Coarse Competitive Equilibrium and Extreme Prices

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Abstract

We introduce a notion of coarse competitive equilibrium (CCE), to study households’ inability to tailor their consumption to the state of the economy. Our notion is motivated by limited cognitive ability (in particular attention, memory, and complexity) and it maintains the complete market structure of competitive equilibrium. Compared to standard competitive equilibrium, our concept yields riskier allocations and more extreme prices (both for consumption and for assets). Thus, limited cognitive ability can produce market data that are usually attributed to heightened degrees of risk aversion. We provide a tractable model that is suitable for general equilibrium analysis as well as asset pricing in dynamic environments.

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1 Introduction

Many of the empirical puzzles in macroeconomics and finance arise from researchers’ inability to reconcile the levels of risk aversion implied by equilibrium models with levels that are observed in other contexts or are reasonable a priori. More specifically, many of these puzzles amount to observing that the degree of risk aversion required to accommodate typical equilibrium outcomes is implausibly high. For example, the equity premium puzzle (Mehra and Prescott, 1985) is the observation that stock returns are too large given reasonable levels of risk aversion and observed variability of aggregate consumption and dividends. Similarly, the home bias (French and Poterba, 1991) refers to the comparatively much higher risk aversion that investors exhibit toward foreign assets by forgoing the diversification opportunities that these markets offer.

We define a notion of competitive equilibrium, *Coarse Competitive Equilibrium* (CCE), that enables the modeler to limit households’ ability to process and react to information. We compare CCE prices to prices in a standard competitive equilibrium (SCE), and show that, holding risk aversion levels constant, the range of CCE prices is much greater range than that of SCE prices. Since equilibrium price variation is an increasing function of the representative agent’s risk aversion, behavioral limitations can be thought of as a substitute for risk aversion: behavioral limitations lead to aggregate prices that are consistent with an economy of agents unaffected by such limitations, but having higher risk aversion.

In the most general terms, we identify behavioral limitations with the cost agents must incur when tailoring their actions to their circumstances. If the circumstance is a history of past actions or events, we interpret the cost as a memory cost: for an event that is too costly to remember, the agent chooses the same action whether or not it happened. If the circumstance is a future contingency, we interpret the cost as a complexity cost: the more finely a given plan depends on future events, the more complex and hence the more costly it is. Finally, if the circumstance is a current variable, we interpret the cost as an information processing or a communication cost: if paying attention to a current event is too costly, the agent chooses the same action whether or not it happens.

We consider both a static and a dynamic exchange economy. We identify circumstance
with the realized price and *action* with a consumption level. Hence, within the classification above, our static model is one of limited information processing ability while our dynamic model imposes both information processing cost and limited memory. For our agents, adjusting their consumptions to price changes is costly. We adopt the following simple cost function: households can only choose *coarse consumption plans*; that is, plans that restrict them to at most $k$ distinct actions. Hence, all coarse plans have zero cost and all others are infinitely costly.\(^1\)

Consider, for example, a household who can distinguish between just two broad categories of economic events. This household may partition circumstances into “good times” and “bad times” and choose one consumption level for each event. Thus, the household makes two decisions: how to define good times/bad times and how much to consume in each contingency. While the second decision is standard, the first one is novel—this is our device for modeling how limited information processing or attention allocation responds to economic incentives. Note that in a stylized model such as ours the mathematical formulation of the behavioral household optimization problem may be more complicated than the standard household decision problem. Optimally partitioning the state space may seem like a difficult task. However, our model is not meant as a description of the household’s reasoning process. Rather, we aim to model a agent who, while unable to react to all price changes, responds to incentives when allocating cognitive resources.

To simplify the exposition, we assume that there is a single physical good. We also assume a continuum of households each with the same state-independent CRRA utility function. The two-period model has a planning period and a consumption period. Households learn the state after the planning period and before the consumption period. In the planning period, each household chooses a coarse (state-contingent) consumption plan. We show that a CCE exists and is Pareto optimal (given the restriction to coarse consumption plans). Notice that a single coarse consumption plan cannot distinguish among all states. Therefore, if all consumers were to choose the same coarse consumption plan, markets would not clear. For example, if

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\(^1\)In the concluding section, we consider alternative formulations of action-circumstances and alternative cost functions and discuss how our results may change with these formulations.
households can only choose two consumption levels (“good times” and “bad times”) then, in
equilibrium, all households cannot classify states into the same two categories. Hence, in a
CCE, ex ante identical consumers may and sometimes must choose distinct plans, otherwise,
the aggregate demand would take only two values. Therefore, consumption is more risky in a
CCE than in a SCE.

To study CCE prices in an economy with a non-atomic endowment distribution, we fix the
households’ utility function and consider a sequence of discrete endowment economies. We
show that, in the non-atomic limit, CCE prices in states near the lowest or the highest possible
endowment realization are extreme; the price of consumption converges to infinity when the
endowment is at or near the lower bound of the distribution while the price converges to zero
when the endowment is near or at the upper bound of the distribution. To see why this result
is suggestive of very high risk aversion, consider a standard economy in which the aggregate
endowment is distributed uniformly on the interval $[1, 2]$. Suppose every household has the
same CRRA utility function $u$. Then, irrespective of the individual endowment distribution,
this economy will behave as if there is a representative agent: it follows from the first order
conditions that equilibrium prices will satisfy

$$u'(x) = \lambda \cdot p(x)$$

for some constant $\lambda$. Differentiating both sides yields

$$u''(x) = \lambda \cdot p'(x).$$

A straightforward manipulation of these two equality reveals that

$$-\frac{xu''(x)}{u'(x)} = -\frac{xp'(x)}{p(x)}.$$

Clearly, the left-hand side of the above equation is the coefficient of relative risk aversion while
the right-hand side is the reciprocal of the elasticity of demand. Hence, the aggregate demand
will have constant elasticity equal to the reciprocal of the coefficient of risk aversion. This
means that the range of observed prices will be an increasing function of the representative
households coefficient of relative risk aversion. An infinite range, which is what a CCE implies,
is suggestive of an infinite coefficient of relative risk aversion.

Our results show how behavioral constraints can create extreme prices in a competitive
model. The intuition behind these results is as follows. The households’ behavioral constraints
imply that paying attention to very rare events is costly. On the other hand, market clearing
and aggregate endowment variability mean that at least some agents distinguish between these
unlikely events. Thus, prices must incentivize a small fraction of households to absorb the
aggregate outcome fluctuation in such rare events. Getting a small fraction of households to
bear the aggregate risk requires extreme prices near unusually low or unusually high endowment
realizations.

We also consider a dynamic economy in which a finite state Markov chain describes the
transitions of the endowment. We show that a stationary CCE exists, which ensures that the
equilibrium of the dynamic economy can be mapped to an equilibrium of an appropriately
defined two-period model. This allows us to describe equilibrium allocations and prices in the
dynamic economy in terms of the prices and allocations of a two-period model. As in the two-
period case, we fix the utility function and consider a sequence of economies converging to a limit
economy with a non-atomic invariant distribution of endowments. The extreme consumption
prices of the two-period model imply extreme asset prices in the dynamic economy. Specifically,
consider an asset that pays off a share of the endowment in all future periods. Because of
stationarity, the price of this asset is a random variable that depends only on the current state
of the economy (the current endowment). When the realized endowment is near or at the lower
bound of possible endowments, the price of the asset is essentially zero. Conversely, as we
approach the upper bound of possible endowment realizations, the price of this asset converges
to infinity.

1.1 Relation to Literature

The game theory literature has developed strategic analogs of coarse equilibrium. Neyman
(1985), Rubinstein (1986), and Abreu and Rubinstein (1988) limit players’ strategies in a re-
peated game to those implementable by finite state automata. Our approach is closest to Neyman (1985) who studies Nash equilibria of a game in which the number of states in the automaton is bounded. Abreu and Rubinstein (1988) also study Nash equilibria, but with a different cost function. Rubinstein (1986) examines a lexicographic cost of complexity and imposes a version of subgame perfection which precludes agents from adopting a different automaton later in the game. In Dow (1991)’s model of search with limited memory, the agent optimally partitions histories. Piccione and Rubinstein (1997) examine the relation between limited memory (i.e., imperfect recall) and time consistency. Wilson (2002) analyzes long-run inference and shows that the optimal use of a limited memory can lead to many well-studied behavioral biases. Mullainathan (2002) studies a model of coarse categorization and its implications for asset returns and their trade volume. In Masatlioglu, Nakajima, and Ozbay (2012) agents make optimal choices subject to an endogenous attention constraint. Jehiel (2005) and Jehiel and Samet (2007) constrain players to respond the same way in similar situations by bundling their decision nodes into analogy classes (by constraining Player i’s beliefs about Player j’s strategies). Despite the differences in modeling details, all these papers, including ours, restrict agents’ ability to tailor their behavior to their environment by imposing additional constraints. We develop a tractable competitive equilibrium model with this type of constraint and analyze its effect on equilibrium prices.

Sims (2003) assumes that agents allocate their attention optimally subject to an information-theoretic constraint. Our cost function is different: we limit the number of possible signal values while Sims (2002) formulates an entropy based constraint. Woodford (2011) modifies Sims’ cost function to address consumer choice anomalies. Mankiw and Reis (2002) study a model that has only a fraction of agents getting new information each period. Hong, Stein, and Yu (2007) study a model in which agents are restricted to a small class of forecasting models that does not include the true model of the world. These papers study how information processing frictions impact asset prices and responses to monetary policy.

There are a number of papers that use rigidities in consumption to close the gap between the level of risk aversion needed to rationalize data and plausible levels of risk aversion. Grossman and Laroque (1990) distinguish liquid and illiquid consumption and assume that agents incur
transaction costs when they sell an illiquid good. Lynch (1996) and Gabaix and Laibson (2002) study a model in which only a fraction of agents can make adjustments at a given time. Guvenen (2009) studies limited stock market participation in a model with heterogeneity in the elasticity of intertemporal substitution in consumption. Chetty and Szeidl (2010) focus on the extent to which consumption rigidities reduce stock market participation. If some households cannot adjust their consumption, the remaining subset must absorb all of the changes in aggregate consumption. As a result, observed prices (or interest rates) are consistent with a lower elasticity of intertemporal substitution or, equivalently, lower relative risk aversion, than in a model in which all households are unrestricted. Unlike Lynch (1996) and Gabaix and Laibson (2002), we do not exogenously fix the fraction of households that can respond to a particular increase in aggregate output. Rather, we make adjustments costly and let households respond optimally. As we discuss in the next section, the fact that households choose their partitions endogenously enhances the impact of their behavioral limitation on equilibrium prices.

2 CCE in a Static Economy

There is a unit mass of households with identical utility functions but different stochastic endowments of the single physical good. Aggregate endowment depends on the realization of the state \( i \in N = \{1, \ldots, n\} \). Let \( \pi_i \) denote the probability of the uncertain state \( i \), \( s_i \) the aggregate endowment in state \( i \) and \( K(s) \) the finite support of \( s \). Let \( a \) be the smallest and \( b \) the largest aggregate endowment realizations. We assume that \( 0 < a < b \). For any set \( X \), let \( |X| \) denote its cardinality; hence \( |K(s)| \) is the number of distinct aggregate endowment realizations.

