) as a function of consumption, capital, and revenue requirements (not taxes).

Write total efficient units of labor as \( E_t = vN^1_1 l^1_t + N^2_1 l^2_t \). The government budget constraint is:

\[ F(K_t, E_t) + K_t = N^1_1 c^1_{yt} + N^2_1 c^2_{yt} + N^1_1 c^1_{ot,t} + N^2_1 c^2_{ot,t} + K_{t+1} + R_t \]

2) Argue why, if there is private information on the types, the government can restrict itself to choosing a menu of allocations for each type. What do these allocations specify for each type?

By the revelation principle, the government must choose a direct revealing mechanism \( (C^1_t, Y^1_t, C^2_t, Y^2_t) \) under a self-selection constraint that states that the skilled do not pretend to be unskilled so as to pay lower taxes.

3) Argue that in general, type 2’s incentive constraint will be binding and write it. You can assume for now that the incentive constraint on type 1 is slack.

Type 2’s incentive constraint will be binding when the government has redistributionary preferences. If it was not, the government could lump-sum tax type 2 in order to transfer resources to type 1, whose social marginal welfare weight is higher. We can write:

\[ U^2(c^2_t, Y^2_t, w_t) = U^2(c^1_t, Y^1_t, w_t) \]

4) The government objective is:

\[ W = \sum_{t=1}^{\infty} W_t = \sum_{t=1}^{\infty} (\mu^1_t N^1_1 u^1_t + \mu^2_t N^2_1 u^2_t) \]

Write the Lagrangian of this problem.

The Lagrangian is:

\[ L_t = W_t + \lambda^2_t \left( U^2 \left( c^2_t, \frac{Y^2_t}{w_t} \right) - U^2 \left( c^1_t, \frac{Y^1_t}{w_t} \right) \right) + \gamma_t \left( F(K_t, E_t) + K_t - N^1_1 c^1_{yt} - N^2_1 c^2_{yt} - N^1_1 c^1_{ot,t-1} - N^2_1 c^2_{ot,t-1} - K_{t+1} - R_t \right) \]

5) Suppose that the government can freely choose \( K_t \) as a control variable. Explain what this
This amounts to assuming that the government has the means to fix the capital stock at its optimal level by issuing public debt. Suppose we are in an economy where debt is contracted for one period and it pays the same interest rate as capital - which must be true in equilibrium since there is no risk. Then private savings finance both investment and public debt and the government can issue debt to fix the total capital stock, which is the aggregate level of intergenerational transfers. This allows to separate the problem of optimizing the aggregate capital stock transferred across periods from the optimal capital allocation within each period.

6) Take the first-order conditions of the government’s maximization problem. Hint: If you need to take the FOC with respect to \( y_t \), it may be easier to do a change of variables and to maximize with respect to \( l_1^1 = \frac{y_t^1}{w_t} \) and \( l_2^2 = \frac{y_t^2}{w_t} \).

FOCs for consumption are:

\[
\frac{\partial L}{\partial c_{2yt}} = (N_t^2 \mu_{2t}^2 + \lambda_t^2) \frac{\partial U^2}{\partial c_{2yt}}(c_{2t}^2, l_t^2) - N_t^2 \gamma_t = 0
\]

\[
\frac{\partial L}{\partial c_{0,t+1}^2} = (N_t^2 \mu_{2t}^2 + \lambda_t^2) \frac{\partial U^2}{\partial c_{0,t+1}^2}(c_{2t}^2, l_t^2) - N_t^2 \gamma_{t+1} = 0
\]

FOCs for labor are:

\[
\frac{\partial L}{\partial l_1^1} = \mu_t^1 N_t^1 \frac{\partial U}{\partial l_1^1}(c_{1t}^1, l_t^1) - \lambda_t^1 \frac{\partial U^2}{\partial l_1^1}(c_{1t}^1, l_t^1) + \gamma_t N_t^1 F_t^1(K_t, E_t) = 0
\]

\[
\frac{\partial L}{\partial l_2^2} = \mu_t^2 N_t^2 \frac{\partial U^2}{\partial l_2^2}(c_{2t}^2, l_t^2) + \lambda_t^2 \frac{\partial U^2}{\partial l_2^2}(c_{2t}^2, l_t^2) + \gamma_t N_t^2 F_t^2(K_t, E_t) = 0
\]

The FOC for capital is:

\[
\frac{\partial L}{\partial K_{t+1}} = \gamma_{t+1} \left( 1 + F'_t(K_t, E_t) \right) - \gamma_t = 0
\]

7) Using the FOC with respect to \( c_{2yt}^2 \) and \( c_{0,t+1}^2 \) and \( K_{t+1} \), show that there should not be a marginal tax on the interest income paid to type 2 (you need to define what interest income means here in terms of primitives, i.e., marginal utility of consumption in different periods). What does this mean and why?