A standard consumption plan is a vector in \( C = \mathbb{R}_+^n \) and a household’s utility of the consumption plan \( c \) is

\[
U(c) = \sum_{i \in N} u(c_i) \pi_i
\]

where \( u \) is any strictly concave CRRA utility index. That is, for \( \rho > 0 \),

\[
u(c_i) = \begin{cases} 
c_i^{1-\rho}/(1-\rho) & \text{if } \rho \neq 1 \\
\ln c_i & \text{if } \rho = 1
\end{cases}
\]
In a CCE, households are restricted to coarse consumption plans; that is, plans with at most \(k\) distinct consumption levels. To avoid trivial cases, we assume throughout that \(1 < k < |K(s)|\).

**Definition 1.** The consumption plan \(c \in C\) is coarse if \(|\{c_i | i \in N\}| \leq k\).

Let \(C_k\) be the set of all coarse consumption plans. Clearly, \(C_k \subset C_{k+1}\) for all \(k\). Note that \(C_k\) is not convex since a convex combination of \(c, c' \in C_k\) would typically require more than \(k\) consumption levels. This non-convexity implies that, in equilibrium, identical households might and typically will choose different consumption plans. For example, if \(k = 2\) and \(K(s) = 3\), some households will consume the same amount in the first two states while others consume equal quantities in the last two states.

An element \(p\) in the \(n-1\)-dimensional simplex \(\Delta(N)\) is a (normalized) price. An endowment \(\omega\) is an element of \(\mathbb{R}^N_+\). At price \(p\), the wealth of a household with endowment \(\omega\) is \(w = p \cdot \omega = \sum_{i \in N} \omega_i \cdot p_i\). That household’s budget set is

\[
B_k(p, w) = \left\{ c \in C_k : p \cdot c \leq w \right\}
\]

We suppress the endowment distribution and call \(E = \{u, k, \pi, s\}\), the economy. A representative household is one with endowment \(s\). We call an economy a representative economy if every household is a representative household. For such a household, we write \(B_k(p)\) rather than \(B_k(p, w)\).

For any set \(X\), let \(\Delta(X)\) denote the set of simple probabilities on \(X\); that is \(\Delta(X)\) is the set of all functions \(\mu \in [0, 1]^X\) such that \(K(\mu) = \{x : \mu(x) > 0\}\) is finite and \(\sum_{x \in X} \mu(x) = 1\). For any \(Y \subset X\), we let \(\mu(Y)\) denote \(\sum_{x \in Y} \mu(x)\). We call \(K(\mu)\) the support of \(\mu\). When \(X\) is finite, we identify \(\Delta(X)\) with the \(|X|-1\) dimensional simplex.

An allocation \(\mu \in \Delta(C)\) is a probability on consumption plans; \(\mu(c)\) represents the fraction of households who choose plan \(c \in C\). An allocation is coarse if \(K(\mu) \subset C_k\). The allocation \(\mu\) is feasible in \(E\) if

\[
\sigma_i(\mu) := \sum_{c \in K(\mu)} c_i \cdot \mu(c) \leq s_i
\]

for all \(i \in N\). For any \(C' \subset C\), let \(M(C')\) be the set of all feasible allocations such that \(K(\mu) \subset C'\).
Hence, $M(C_k)$ is the set of feasible allocations for our behavioral economy. Henceforth, $\mu$ is feasible means $\mu \in M(C_k)$.

The coarse consumption plan $c \in B_k(p, w)$ is optimal at prices $p$ and wealth $w > 0$ if $U(c) \geq U(c')$ for all $c' \in B_k(p, w)$. Because we assume CRRA utility, optimal consumption plans are homogenous in the household’s wealth; a coarse consumption $c$ is optimal for a household with wealth $w$ at price $p$ if and only if $\frac{p}{w}c$ is optimal for the representative agent at price $p$. Therefore, aggregate demand is independent of the wealth distribution and so are CCE equilibrium prices. Moreover, we can convert the CCE allocation of the representative economy into a CCE of $E$ with the same aggregate endowment by evaluating the wealth of each household at the equilibrium price and distributing the allocation among households in proportion to their wealth. Finally, we can rescale any CCE allocation for $E$ to derive a CCE allocation for the representative economy with the same aggregate endowment.

Henceforth, we describe the set of CCE in terms of its representative economy counterpart and appeal to these observations to justify suppressing the wealth distribution.

**Definition 2.** The price-allocation pair $(p, \mu)$ is a CCE of $E$ if (i) $\mu$ is feasible for $E$ and (ii) $\mu(c) > 0$ implies $c$ is optimal for the representative agent at prices $p$.

We call $p$ ($\mu$) a CCE price (allocation) if $(p, \mu)$ is a CCE for some $\mu$ ($p$). Two consumption plans $c, c'$ are conformable if $c_i = c_j$ if and only if $c'_i = c'_j$; i.e., two consumption plans are conformable if they induce the same partition of $N$. We write $c \sim c'$ if $c$ and $c'$ are conformable.

**Definition 3.** An allocation is simple if $\mu(c) \cdot \mu(c') > 0$ and $c \sim c'$ implies $c = c'$. An allocation is fair if $\mu(c) \cdot \mu(c') > 0$ implies $U(c) = U(c')$.

A fair allocation yields the same utility to every household. In a simple allocation, each partition of $N$ has at most one consumption plan associated with it. Thus, if $\mu$ is simple, the cardinality of $K(\mu)$ is at most equal to the number of partitions of $N$ with $k$ or fewer elements. There are two other properties of allocations, monotonicity and measurability, that play a role in our characterizations of CCE equilibria. Monotonicity requires that consumption is weakly increasing in the aggregate endowment. Measurability means that the allocation remains feasible if states with identical endowments are combined into a single state.
Definition 4. The plan $c$ is **monotone** if $c_i \geq c_j$ whenever $s_i > s_j$. The plan $c$ is **measurable** if $c_i = c_j$ whenever $s_i = s_j$. The allocation $\mu$ is **monotone/measurable** if all $c \in K(\mu)$ are monotone/measurable.

The mean utility, $W(\mu)$, of allocation $\mu$ is

$$W(\mu) = \sum_c U(c) \cdot \mu(c)$$

We say that $\mu \in M(C_k)$ solves the **planner’s problem** if $W(\mu) \geq W(\mu')$ for all $\mu' \in M(C_k)$. The main technical result of this section is Lemma 1 below which identifies properties of the solutions to the planners problem. In Theorem 1, we take advantage of these properties to relate the planner’s problem to CCE equilibria of the representative economy. In an economy without our behavioral constraints, simplicity, fairness, monotonicity and measurability of solutions to the planner’s problem would follow immediately from the strict concavity of the household utility function. The argument for simplicity is unaffected by behavioral constraints. But none of the remaining properties hold for a general strictly concave utility function given the coarseness constraint. Lemma 1 shows that they do hold with a strictly concave CRRA utility function.

**Lemma 1.** (i) There is a solution to the planner’s problem (ii) Every solution to the planner’s problem is simple, fair, monotone, and measurable.

Household preferences in our model satisfy local non-satiation and, therefore, by the first welfare theorem, CCE allocations are Pareto efficient. (Of course, Pareto efficiency is defined relative to allocations in $M(C_k)$.) In a representative CCE, allocations must be fair and, by Lemma 1, the planner’s problem has a fair solution. Thus, if a CCE allocation did not solve the planner’s problem there would exist a fair allocation that yields higher household utility, contradicting the Pareto efficiency of CCEs. Hence, every CCE allocation must solve the planner’s problem. Theorem 1, below, establishes the converse. Theorem 1 and Lemma 1 together establish the existence and Pareto-efficiency of CCE. They also show that every CCE allocation is simple, fair, monotone, and measurable.

**Theorem 1.** An allocation solves the planner’s problem if and only if it is a CCE allocation.
Existence of a CCE and its Pareto-efficiency relies neither on CRRA preferences nor on the correspondence between solutions to the planner’s problem and equilibria; it is possible to establish existence (and Pareto-efficiency) using standard fixed-point arguments and relying only on the concavity of the household utility function. Such a proof would not yield the monotonicity and measurability of equilibrium allocations and the related monotonicity of CCE prices. As we illustrate in the following section, without CRRA utility, it is possible to construct examples of CCE that do not satisfy these properties.

For any \( r \in \{s_i : i \in N\} \), let \( p(r) = \sum_{i : s_i = r} p_i \) and \( \pi(r) = \sum_{i : s_i = r} \pi_i \). Two prices \( p, \hat{p} \) are equivalent if \( p(r) = \hat{p}(r) \) for all \( r \). In a pure endowment economy the realized endowment resolves all uncertainty; that is, \( s \) is one-to-one. In that case, \( p(s_i) = p_i \) and \( \pi(s_i) = \pi_i \) for all \( i \). Hence, for a pure endowment economy, \( p, \hat{p} \) are equivalent if and only if \( p = \hat{p} \).

**Definition 5.** The price \( p \) is monotone if \( r > \hat{r} \) implies \( p(r)/\pi(r) \leq p(\hat{r})/\pi(\hat{r}) \).

We say that the CCE price is essentially unique if and only if all CCE prices are equivalent. Theorem 2 below shows that the CCE price is essentially unique and monotone. Hence, a pure endowment economy has a unique CCE price.

**Theorem 2.** The CCE price of any economy is essentially unique and monotone.

Next, we illustrate Theorems 1 and 2 in a simple example. There are four equally likely states with endowments between 1 and 2. Specifically,

\[
s_1 = 1, s_2 = 4/3, s_3 = 5/3, s_4 = 2
\]

The utility function is logarithmic and households must choose 2—coarse plans. As a benchmark, first consider a standard competitive equilibrium without the coarseness constraint. In that case, the representative household consumes her endowment and the prices are given by \( p^* \) where

\[
p_1^* = .35; p_2^* = .26; p_3^* = .21; p_4^* = .18
\]

Next, we describe the CCE with \( k = 2 \), that is, all households are restricted to 2—coarse plans. Lemma 1 implies that equilibrium consumption plans are characterized by a cutoff state; that
is, each household chooses a state $j \in \{1, 2, 3, 4\}$ such that states $i \leq j$ are associated with low consumption ("bad times") and states $i > j$ are associated with high consumption ("good times"). There are three distinct plans chosen by consumers; around 39% of consumers identify states 2, 3 and 4 as "good times" and single out state 1 for low consumption; consumption levels in that plan are $c_1 = .85, c_2 = c_3 = c_4 = 1.7$. Around 36% of households choose identify states 1 and 2 as bad times and states 3 and 4 as good times; the corresponding consumption levels are $c_1 = c_2 = 1.04$ and $c_3 = c_4 = 1.96$. Finally, the remaining 25% of identify states 1, 2 and 3 as bad times and single out state 4 for high consumption; their consumption levels are $c_1 = c_2 = c_3 = 1.18$ and $c_4 = 2.56$. The equilibrium price is $p$ where

$$p_1 = .4, p_2 = .25, p_3 = .21, p_4 = .133$$

Notice that the largest difference between the equilibrium price in a standard economy and the CCE price is in the states with the highest and lowest endowments. The ratio of $p_1^*$ and $p_4^*$ is equal to 2, the ratio of the aggregate endowment in those two states. By contrast, the ratio of $p_1$ and $p_4$ is 3. As we will show in the next section, this is no accident. In any CCE with many states, the prices in states with endowments near the upper or lower bounds exhibit the greatest departure from standard equilibrium prices.

Monotonicity of CCE consumption plans play a key role in the subsequent sections when we derive our pricing result. Next, we illustrate how those results can fail if the utility function is not CRRA: the endowments are the same as in the example above. The utility index is:

$$u(z) = \begin{cases} 
2z & \text{if } z \leq 1 \\
1 + z & \text{if } z \in [1, 2] \\
2 + z/2 & \text{if } z > 2
\end{cases}$$

For this utility function the unique CCE allocation consists of two consumption plans; 2/3 of consumers choose the plan $c_1 = c_2 = 1, c_3 = c_4 = 2$ while 1/3 of consumers choose the plan $c_1 = c_3 = 1$ and $c_2 = c_4 = 2$. The CCE price is $p = (\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4})$. Notice that the second
consumption plan is not monotone establishing that with general risk averse utility functions monotonicity may fail.\textsuperscript{2}

3 Consumption Risk and Price Variation in a CCE

In this section, we compare the CCE for economy $E$ to the standard equilibrium for $E$. In particular, we show that consumption is riskier and prices are more extreme in a CCE than in a standard competitive equilibrium (SCE).

We first establish that CCE yields greater consumption risk. Given $\pi$ and $z \in \mathbb{R}^N$, we let $F_z$ denote the cumulative distribution of the random variable $z$; that is, $F_z(x) = \sum_{i : z_i \leq x} \pi_i$. Hence, $F_s$ is the cdf of the endowment and $F_c$ is the cdf of consumption associated with the plan $c$. Then, $G_\mu$, the cdf of consumption given the allocation $\mu$ is

$$G_\mu(x) = \sum_c F_c(x) \cdot \mu(c).$$

Consider a CCE equilibrium $(p, \mu)$ for the representative economy. We can think of (random) consumption as the result of a two stage lottery; the first stage reveals the state $i \in N$ and the second stage reveals the consumption in state $i$. Since $k < |K(s)|$, consuming the endowment in every state is not feasible and there are $c \in K(\mu)$ with $c_i \neq s_i$ for some $i \in N$. Moreover, $s_i \geq \sigma_i(\mu)$ for all $i$ since $\mu$ is feasible and hence the expected value of consumption in state $i$ is less than or equal to $s_i$, the SCE consumption in state $i$. This implies that SCE consumption second order stochastically dominates CCE consumption and hence welfare (i.e., mean utility) in a CCE is strictly less than welfare in a SCE. Put differently, the fact that $s$ is not in $C_k$, by itself, ensures that the representative household bears greater consumption risk in a CCE than in a SCE.

More generally, when different households have different endowments, a CCE may result in less income inequality than an SCE. In this case, it is difficult to conclude that ex ante welfare

\textsuperscript{2}The utility function in the example is not strictly concave. However, it is straightforward to show that a strictly concave approximation of the utility function in this example would also lead to non-monotone equilibrium consumption plans.
will be lower in a CCE than in an SCE; prices will not be the same in the two different types of equilibrium and given a fixed endowment distribution, different prices will imply different wealth distributions. A tighter wealth distribution will imply higher ex ante welfare, higher possibly than in a SCE.

We now show that for any fixed range of the endowment realization \([a, b]\), the equilibrium price \(p \in \Delta(N)\) in a CCE (normalized by the probability of the state) can be arbitrarily large or arbitrarily close to zero. Thus, CCE exhibits extreme prices. Specifically, extreme prices emerge when the endowment has many possible realizations and \(F_s\) approaches a nonatomic, continuous distribution. The next subsection defines this notion precisely and establishes the main result.

### 3.1 Convergent Economies and Extreme Prices

Let \(E^n\) be a pure endowment economy with \(n \geq k + 1\) states and order states so that \(s_i < s_j\) if \(i < j\). We define a sequence of economies converging to a limit economy with a continuous endowment distribution.

**Definition 6.** A sequence of economies \(\{E^n\} = \{(u, k, \pi^n, s^n)\}\) is convergent if \(s^n\) converges in distribution to a random variable with a continuous and strictly positive density on \([a, b]\).

Let \(p^n\) be the equilibrium price of \(E^n\). The *pricing kernel* \(\kappa^n \in \mathbb{R}^n_+\) is the equilibrium price normalized by the probability of the state:

\[
\kappa^n_i = \frac{p^n_i}{\pi^n_i}
\] (1)

Theorem 3, below, characterizes \(\{p^n\}\), the CCE price sequence, and \(\{\kappa^n\}\), the corresponding sequence of pricing kernels, for a convergent sequence of economies. Together, we call these two sequence a CCE sequence. For any real-valued function \(X\) on \(\{1, \ldots, n\}\) and \(A \subseteq \mathbb{R}\), let \(\Pr(X \in A)\) denote the probability that \(X\) takes a value in \(A\); that is,

\[
\Pr(X \in A) := \sum_{\{i: X_i \in A\}} \pi_i
\]
For any compact set $X$, every sequence $\{x^n\} \subset X$ has a convergent subsequence. In the statements below, we use this fact and write $\lim x^n$ to denote any such limit.

**Theorem 3.** For any CCE sequence $\{(p^n, \kappa^n)\}$, $K$ and $\epsilon > 0$, (i) $\lim p^n_1 > 0$; if $\rho \geq 1$, then $\lim \Pr(\kappa^n > K) > 0$ and (ii) $\lim \Pr(\kappa^n < \epsilon) > 0$; if $\rho > 1$, then $\lim \Pr(p^n = 0) > 0$.

For the state with the lowest endowment, Theorem 3(i) establishes that the limit price is greater than zero even though the limit probability of that state is zero. Thus, consumption in the lowest endowment state is extremely expensive. Clearly, this implies that the pricing kernel in state 1 converges to infinity but since the probability of state $n$ converges to zero, this leaves open the question of whether there is a positive limit probability of an arbitrarily high pricing kernel. The second part Theorem 3(i) shows that this is the case if the household is sufficiently risk averse, with a parameter of relative risk aversion greater or equal to 1.

Part (ii) of Theorem 3 establishes that there is a positive limit probability that the pricing kernel is arbitrarily close to zero. By Theorem 2 this occurs when the endowment realization is near its upper bound $b$. Finally, if relative risk aversion is above 1, a stronger result is true: the limit price is zero in a nontrivial interval of states near the highest endowment realization.

To prove Theorem 3, we first establish the following dominance lemma (Lemma 13): let $L^n_k$ be the maximal utility of a representative household in $E^n$ at the CCE price when restricted to $k$-coarse plans. Let $L^n_{k-1}$ be the maximal utility that the same household would enjoy at the $k$-coarse CCE price under if it were restricted to $k-1$-coarse consumptions. Clearly, $L^n_{k-1} \leq L^n_k$. The dominance lemma shows that $L^n_{k-1}$ is uniformly bounded away from $L^n_k$ for all $n$; hence $k-1$-coarse plans do uniformly worse than equilibrium plans.

To see the argument for the first part of Theorem 3, assume that the equilibrium price in state 1 converges to zero. In equilibrium, some households must choose a lower consumption in state 1 than in all other states because aggregate consumption is lower in state 1 than in all other states and because all equilibrium plans are monotone. An alternative plan for these consumers would be to set consumption in state 1 equal to consumption in state 2 while reducing all consumption a bit so as to satisfy the budget constraint. If the price in state 1 goes to zero, then this plan yields essentially the same utility as the original plan. But since the
new plan is \( k - 1 \) coarse we have established a contradiction to the dominance lemma. Hence, the price in state 1 must stay bounded away from zero.

A similar application of the dominance lemma shows that consumption in the highest endowment states must be very cheap so that consumers find it worthwhile to single them out; so cheap that the probability-weighted utility in those states stays bounded away from zero. As a consequence, the pricing kernel must be close to zero. For the final part of Theorem 3 note that utility is bounded above if \( \rho > 1 \) and, therefore, consumers are unwilling to single out very unlikely low-price events no matter how low the price. In that case, part of the aggregate endowment near \( b \) is not consumed and prices are zero.

### 3.2 Limit Prices

When characterizing equilibrium prices with many possible endowment realizations, it is easier to work with cumulative prices: a nondecreasing, right-continuous function \( H : \mathbb{R} \to [0, 1] \) is a cumulative price if \( H(x) = 0 \) for \( x < 0 \) and \( H(1) = 1 \). Given any \( E^n \) and \( x \in [\pi^n_1, 1] \), let \( j^n_x = \max\{j : \sum_{i \leq j} \pi^n_i \leq x\} \). If \( x \in [0, \pi^n_1] \), we let \( j^n_x = 1 \). Then, for any price \( p^n \) of \( E^n \), define its cumulative, \( P^n \) as follows:

\[
P^n(x) := \sum_{i \leq j^n_x} p^n_i
\]

Note that \( P^n(F^n_\pi(s^n_i)) \) is the cost of one unit of consumption in the event \( \{1, \ldots, i\} \). In our discrete economies, the CCE cumulative price is necessarily a step function.

Let \( \mathcal{H} \) be the set of all cumulatives. For any sequence \( H^n \in \mathcal{H} \), we say that \( H \in \mathcal{H} \) is its limit if the restriction of \( H \) to the unit interval is continuous and \( \lim_n H^n(x) = H(x) \) for all \( x \in (0, 1] \). We say that \( P \in \mathcal{H} \) is a limit CCE price for \( E^n \) if \( E^n \) has a subsequence of CCE cumulative prices that converge to \( P \). If, in addition, the restriction of \( P \) to the unit interval is concave, then we say \( P \) is a concave limit price.

**Lemma 2.** Every convergent sequence of economies has a concave limit CCE price.

Below, we pass to the relevant subsequences of economies and CCE prices and refer to a limit CCE price simply as the limit CCE price. As a benchmark, consider the limit price of
standard economies. The unique SCE price \( p \in \Delta(N) \) is

\[
p^*_i = \frac{\pi_i u'(s_i)}{\sum_{j=1}^{n} \pi_j u'(s_j)}
\]

Since \( s_i \in [a, b] \) it follows that for all \( i \in N \) the pricing kernel \( \kappa^*_i \) of a standard economy satisfies:

\[
\kappa^*_i = \frac{p^*_i}{\pi_i} \in \left[ \frac{u'(b)}{u'(a)}, \frac{u'(a)}{u'(b)} \right]
\]

Thus, the SCE pricing kernel is bounded away from zero and infinity. Moreover, if \( b - a \) is small then \( u'(b)/u'(a) \) is close to 1 and therefore the range of the pricing kernel is small. It is straightforward to show that there is a unique equilibrium limit price \( P^* \) along for any convergent sequence of SCE prices. This limit price is

\[
P^*(x) = \frac{\int_{0}^{x} u'(F^{-1}(r))dr}{\int_{0}^{1} u'(F^{-1}(r))dr}
\]

where \( F \) is the cdf of the limit endowment. Hence, \( P^* \) is continuous and differentiable on \([0, 1]\) with a derivative equal to the limit pricing kernel and uniformly bounded away from zero and infinity. If the variation in limit endowment, \( b - a \), is small, then this derivative is nearly constant and hence bounded above and away from 0. The following theorem shows that limit CCE prices have neither of these properties.