Combining the equations above we get:
\[
\frac{\partial U^2}{\partial c^o_{t+1}}(c^1, l^1_t) = \frac{\gamma_t}{\gamma_{t+1}} = 1 + F'_{K}(K_{t+1}, E_{t+1})
\]

The left-hand side is just the marginal rate of substitution of type 2 between current and future consumption, and this is equal to one plus the after-tax interest rate at the optimum of the consumers optimization program. On the other hand, the marginal productivity of capital on the right-hand side equals the before-tax interest rate at the optimum of the producers program. It follows immediately that at the social optimum, the government should not tax the interest income paid to type 2.

At this stage this result brings to mind the fact that the marginal tax rate on the labor income of the most productive agent is zero, which is not very surprising. It is still possible for the interest income paid to type 1 to be taxed. This is where the weak separability of utility functions comes into play.

8) Take the FOCs with respect to \(c^1_{yt}\) and \(c^1_{o,t+1}\). Impose a weak separability assumption, i.e.,

\[u_i(c^1_t, l^1_t) = \tilde{u}_i(h(c^1_t), l^1_t)\]

for some function \(h()\).

Show that we can rewrite the FOC with respect to \(c^1_{yt}\) so as to make the FOC for \(c^1_{o,t+1}\) appear in it. This will help you argue that the tax on type 1’s capital income should also be zero.

The first-order conditions in \(c^1_{yt}\) and \(c^1_{o,t+1}\) are:

\[
\frac{\partial L}{\partial c^1_{yt}} = N^1_t \mu^1_t \frac{\partial U^1}{\partial c^1_{yt}}(c^1_t, l^1_t) - \lambda^2_t \frac{\partial U^2}{\partial c^o_{yt}}(c^1_t, l^1_t) - N^1_t \gamma_t = 0
\]

\[
\frac{\partial L}{\partial c^1_{o,t+1}} = N^1_t \mu^1_t \frac{\partial U^1}{\partial c^1_{o,t+1}}(c^1_t, l^1_t) - \lambda^2_t \frac{\partial U^2}{\partial c^o_{o,t+1}}(c^1_t, l^1_t) - N^1_t \gamma_{t+1} = 0
\]

If we impose weak separability

\[U^i(c^1_t, l^1_t) = \tilde{U}^i(h(c^1_t), l^1_t)\]

then for all \(l\) and for each \(i = 1, 2\)

\[
\frac{\partial U^i}{\partial c^1_{yt}}(c^1_t, l^1_t) = \frac{\partial h(c^1_t)}{\partial c^1_{yt}} / \frac{\partial h(c^1_t)}{\partial c^1_{o,t+1}}
\]
The equation

\[
\frac{\partial L}{\partial c_{yt}} = 0
\]

can then be rewritten as

\[
\frac{\partial h(c_{it})}{\partial c_{i,t}} \left( N_t \mu_t \frac{\partial U^1}{\partial c_{o,t+1}}(c_{t}^1, l_{t}^1) - \lambda_t^2 \frac{\partial U^2}{\partial c_{o,t+1}}(c_{t}^1, l_{t}^1 v) \right) = N_t \gamma_t
\]

and we get by substituting in the other first-order condition

\[
\frac{\partial h(c_{i,t})}{\partial c_{i,t}} = \frac{\gamma_t}{\gamma_{t+1}} = 1 + F'_K(K_{t+1}, E_{t+1})
\]

This last equation means that the marginal rate of substitution of the consumer and its equivalent for the firm are equal. Thus at the social optimum, type 1 should not be taxed on his interest income, no more than type 2. Therefore the optimal taxation of interest income is zero.

9) What is driving the result in 8)? Think of several things and try to reason through them carefully. (Hints: you know what made the results go through in the FOCs mechanically. Stare at those FOCs and think what happens if you relaxed some assumptions. Question all FOCs you used!)

The result is due to Atkinson Stiglitz preferences. As long as the government can choose the optimal level of capital, the level of consumption in the two periods is not a good tag for the type and therefore there is not reason to tax it.