**Theorem 4.** For any limit CCE price \( P \), (i) \( P(0) > 0 \); if \( \rho \geq 1 \), then \( P'(0) = \infty \) and (ii) \( P'(1) = 0 \); if \( \rho > 1 \), then \( P(x) = 1 \) for some \( x < 1 \).

Theorem 4 translates Theorem 3 into properties of the limit price: \( P(0) > 0 \) follows from the fact that \( \lim p^n_i > 0 \) and \( P'(0) = \infty \) for \( \rho \geq 1 \) follows from the fact that the pricing kernel is arbitrarily large in the limit; the fact that the pricing kernel is arbitrarily close to zero yields \( P'(1) = 0 \); the final part of the theorem follows from the fact that the limit price is zero with positive probability whenever \( \rho > 1 \).

We can relate the quantity \( P(0) \) to the value of relaxing the coarseness constraint. Suppose, at a cost \( \tau \), the household can relax the coarseness constraint from \( k \) to \( k + 1 \). Thus, after this
trade the household has smaller wealth (by $\tau$) but can choose a plan in $C_{k+1}$. Suppose that $
abla < a \cdot P(0)$. In equilibrium, some households must consume at least $a$ in the lowest state. For such a household, consider a new plan that isolates state 1 and consumes $\epsilon > 0$ in that state. In the limit, this plan relaxes the budget constraint by $P(0)(a - \epsilon) > \tau$ for $\epsilon$ sufficiently small. The utility of this plan is only slightly lower than the original utility and this difference vanishes as $n$ goes to infinity since the probability of state 1 goes to zero. Thus, $aP(0)$ is bounded above by the shadow price of the coarseness constraint. It follows that $P(0)$ converges to zero as $k$ goes to infinity: the equivalence of equilibrium allocations and solutions to the planner’s problem established in Theorem 1 ensures that the value of an additional partition element must go to zero eventually. Since $aP(0)$ is smaller than the shadow price of additional partition elements, $P(0)$ must converge to zero as $k$ goes to infinity.

3.3 An Example

Let $E^n = (u, k, \pi^n, s^n)$ be a convergent sequence of pure endowment economies such that $u(x) = \ln x$ and suppose that the limit endowment is uniformly distributed on $[a, b]$. Let $k = 2$, i.e., each household is confined to at most two different consumption levels. Theorem 1 and Lemma 1 together imply that for each $n$, a CCE $(p^n, \mu^n)$ exists and involves households choosing monotone consumption plans. For $k = 2$, monotonicity means that for each household, there is a cutoff state $j$ and two consumption levels $x < y$ such that the household consumes $x$ if the state is lower than $j$ and consumes $y$ otherwise. Since the utility function is unbounded, the price must be strictly greater than zero in every state, which implies that the feasibility constraint holds with equality. This, in turn, implies that for every $j$, there is a consumption plan in the support of $\mu^n$ with cutoff $j$; otherwise aggregate consumption would be the same in two consecutive states and since aggregate endowment is strictly increasing, feasibility would not be satisfied with equality.

Figure 1 depicts the CCE limit price ($P$—solid line) and the standard limit price ($P^*$—dashed line). By Lemma 2, the limit cumulative price $P$ is concave. By Theorem 4, $P(0) > 0$; that is, the limit price has a mass point at 0.
4 CCE in a Dynamic Economy

In this section, we extend our analysis to a dynamic Lucas-tree economy. We show that there is a one-to-one correspondence between the stationary CCE of our dynamic economy and the CCE of a corresponding static economy. This correspondence enables us to relate the extreme consumption prices analyzed in Theorem 3 to extreme asset prices.

As in the static economy, \( N = \{1, \ldots, n\} \) is a finite set of states and each state \( i \in N \) implies a dividend (endowment) realization \( s_i \in [a, b] \). A \( t \)-period history \( h \) is a vector \((i_1, \ldots, i_t) \in N^t\); we write \( hi \) for the history \((h, i)\). We call \( H^t \) the set of all \( t \)-period histories and \( H = \bigcup_{t \geq 1} N^t \) the set of all histories. Given any history \( h = (i_1, \ldots, i_t) \), we let \( j(h) = i_t \) and let \( H^t_i = \{h \in N^t : j(h) = i\} \) be the set of all \( t \)-period histories that end in \( i \).
A matrix of transition probabilities, $\Phi$, describes the evolution of the state; $\Phi_{ij}$ is the probability that the state at date $t + 1$ is $j$ given that it is $i$ on date $t$. We assume that $\Phi$ has a stationary distribution $\pi$; that is,

$$\pi = \pi \cdot \Phi$$

(3)

The initial state (the period 1 history) is drawn from the stationary distribution $\pi$. Therefore, the probability of history $h = (i_1, \ldots, i_t) \in H^t$ is

$$\lambda_h = \pi_{i_1} \cdot \Phi_{i_1i_2} \cdots \Phi_{i_{t-1}i_t}$$

(4)

Households choose a consumption plan before learning the initial state. The assumption that the initial state is chosen according to the invariant distribution means that we can ignore transitory effects of the initial condition. As we show below, the economy has a stationary equilibrium allocation and stationary equilibrium prices. Moreover, we can map the dynamic economy to the two-period economy analyzed in the previous section.

A function $d \in \mathbb{R}_+^H$ is a (dynamic) consumption plan and $\mathcal{D}$ is the set of all consumption plans. The definition of coarse consumption plans mirrors the corresponding definition for the static economy:

**Definition 7.** The consumption plan $d \in \mathcal{D}$ is coarse if $|\{d_h | h \in H\}| \leq k$.

Let $\mathcal{D}_k$ be the set of all coarse consumption plans. As in the static economy, we assume that each household chooses its coarse consumption plan before the initial state is realized. Hence, each household is restricted to at most $k$ different levels of consumption throughout its entire lifetime.\(^3\)

The household’s utility from the consumption plan $d$ is

$$V(d) = (1 - \beta) \sum_{t \geq 1} \sum_{h \in N^t} u(d_h) \beta^{t-1} \lambda_h$$

(5)

\(^3\)An alternative version of the model in which households face a weaker constraint and are allowed to choose $k$ different consumption levels each period is also possible. If the state process is iid, the alternative model would have exactly the same set of equilibria as our current model. If the Markov process is not iid, then the alternative model would be more difficult to analyze but we conjecture that extreme asset prices would emerge in that version as well.
where $\beta \in (0, 1)$ is the discount factor. The sextuple $E^* = (u, \beta, k, \pi, s, \Phi)$ is a dynamic economy.

An allocation is a probability distribution on $D$. It is coarse if its support is contained in the set of coarse consumption plans. Thus, the set of dynamic allocations is $\Delta(D)$ and the allocation $\nu \in \Delta(D)$ is coarse if $K(\nu) \subseteq D_k$. The allocation $\nu$ is feasible in $E^*$ if

$$\sigma^*_h(\nu) := \sum_{d \in K(\nu)} d_h \cdot \nu(d) \leq s_j(h)$$

for all $h \in H$. For any $D' \subset D$, let $M^*(D')$ be the set of all feasible allocations $\nu$ such that $\nu(D') = 1$. Hence, $M^*(D_k)$ is the set of feasible coarse allocations for $E^*$. Henceforth, $\nu$ is feasible means $\nu \in M^*(D_k)$.

A function $q \in \mathbb{R}^H_+$ is a (dynamic) price if $\sum_H q_h = 1$. The representative household’s budget is

$$B^*_k(q) = \left\{ d \in D_k : \sum_{h \in H} q_h [d_h - s_j(h)] \leq 0 \right\}$$

(6)

The coarse consumption plan $d \in B^*_k(p)$ is optimal at prices $q$ if $V(d) \geq V(d')$ for all $d' \in B^*_k(q)$.

**Definition 8.** The price-allocation pair $(q, \nu)$ is a CCE of $E^*$ if (i) $\nu$ is feasible for $E^*$ and (ii) $\nu(d) > 0$ implies $d$ is optimal at prices $q$.

Fix a dynamic economy $E^* = (u, \beta, k, \pi, s, \Phi)$ and consider the static economy $E = (u, k, \pi, s)$. The two economies share the same utility function and coarseness constraint. In both economies the initial endowment is chosen according to the distribution $\pi$. In the dynamic economy, the endowment evolves according to a Markov process while in the static economy the endowment stays fixed. Since $\pi$ is the stationary distribution of the Markov process with transition matrix $\Phi$ it follows that

$$\sum_{h \in H^*_i} \lambda_h = \pi_i$$

(7)

for all $t \geq 1$. Hence, the ex ante probability of state $i$ in period $t$ is $\pi_i$ for every $t$. Since $(1 - \beta) \sum_t \sum_{h \in H^*_i} \beta^{t-1} \lambda_h = \pi_i$, the dynamic economy can be thought of as a version of the static economy in which each state $i$ is split into many identical states corresponding to the
branches of the event tree that end with \( i \). We refer to \( E = (u, k, \pi, s) \) as the static economy for \( E^* = (u, \beta, k, \pi, s, \Phi) \). The measurability of CCE consumption plans in \( E \) will yield the stationarity of CCE consumption plans in \( E^* \).

A consumption plan is stationary if consumption depends only on the current state. We can associate stationary consumption plans of the dynamic economy \( E^* \) with consumption plans of the corresponding static economy \( E \). Formally, the plan \( d \) is stationary if there exists a consumption plan, \( c \), for the static economy \( E \) such that \( d_h = c_{j(h)} \) for all \( h \in H \). Let \( \mathcal{D} \) denote the set of stationary plans and let \( T_1 : \mathcal{C} \to \mathcal{D} \) be the above one-to-one mapping between static consumption plans and stationary dynamic plans. Thus, \( d = T_1(c) \) is the dynamic consumption plan in which a household consumes \( c_i \) whenever the state \( i \) occurs. The set of stationary allocations is \( \Delta(\mathcal{D}) \). Let \( T_3 : \Delta(\mathcal{C}) \to \Delta(\mathcal{D}) \) be the one-to-one mapping between allocations in the static economy and stationary allocations in the dynamic economy defined by \( \nu(d) = \mu(T_1^{-1}(d)) \) for \( d \in \mathcal{D} \) and \( \nu(d) = 0 \) for \( d \notin \mathcal{D} \).

A price is stationary if the price after history \( h \) depends only on the current state \( j(h) \) and on the discounted probability of history \( h \) appropriately normalized. More precisely, \( q \) is stationary if there is a static price \( p \) (and a corresponding pricing kernel \( \kappa \)) such that for all \( t \geq 1 \) and all \( h \in N^t \)

\[
q_h = \lambda_h (1 - \beta) \beta^{t-1} \frac{p_{j(h)}}{\pi_{j(h)}}
= \lambda_h (1 - \beta) \beta^{t-1} \kappa_{j(h)}
\]

Equations (7) and (8) imply

\[
\sum_{t=1}^{\infty} \sum_{h \in H^t_i} q_h = p_i
\]

for all \( i \) and hence \( \sum_{h \in H} q_h = \sum_{i=1}^{n} p_i = 1 \). Hence, for each static price \( p \) there is a corresponding stationary dynamic price and, conversely, each stationary dynamic price can be mapped to a static price.

Let \( \tilde{Q} \subset \Delta(H) \) be the set of stationary prices. Let \( T_2 : \Delta(N) \to \tilde{Q} \) be the one-to-one mapping between prices in the static economy and stationary prices in the dynamic economy.
defined above. To summarize: \( T_1 : \mathcal{C} \xrightarrow{1-1} \mathcal{D} \) is the mapping that identifies the unique stationary consumption plan associated with each static consumption plan, \( T_2 : \Delta(N) \xrightarrow{1-1} \mathcal{Q} \) defines the unique stationary price associated with each static price and \( T_3 : \Delta(\mathcal{C}) \xrightarrow{1-1} \Delta(\mathcal{D}) \) defines the unique stationary allocation associated with each static allocation.

Equation (7) implies that
\[
V(T_1(c)) = (1 - \beta) \sum_{t\geq 1} \sum_{h \in N} u(c_{j(h)}) \beta^{t-1} \lambda_h = U(c)
\]
and Equation (9) implies that \( c \in B_k(p) \) if and only if \( T_1(c) \in B_k^*(T_2(p)) \cap \mathcal{D} \). Finally, note that \( \mu \in \Delta(\mathcal{C}) \) is feasible in \( E \) if and only if \( T_3(\mu) \in \Delta(\mathcal{D}) \) is feasible in \( E^* \).

Theorem 5 below connects CCEs of the dynamic economy and CCEs of the corresponding static economy. An equilibrium allocation of the static economy yields a stationary equilibrium allocation of the corresponding dynamic economy and an equilibrium price of the static economy yields a stationary equilibrium price of the dynamic economy.

**Theorem 5.** (i) If \((p, \mu)\) is a CCE of \( E \), then \((T_2(p), T_3(\mu))\) is a CCE of \( E^* \). (ii) If \( \nu \) is a CCE allocation for \( E^* \), then \( \nu \in \Delta(\mathcal{D}) \) and \( T_3^{-1}(\nu) \) is a CCE allocation for \( E \).

Theorem 5 leaves open the possibility of a non-stationary CCE price (supporting a stationary CCE allocation). Theorem 5 relies on the assumption that the initial state is chosen according to the stationary distribution \( \pi \). Without this assumption, there might still be an analogue of Theorem 5 but the mapping between the dynamic and the static economy would be more complicated.

## 5 Extreme Asset Prices and the Safe Haven Premium

Recall that a static pure endowment static economy \( E = (u, k, \pi, s) \) has a unique CCE price. In this section, we relate this price \( p \) to asset prices in the dynamic economy. To simplify the exposition, we restrict ourselves to iid transitions: \( \Phi_{ij} = \pi_j \) for all \( i, j \) and refer to a dynamic economy with constant transition probabilities as an iid economy. In addition, we assume \( \rho = 1 \) so that \( u(z) = \ln z \). These restrictions are made for expositional ease. All results below can
be extended to arbitrary Markov transitions provided that all ratios of transition probabilities stay bounded. For $\rho > 1$, the equilibrium price of consumption may be zero; this would allow us to strengthen some of the results below but at the cost of a more cumbersome notation. For $\rho < 1$, we would need to slightly weaken the result on extremely low asset prices.

Consider an asset $z = (z_1, \ldots, z_n)$ in zero net supply that delivers $z_i$ units of the consumption good next period in state $i$. Let $r_h(z)$ be the CCE price of this asset in terms of current period consumption after history $h$. Recall that $j(h) \in N$ is the last state of history $h$. A standard no-arbitrage argument implies that

$$r_h(z) = \frac{\sum_{i \in N} q_{hi} z_i}{q_h}$$

(11)

where the numerator is the expected value of the return $z$ after history $h$ and the denominator is the price of consumption after history $h$. Since $q$ is stationary, the price of the asset depends only on $j(h)$ and therefore $r_h(z) = r_{j(h)}(z)$. Formulas (8) and (11) imply that

$$r_h(z) = r_{j(h)}(z) = \beta E[\kappa z]$$

(12)

where $E[\xi]$ is the expectation of the random variable $\xi : N \to \mathbb{R}$ with respect to $\pi$. Since $\kappa_i > 0$ for all $i \in N$, the asset price is well defined and since $r_h$ depends only on the most recent state we write $r(z) = (r_1(z), \ldots, r_n(z))$ for the vector of asset prices.

As in Theorem 3, we consider a sequence of economies that converges to a limit economy with a continuous distribution of endowments. Formally, we say that a sequence of iid-economies $\{E^{*n}\} = \{(u, \beta, k, \pi^n, s^n)\}$ is convergent if the corresponding sequence of static economies $\{E^n\} = \{(u, k, \pi^n, s^n)\}$ is convergent. Consider a sequence of asset returns $\{z^n\}$ with $z^n \in \mathbb{R}^n$. We say that $\{z^n\}$ is bounded if there are $0 < \gamma_1 < \gamma_2 < \infty$ such that $z_i^n \in [\gamma_1, \gamma_2]$ for all $i$ and $n$.

Theorem 6 shows that limit equilibrium asset prices are extremely high if the endowment is near its upper bound and extremely low if the endowment is near its lower bound.

**Theorem 6.** Let $\{z^n\}$ be a bounded sequence of asset returns, let $\{E^{*n}\}$ be a convergent se-
quence of iid economies and let \( r^n(z^n) \) be the equilibrium asset price of \( z^n \) in \( E^{*n} \). Then, 
\[
\lim \text{Prob} (r^n(z^n) < \epsilon) > 0 \text{ for all } \epsilon > 0 \text{ and } \lim \text{Prob} (r^n(z^n) > K) > 0 \text{ for all } K.
\]

Theorem 6 is a corollary of Theorem 3 and Equation (12) above. Theorem 3 can be used to derive another asset pricing implication that relates risk-free and nearly-riskfree asset prices. Let \( e^n = (1, \ldots, 1) \) be a risk-free asset and let \( e^{n\epsilon} \) be the following nearly-risk-free asset:

\[
e^{n\epsilon}_i = \begin{cases} 
1 & \text{if } i \geq j^n_\epsilon \\
0 & \text{if } i < j^n_\epsilon
\end{cases}
\]

Recall that \( j^n_x = \max\{j : \sum_{i \leq j} \pi_i \leq x\} \) and hence \( e^{n\epsilon} \) yields 1 in all but the \( \epsilon \)-fraction of states with the lowest endowment. Let \( \{E^{*n}\} \) be a convergent sequence of iid-economies and let \( r^n(e^n) \) and \( r^n(e^{n\epsilon}) \) be the equilibrium asset price of \( e^n \) and \( e^{n\epsilon} \) in \( E^{*n} \).

**Theorem 7.** There is \( \delta > 0 \) such that 
\[
\lim r^n_i(e^{n\epsilon}) / r^n_i(e^n) \leq 1 - \delta \text{ for all } \epsilon > 0.
\]

Theorem 7 shows that the risk-free premium does not converge to zero as the returns of the nearly riskfree asset converges in distribution to the returns of the riskfree asset. This safe-haven premium comes about because the price of consumption in the lowest endowment state is bounded away from zero and, therefore, the risk-free asset is always more costly than the nearly risk free asset.

### 6 Conclusion

In this paper, we analyze the implication of coarse consumption plans on equilibrium asset prices in a standard Lucas-style endowment economy. We assumed that agents’ consumption plans must be \( k \)-coarse, that is, may take on at most \( k \) distinct values. We show that the coarseness constraint leads to extreme and volatile prices when the endowment realization is near its upper or lower bound. This result is robust in the sense that equilibrium prices are extreme also in an economy that includes a small fraction of unconstrained agents, provided that they are either risk averse or cannot consume negative amounts. The demand of a small
mass of such agents is not going to affect the aggregate quantities significantly enough to change the prices.

In settings where agents are cognitively constrained, the market mechanism acquires a new function: it allocates agents’ scarce attention. For markets to clear, the equilibrium prices have to accentuate the relevant events to attract the households’ attention. Since it is particularly costly to pay attention to tail events, the price variation in those events has to be large enough to make them salient.

In our formulation, the household’s consumption decision is coarse, i.e., insensitive to some details of the underlying economic conditions. On the other hand, the household financing decisions are unconstrained and typically a household’s net trades will depend on the exact state of the world. In contrast, in models of incomplete markets, put restrictions on the trading strategies of the agent, so the set of achievable plans is changing as his endowment changes, whereas in our model it’s fixed as long as the total wealth is constant. An alternative model would impose coarseness constraints on the financial transactions. For example, assuming household’s net trades are 2-coarse amounts to assuming that states are partitioned into “borrowing states” and “lending states” and the household borrows some fixed amount $x$ whenever it finds herself in a borrowing state and lends a fixed amount $y$ if she finds herself in a lending state. More generally, assuming $k$-coarse net trades would allow us to capture households that are constrained in their financial transactions.

There is a modeling tradeoff between the two types of constraints. If net trades must satisfy a coarseness-constraint then – for a generic choice of the endowment – the consumption choice will not be coarse. Which constraint is appropriate depends on the particular application; specifically, it depends on what is the household’s active decision, i.e., the decision that is the focus of the analysis, and what are the residual decisions, i.e., decisions that the household is not focusing on, but are automatically implied by the active decision and the budget constraint. Our model describes the allocation of real resources and in this context the focus of the household’s decision is the consumption choice.
A Appendix: Proofs

If $K(\mu) \subset \{c_1, \ldots, c_m\}$, we write $\mu = (a, c)$ where $a = (\alpha_1, \ldots, \alpha_m)$, $c = (c_1, \ldots, c_m)$ and $\mu(c_l) = \alpha_l$ for all $l$. It will be understood that $a = (\alpha_1, \ldots, \alpha_m)$, $\hat{a} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)$, and so forth. We follow the same convention with $c, \hat{c}$ etc. If $\{c_1, \ldots, c_m\}$ contains exactly one representative from each equivalence class of $\sim$, we say that $\mu = (a, c)$ is in simple form. Thus, $\mu$ can be expressed in simple form if and only if it is simple. We write $\delta_c$ for the allocation $\mu$ where all households consume $c \in \mathcal{C}_k$.

A.1 Proof of Lemma 1

Lemma 3. If $\mu$ is feasible and not simple, then there is a simple and feasible $\mu'$ such that $W(\mu') > W(\mu)$.

Proof. Let $\mu = (a, c)$. If $\mu$ is not simple, there is $c, c' \in K(\mu)$ such that $c \sim c'$. Let $c^* = \gamma \cdot c + (1 - \gamma) c'$ where $\gamma = \frac{\mu(c)}{\mu(c) + \mu(c')}$ and let $\mu^*$ be the allocation derived from $\mu$ by replacing $c, c'$ with $(\mu(c) + \mu(c')$ probability of) $c^*$. Since, $c \sim c'$ are coarse, so is $c^*$ and $\mu^*$. Since $u$ is strictly concave, $W(\mu^*) > W(\mu)$. Note that $|K(\mu^*)| < |K(\mu)|$. If $\mu^*$ is simple, we are done. Otherwise, repeat the above argument. Since $K(\mu)$ is finite, this process must terminate with a simple allocation. \hfill \Box

Lemma 4. If $\mu$ is feasible, simple but not fair, then there is a feasible, simple and fair $\mu'$ such that $W(\mu') > W(\mu)$ and $|K(\mu')| \leq |K(\mu)|$.

Proof. Let $\mu = (a, c)$, let $x^l$ be the certainty equivalent of $c_l$ and $\bar{x}^l$ be the corresponding constant consumption plan; that is, $u(x^l) = U(c^l)$ and $\bar{x}^l = x^l$ for all $i, l$. Also, let $x = \sum_{l=1}^{m} \alpha^l x^l$ and let $\bar{x}$ be the corresponding constant consumption plan. Let $\hat{\mu} = (\hat{a}, \hat{c})$ such that $\hat{\alpha}^l = \frac{\alpha^l x^l}{x}$ and $\hat{c}^l = \frac{x c^l}{x^l}$ for all $l$. Finally, let $\bar{\mu} = (\bar{a}, \bar{c})$ such that $\bar{\alpha}^l = \alpha^l$ and $\bar{c}^l = \bar{x}^l$ for all $l$. Since $u$ is strictly concave and $\mu$ is not fair, $W(\delta_x) > W(\hat{\mu})$. Since $u$ is CRRA,

$$u^{-1}(U(c^l')) = \frac{x}{x^l} u^{-1}(U(c^l)) = \frac{x}{x^l} x^l = x;$$
hence, \( W(\hat{\mu}) = W(\delta_\varepsilon) \). By definition, \( W(\hat{\mu}) = W(\mu) \). Hence, \( W(\hat{\mu}) > W(\mu) \). By construction \( \hat{\mu} \) is fair. It is easy to verify that \( \sum_i c_i' \alpha^l = \sum_i c_i \alpha^l \) for all \( i \in N \) and hence \( \hat{\mu} \) is feasible. Clearly, \( |K(\hat{\mu})| \leq |K(\mu)| \).

\[ \square \]

**Lemma 5.** A solution to the planner’s problem exists and every solution to the planner’s problem is simple and fair.

**Proof.** The allocation \( \delta_\varepsilon \) such that \( c_i = \min_i s_i \) for all \( i \) is feasible. Thus, \( M(C_k) \) is non-empty. Since \( \delta_\varepsilon \) second order stochastically dominates any feasible \( \mu \) it follows that \( W(\mu) < W(\delta_\varepsilon) \) for every feasible \( \mu \in M(C_k) \). Hence,

\[ W_k = \sup_{\mu \in M(C_k)} W(\mu) \]

is well-defined. By Lemmas 3 and 4, there exists a sequence of feasible, simple, and fair allocations \( \mu^t = (a^t, c^t) \) such that \( W(\mu^t) \geq W_k - 1/t \) and \( a^t \in \mathbb{R}^m_+ \) for all \( t \), where \( m \) is the cardinality of the set of all partitions of \( N \) with \( k \) or fewer elements.

By passing to a subsequence, \( a^t = (\alpha^t_1, \ldots, \alpha^t_m) \) converges to some \( a \in \Delta(\mathbb{R}^m_+) \). If \( c^t_l \) is unbounded for some \( l \), we must have \( \alpha^t_l = 0 \). Let \( A \subset N \) be the set of \( l \) such that \( \alpha^t_l \neq 0 \). Then, \( A \neq \emptyset \) and \( c^t_l \) is bounded for all \( l \in A \). Hence, there exists a subsequence of \( \mu^t \) along which \( c^t_l \) converges to some \( c^l \in C_k \) for every \( l \in A \).

Let \( \mu = (a, c) \) where \( a = (\alpha^l)_{l \in A} \) and \( c = (c^l)_{l \in A} \). Since \( \lim W(\mu^t) = W_k \) and each \( \mu^t \) is fair, \( U(c^t_l) = W(\mu^t) \). So, by the continuity of \( u \), we have \( U(c^t_l) = W_k \) for all \( l \in A \) and therefore \( W(\mu) = W_k \). Finally, \( \sum_{l \in A} \alpha^t_l c^t_l \leq \sum_{l=1}^m \alpha^t_l c^t_l \leq s_i \) for all \( i, l, t \) and so \( \sum_{l \in A} \alpha^l c^l_i \leq \sum_{l=1}^m \alpha^l c^l_i \leq s_i \) for all \( i, l, \). Hence \( \mu \) is feasible and therefore \( \mu \) solves the planner’s problem. Then, Lemmas 3 and 4 imply that \( \mu \) must be simple and fair.

\[ \square \]

**Lemma 6.** Let \( E = (u, k, \pi, s) \), \( \hat{E} = (u, k, \hat{\pi}, \hat{s}) \) be such that \( F_s = F_{\hat{s}} \) and let \( W_k \) and \( \hat{W}_k \) be the maximal mean utility attainable in \( E \) and \( \hat{E} \) respectively. Then, \( W_k = \hat{W}_k \).

**Proof.** Without loss of generality, we assume that \( E \) is a pure endowment economy; that is, \( N = \{1, \ldots, n\} \), \( s \) is one-to-one, \( \hat{N} = \{ij : i \in N, j \in N_i\} \) for some collection of \( N_i \)'s such that \( N_i = \{1, \ldots, n_i\} \) for all \( i \), \( s_i = s_{ij} \) for all \( i \) and \( j \in N_i \) and \( \sum_{j \in N_i} \pi_{ij} = \pi_i \). We will show that
(i) For any feasible allocation in $E$, there is a feasible allocation in $\hat{E}$ that achieves the same mean utility and (ii) vice versa. Then, $5$ ensures that $W_k = \hat{W}_k$.

(i) For any feasible allocation $\mu = (a, c)$ in $E$ define the allocation $\hat{\mu} = (\hat{a}, \hat{c})$ for the economy $\hat{E}$ as follows: $\hat{c}^i_l = c^i_{ij}$ for all $i$ and $j \in N_i$. Clearly, $\hat{W}(\hat{\mu}) = W(\mu)$ and $\hat{\mu}$ is feasible (for $\hat{E}$).

(ii) We will first show that given any consumption $\hat{c}$ for $\hat{E}$, there is an allocation, $\mu = (a, c)$, for $E$ such that $W(\mu) = \hat{U}(\hat{c})$ and

$$\sigma_i(\mu) = \frac{1}{\pi_i} \sum_{j \in N_i} \pi_{ij} \hat{c}_{ij}$$

for all $i \in N$. Let $L = \{(j_1, \ldots, j_n) : j_i \in N_i \forall i \in N\}$ and let $\gamma^{ij} = \frac{\pi_{ij}}{\pi_i}$ for all $i \in N$ and $j \in N_i$. For each $l = (j_1, \ldots, j_n) \in L$, let $\alpha^l = \gamma^{1j_1} \cdot \gamma^{2j_2} \cdots \gamma^{nj_n}$ and $c^l_i = \hat{c}_{ij}$. Since $\hat{c}$ is coarse and $\{c^l_i : i \in N\} \subset \{\hat{c}_{ij} : i \in N, j \in N_i\}$, each $c^l_i$ is coarse. Verifying that $W(\mu) = U(\hat{c})$ and that the equation displayed above holds for all $i \in N$ is straightforward.

To conclude, let $\hat{\mu} = (\hat{a}, \hat{c})$ be a feasible allocation in $\hat{E}$. For each $c^l \in K(\hat{\mu})$, choose $\mu^l$ as described above. Let $\mu = \sum_l \alpha^l \cdot \mu^l$. Then, $\mu$ is feasible since $\sigma_i(\mu) \leq \max_{j \in N_i} \sigma_{ij}(\hat{\mu}) \leq s_i$ for all $i \in N$. Since $W(\mu^l) = \hat{U}(c^l)$ for all $l$, we have $W(\mu) = \hat{W}(\hat{\mu})$ as desired. \(\square\)

To conclude the proof of Lemma 1, we will show that if $\mu = (a, c)$ is a solution to the planner’s problem, then it is also monotone and measurable. Assume $\mu$ is not measurable; that is, $c^l_i > c^l_j$ and $s_j = s_k$ some $l$ with $\alpha^l > 0$. Then, let $\hat{\mu} = (\hat{a}, \hat{c})$ where $\hat{a} = (\alpha^1, \ldots, \alpha^{l-1}, \alpha^l, \alpha^{l+1}, \ldots, \alpha^m)$, $\hat{c} = (c^1, \ldots, c^{l-1}, c^l, c^{l+1}, \ldots, c^m)$ and $\alpha^l = \frac{\delta^\pi_j}{\pi_j + \pi_k}$, $\alpha^k = \alpha^l - \alpha^k$, $c^l_i = c^l_j = c^l_l$ for all $i \neq j$, $c^l_j = c^l_k = c^l_l$ and $c^l_j = c^l_k = c^l_l$. Then, $\hat{\mu}$ achieves the same mean utility as $\mu$ and is feasible but not simple contradicting Lemma 4.

To prove that $\mu$ satisfies monotonicity, we note that if $s_i \geq s_j$, then $\sigma_i(\mu) \geq s_j(\mu)$. If not, $\sigma_i(\mu) < s_i$ and there must be some $l$ such that $c^l_i < c^l_j$. Then, let $\hat{c}_i = c^l_i$ for $l \neq i$ and $\hat{c}_i = c^l_i$. Note that $\hat{c}$ is coarse and yields a strictly higher utility than $c^l$. Since $\sigma_i(\mu) < s_i$, mean utility can be increased by replacing $c^l$ with $\hat{c}$ for a small fraction of households, contradiction the optimality of $\mu$.

So, assume $\mu$ fails monotonicity: then, $c^l_i > c^l_j$ for some $i, j$ such that $s_i < s_j$. Without loss of generality, we assume $i = 1, j = 2$ and $l = 1$. We showed in the previous paragraph
that $\sigma_2(\mu) \geq \sigma_1(\mu)$, so we must have $c^o$ such that $c_2^o > c_1^o$; again without loss of generality, we assume $o = 2$ and that $\pi_i \geq \pi_j$ (an obvious adjustment is needed if this last inequality is reversed). To summarize: $c_1 > c_2^1; c_1^2 < c_2^2$ and $\pi_1 \geq \pi_2$.

Construct a new state space $\hat{N} = N \cup \{n + 1\}$ and $\hat{s} \in [a, b]^{n+1}$ such that $\hat{s}_t = s_t$ for all $t \in N$ and $\hat{s}_{n+1} = s_1$. Also, set $\hat{\pi}_t = \pi_t$ for all $t \neq 1, n + 1$, $\hat{\pi}_1 = \pi_2$ and $\hat{\pi}_{n+1} = \pi_1 - \hat{\pi}_2$. If $\pi_{n+1} = 0$, then ignore state $n + 1$ in the argument below.

Next we construct a feasible allocation $\hat{\mu} = (\hat{a}, \hat{c})$ for $\hat{E}$ such that (i) $\hat{W}(\hat{\mu}) = W(\mu)$ and (ii) $\hat{\mu}$ is not fair; that is, there is $i, j$ such that $U(c_i) \neq U(c_j)$. By Lemma 6, $\hat{W}(\hat{\mu}) = W(\mu)$ implies that $\hat{\mu}$ solves the planner’s problem for $E$. Since $\hat{\mu}$ is not fair this contradicts Lemma 5 and therefore completes the proof.

The allocation $(\hat{a}, \hat{c})$ has $m + 4$ distinct consumption plans; for $1 \leq l \leq m$

$$\hat{c}_i^l = \begin{cases} c_i^l & \text{if } 1 \leq i \leq n \\ c_1^l & \text{if } i = n + 1 \end{cases}$$

for $l = m + 1, m + 2$

$$\hat{c}_i^{m+1} = \begin{cases} c_1^1 & \text{if } i = 1, 2 \\ c_1^2 & \text{if } 2 \leq i \leq n \\ c_1^1 & \text{if } i = n + 1 \end{cases}$$

and $\hat{c}_i^{m+2} = \begin{cases} c_1^1 & \text{if } i = 1, 2 \\ c_1^2 & \text{if } 2 \leq i \leq n \\ c_1^1 & \text{if } i = n + 1 \end{cases}$

for $l = m + 3, m + 4$

$$\hat{c}_i^{m+3} = \begin{cases} c_1^2 & \text{if } i = 1, 2 \\ c_1^2 & \text{if } 2 \leq i \leq n \\ c_1^2 & \text{if } i = n + 1 \end{cases}$$

and $\hat{c}_i^{m+4} = \begin{cases} c_1^2 & \text{if } i = 1, 2 \\ c_1^2 & \text{if } 2 \leq i \leq n \\ c_1^2 & \text{if } i = n + 1 \end{cases}$

Define

$$\gamma := \frac{c_2^2 - c_1^2}{c_2^2 - c_1^2 + c_1^2 - c_2^1}$$
and note that $0 < \gamma < 1$. Choose $0 < \epsilon < \min\{\alpha^1, \alpha^2\}$ and let
\[
\hat{\alpha}^l = \begin{cases} 
\alpha^l & \text{if } 3 \leq l \leq m \\
\alpha^1 - \epsilon(1 - \gamma) & \text{if } l = 1 \\
\alpha^2 - \epsilon\gamma & \text{if } l = 2 \\
\epsilon\gamma/2 & \text{if } l = m + 1, m + 2 \\
\epsilon\gamma/2 & \text{if } l = m + 3, m + 4
\end{cases}
\]

It is straightforward to check that the allocation is feasible. Moreover, since $\hat{\pi}_1 = \hat{\pi}_2$ it follows that $\hat{W}(\hat{\mu}) = W(\mu)$. Since $U(\hat{c}_{m+1}) > U(\hat{c}_{m+2})$ it follows that $\hat{\mu}$ is not fair as desired. \qed

A.2 Proof of Theorem 1

Let $Z = \mathbb{R}_{++}^N$. Then, for all $z \in Z$, let $M^z(C')$ be the set of all allocations with support contained in $C'$ that are feasible for the economy $E = (u, k, \pi, z)$. Let
\[
W_k(z) = \max_{\mu \in M^z(C_k)} W(\mu).
\]

Hence, $W_k$ is the planner’s value as a function of the endowment. Let $Z^* = \{z \in Z : W_k(z) > W_k(s)\}$.

Clearly, $W_k(z) > W_k(y)$ whenever $z_i > y_i$ for all $i \in N$ since we can take the optimal allocation for $y$ and increase every consumption in every state by a small constant amount. Hence, $Z^*$ is nonempty.

Suppose $|z_i - y_i| < \epsilon$ for all $i$. Let $y_i^+ = \max\{y_i, z_i\}$, $y_i^- = \min\{y_i, z_i\}$ for all $i$ and let $\mu = (a, c)$ be optimal for $(u, \pi, y^+)$. Then, $(a, (1 - \frac{\epsilon}{a})c)$ is feasible for $(u, \pi, y^-)$. Since $W(a, (1 - \frac{\epsilon}{a})c)$ is continuous in $\epsilon$ at $\epsilon = 0$ and $W_k(y)$ is nondecreasing in each coordinate, for $\epsilon' > 0$, there exists $\epsilon > 0$ such that
\[
|W_k(y) - W_k(z)| \leq |W_k(y^+) - W_k(y^-)| < \epsilon'
\]
proving that $W$ is continuous at $y$ and hence $Z^y$ is open.

We note that since $W$ is a concave function of $\mu$, $W_k$ is a concave function of $z$ and hence the set $Z^s$ is convex. To see that, fix $z^1, z^2 \in Z^s$ and choose $\mu^i \in M^z(C_k)$ such that $W(\mu^i) = W_k(z^i)$ for $i = 1, 2$. By Lemma 1, such $\mu^i$ exist. Clearly, $\gamma\mu^1 + (1-\gamma)\mu^2 \in M^z(C_k)$ for $\hat{z} = \gamma z^1 + (1-\gamma)z^2$ and hence $W_k(\gamma z^1 + (1-\gamma)z^2) \geq W(\gamma\mu^1 + (1-\gamma)\mu^2) = \gamma W(\mu^1) + (1-\gamma)W(\mu^2) = \gamma W_k(z^1) + (1-\gamma)W_k(z^2)$.

Since $Z^s$ is nonempty, open, and convex, and $s \notin Z^s$, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^n$ such that $p_i \neq 0$ for some $i$ and $\sum_i p_i \cdot z_i > \sum_i p_i \cdot s_i$ for all $z \in Z^s$. Since $W_k$ is nondecreasing in each coordinate, we must have $p_i \geq 0$ for every $i \in N$ and hence we can normalize $p$ to ensure that $p \in \Delta(N)$.

Let $\mu = (a, c)$ be a solution to the planner’s problem, where $a^l > 0$ for all $l$. The argument establishing that each $c^l$ must maximize $U$ given budget $B(p)$ is standard and omitted, as is the proof of the following lemma:

**Lemma 7.** If $(p, \mu)$ is a CCE, then $\mu$ is Pareto-efficient.

Finally, to see that if $(p, \mu)$ is a CCE, then $\mu$ must be a solution to the planner’s problem, note that since every household has the same endowment, $\mu$ must be fair. But then, if $\mu$ did not solve the planner’s problem, the solution to the planner’s problem would Pareto-dominate it, contradicting Lemma 7.

\[ \square \]

### A.3 Proof of Theorem 2

**Lemma 8.** The CCE price of a pure endowment economy is unique.

**Proof.** First, we show that for all $c$ in the support of $\mu$, $c_i > 0$ for all $i$. For any $c$ in the support of $\mu$, let $A = \{i : c_i = 0\}$. If $A \neq \emptyset$, household optimality implies $\sum_{i \in A} p_i = 1$ and $\sum_{i \in N \setminus A} p_i = 0$; otherwise consumption can be raised by $\epsilon$ on the set $A$ and lowered by $\epsilon \sum_{i \in N \setminus A} p_i$ on the set $N \setminus A$ resulting in an overall increase of utility for small $\epsilon$. It follows that $c$ costs the same as $2c$ and since $c_i > 0$ for some $i$ and $u$ is strictly increasing, $c$ cannot be optimal if $A \neq \emptyset$. 

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Since $E$ is a pure endowment economy, assume without loss of generality that $s_i < s_{i+1}$. For any $\mu$, let $I(\mu) = \{i < n : c_i < c_{i+1} \text{ for some } c \in K(\mu)\}$. Since every CCE allocation solves the planner’s problem and $k > 1$ (i.e., household can have at least two distinct consumption levels), $I(\mu) \neq \emptyset$. Hence, for any competitive allocation $\mu$, let $J(\mu) = \max I(\mu)$. Let $(\mu^l, p^l)$ for $l = 1, 2$ be two CCE.

We claim that $i \notin I(\mu^l)$ implies $i + 1 \notin I(\mu^l)$. To see why this is the case, note that if $i \notin I(\mu^l)$, then $\sigma_i(\mu^l) = \sigma_{i+1}(\mu^l)$ and therefore $\sigma_{i+1}(\mu^l) < s_{i+1}$ and hence $p_{i+1} = 0$. Then, if $c_{i+2} > c_{i+1}$ for any $c \in K(\mu^l)$, define $\hat{c}_j = c_j$ for $j \leq i$, $\hat{c}_j = c_j$ for $j \geq i + 2$ and $\hat{c}_j = c_{i+1}$ and note that $\hat{c}$ is coarse, costs the same as $c$ but yields strictly higher utility, contradicting the fact that $\mu^l$ is a CCE allocation.

Next, we claim that $J(\mu^1) = J(\mu^2)$. If not, assume without loss of generality that $J(\mu^1) > J(\mu^2)$. Define $\hat{s}_j = s_j$ for all $j \leq J(\mu^2)$ and $\hat{s}_j = s_j + 1$ for all $j \geq J(\mu^2) + 1$. Then, since we established in the preceding paragraph that $p_j = 0$ for all $j > J(\mu^2)$, we conclude that $(p^2, \mu^2)$ is a CCE for the economy with endowment $\hat{s}$. Therefore, by Theorem 1, $W_k(s) = W_k(\hat{s})$. But, since $i := J(\mu^2) < J(\mu^1)$, the previous claim implies $i \in I(\mu^1)$. Hence, there exist $c \in K(\mu^1)$ such that $c_i < c_{i+1}$. Since $c$ is monotone (by Lemma 1), $\hat{c}$ defined by $\hat{c}_j = c_j$ for all $j \leq i$ and $\hat{c}_j = c_j + 1$ for all $j \geq k + 1$ is coarse. Let $\hat{\mu}$ be the allocation derived from $\mu^1$ by replacing $c$ with $\hat{c}$. Note that $\hat{\mu}$ yields strictly higher mean utility than $\mu^1$ and is feasible for the economy with endowment $\hat{s}$, contradicting $W_k(s) = W_k(\hat{s})$.

Note that if $J(\mu^1) = J(\mu^2) = 1$, then $p_1 = p_2 = 1$ and hence $p^1 = p^2$ as desired. So, henceforth we assume $J(\mu^1) = J(\mu^2) > 1$. By Theorem 1, both $\mu^1, \mu^2$ solve the planner’s problem. Then, the linearity of $W$ ensures that $\mu = .5\mu^1 + .5\mu^2$ also solves the planner’s problem and hence by Theorem 1, there exists some $p$ such that $(p, \mu)$ is a CCE. Then, the previous claim ensures that $J := J(\mu) = J(\mu^1) = J(\mu^2) > 1$.

For any $c$ such that $c_j > 0$ for all $j$ and for any $i = 1, \ldots, n - 1$, define

$$MRS_i(c) = \frac{\sum_{j \leq i} \pi_j u'(c_j)}{\sum_{j > i} \pi_j u'(c_j)}$$

For the price $p$ defined above, define $q \in \mathbb{R}^n$ such that $q_i = \sum_{j \leq i} p_i$. Define $q^1, q^2$ in an analogous
for all states $j \leq i$ by $\epsilon$ and in all states $j > i$ by $\epsilon'$ in a budget neutral manner. The optimality of $c^i$ ensures that this alternative plan cannot increase utility which means:

$$q^1_i = MRS_i(c^i)(1 - q^1_i)$$

for all $i \leq J$. But since $K(\mu^1) \subset K(\mu)$, the equations above also hold for $q$ proving that $q_j = q^1_j$ for all $j$ and hence $p^1 = p$. A symmetric argument ensures that $p^2 = p$. $\square$

Let $E = (u, k, \pi, s)$, $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be two static economies. We say that $\hat{E}$ is noisier than $E$ if there is a function $g : \hat{N} \to N$ such that $\hat{s}_j = s_i$ whenever $g(j) = i$ and $\sum_{j: g(j) = i} \pi_j = \pi_i$ for all $i$. We call such a $g$ a homomorphism. The homomorphism $g$ is rational (uniform) if $i = g(j)$ implies $\hat{\pi}_j = \pi_i$ is a rational number ($g(j) = g(j')$ implies $\pi_j = \pi_{j'}$). Clearly, if $g$ is uniform, then it is rational. When $\hat{E}$ is noisier than $E$, we write $[\hat{E} \parallel E]$; if there exists a rational (uniform) homomorphism from $\hat{E}$ to $E$, then we say $[\hat{E} \parallel E]$ is rational (uniform).

For any consumption $c$, price $p$ and allocation $\mu = (a, c)$ in $E$, define the corresponding consumption $\hat{c}$, price $\hat{p}$ and allocation $\hat{\mu} = (a, \hat{c})$ for $\hat{E}$ as follows: $\hat{c}_j = c_{g(j)}$; $\hat{p}_j = \frac{\hat{\pi}_j p_{a(j)}}{\pi_{g(j)}}$ for all $j$; $\hat{c}$ is such that $\hat{c}^l = \theta_1(c^l)$ for all $l$. For any $c$ let $\theta_1(c)$ be the $\hat{c}$ defined above; let $\theta_2(p)$ be the $\hat{p}$ defined above and for $\theta_3(\mu)$ be the $\hat{\mu}$ defined above. Let $\Theta^g = (\theta_1, \theta_2, \theta_3)$. Also, let $D(p)$ be the set of solutions to a households utility maximization problems at price $p$ in the economy $E$. We use $\hat{D}$, $\tilde{D}$ etc. for $\hat{E}$, $\tilde{E}$ etc.

**Lemma 9.** If $[\hat{E} \parallel E]$ is rational and $c \in D(p)$, then $\theta_1(c) \in \hat{D}(\theta_2(p))$.

**Proof.** The following assertions are easy to verify: (1) $U(c) = \hat{U}(\theta_1(c))$ and (2) $c \in B(p)$ implies $\theta_1(c) \in \hat{B}(\hat{p})$.

Since $[\hat{E} \parallel E]$ is rational, there exists $\tilde{E}$ such that $[\hat{E} \parallel \tilde{E}]$ and $[\tilde{E} \parallel E]$ are uniform. Let $g$ be a $[\tilde{E} \parallel E]$-homomorphism and $\Theta^g = (\theta_1, \theta_2, \theta_3)$; let $\hat{g}$ be a uniform $[\tilde{E} \parallel \hat{E}]$-homomorphism and $\Theta^{\hat{g}} = (\theta_1, \theta_2, \theta_3)$. Then, $\tilde{g} = g \circ \hat{g}$ is a uniform $[\hat{E} \parallel E]$-homomorphism and let $\Theta^{\tilde{g}} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$; $\tilde{\theta}_l = \hat{\theta}_l \circ \theta_l$ for $l = 1, 2, 3$. The two assertions above imply that we are done if we can show $\tilde{c} \in D(\tilde{\theta}_2(p))$ implies $\tilde{c} = \tilde{\theta}_1(c)$ for some $c \in D(p)$. 

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Suppose not and assume without loss of generality that for some \( \tilde{c} \in \tilde{D}(\tilde{\theta}_2(p)) \), \( \tilde{c}_1 \neq \tilde{c}_2 \) despite \( \tilde{g}(1) = \tilde{g}(2) \); that is, despite \( \tilde{p}_1 = \tilde{p}_2 \). Consider the endowment \( \tilde{s} \) such that \( \tilde{s}_1 = \tilde{s}_2 = (\tilde{c}_1 + \tilde{c}_2)/2 \) and \( \tilde{s}_i = \tilde{c}_i \) for all \( i > 2 \). Let \( c'_1 = c_2 \) and \( c'_2 = c_1, c'_i = c_i \) for all \( i > 2 \). Note that \( \tilde{U}(c') = \tilde{U}(\tilde{c}) \) and therefore \( c' \in \tilde{D}(\tilde{p}) \). Hence, \( \mu = 5\delta_e + 5\delta_{c'} \) is a CCE for the economy \( (u, \pi, \tilde{s}) \). Therefore, \( \mu \) is a solution to the planner’s problem (by Theorem 1). But \( \mu \) is not measurable, contradicting Lemma 1.

**Lemma 10.** For any \( \hat{E}, \hat{E}' \) such that \( [\hat{E}] \preceq [E], c \in D(p) \) implies \( \theta_1(c) \in \hat{D}(\hat{\theta}_2(p)) \).

**Proof.** Let \( \hat{E} = (u, \pi, \hat{s}) \) where \( \pi \in \Delta(\hat{N}) \) and let \( g \) be the \([\hat{E}] \preceq [E]\)-homomorphism. We can construct a sequence \( \hat{\pi}^m \) converging to \( \hat{\pi} \) such that for all \( j, \frac{\hat{\pi}_j}{\pi_{g(j)}^m} \) is a rational number. Then, \( g \) is a rational homomorphism from \( E^m = (u, \pi^m, \hat{s}) \) to \( E \). Let \( c \in D(p) \) and \( \hat{c} = \theta_1(c), \hat{p} = \theta_2(p) \). Then, by Lemma 9,

\[
\sum_{i \in N} u(\hat{c}_i) \cdot \hat{\pi}_i^m \geq \sum_{i \in N} u(c'_i) \cdot \hat{\pi}_i^m
\]

for all \( c' \in B(\hat{p}) \) and therefore \( \sum_{i \in \hat{N}} u(\hat{c}_i) \hat{\pi}_i \geq \sum_{i \in \hat{N}} u(c'_i) \hat{\pi}_i \) for all \( c' \in B(\hat{p}) \) as desired. 

**Lemma 11.** The CCE price of a pure endowment economy is monotone.

**Proof.** Suppose \( (p, \mu) \) is the CCE of some pure endowment economy \( E = (u, k, \pi, s) \) and \( \frac{\mu_i}{\pi_i} > \frac{\mu_j}{\pi_j} \) for some \( s_i > s_j \). Assume \( \pi_i > \pi_j \) (if the opposite inequality holds, a symmetric argument applies; in case of equality ignore \( s_{n+1} \)). Let \( \hat{s}_{n+1} = s_i \) and \( \hat{\pi}_{n+1} = \pi_i - \pi_j, \hat{s}_l = s_l \) for all \( l \leq n, \hat{\pi}_i = \pi_j \) and \( \hat{\pi}_i = \pi_l \) for all \( l \neq i, n + 1 \). Then, \([\hat{E}] \preceq [E]\) and hence, by Lemma 10, \( (\hat{p}, \hat{\mu}) := (\theta_2(p), \theta_3(\mu)) \) is a CCE of \( \hat{E} \). By assumption \( \hat{p}_i > \hat{p}_j \) and \( \hat{\pi}_i = \hat{\pi}_j \) which means no household will consume more in state \( i \) than in state \( j \). Since \( s_i > s_j \), this implies \( \hat{p}_i = 0 \), a contradiction.

**Lemma 12.** The CCE price of any economy is essentially unique and monotone.

**Proof.** Let \( \hat{E} = (u, k, \hat{\pi}, \hat{s}) \) be any economy and let \( E = (u, k, \pi, s) \) be the corresponding pure endowment economy; that is \([\hat{E}] \preceq [E]\). For any plan measurable plan \( \hat{c} \) for \( \hat{E} \), define the plan \( c' \) for \( E \) as follows \( c'_i = \hat{c}_j \) for some \( j \) such that \( s_i = g(\hat{s}_j) \). Then, define \( \mu' \) in the obvious way: \( \mu'(c') = \hat{\mu}(\hat{c}) \). Given a price \( \hat{p} \) for \( E \), define \( p' \) for \( E \) as follows: \( p'_i = \sum_{j: g(j) = i} \hat{p}_j \) for all \( i \).
Suppose there are two CCE for \( \hat{E} \), \((\hat{p}^l, \hat{\mu}^l)\) for \( l = 1, 2 \) such that \( \hat{p}^1 \) and \( \hat{p}^2 \) are not equivalent. Then, since \( \hat{\mu}^l \)’s are measurable, (Lemma 1) the corresponding \( \mu^l \) are well-defined allocations for \( E \). It is easy to see that \((p^l, \mu^l)\) are CCE equilibria for \( E \). But since \( \hat{p}^1 \) and \( \hat{p}^2 \) are not equivalent, \( p^1 \neq p^2 \), contradicting Lemma 8.

Finally, take CCE of \( E \) and recall that by Lemma 10, \((\hat{p}, \hat{\mu}) := (\theta_2(p), \theta_3(\mu))\). By Lemma 11, \((p, \mu)\) is monotone and therefore, so is \((\hat{p}, \hat{\mu})\). Then essential uniqueness ensures that all CCE prices of \( \hat{E} \) are monotone. \( \Box \)

### A.4 Dominance Lemma

Let \( \{E^n\} \) be a convergent sequence of economies where \( E^n = (u, k, \pi^n, s^n) \). Let \((p^n, \mu^n)\) be a CCE of \( E^n \), let \( w^n = p^n \cdot s^n \) be the consumer’s equilibrium wealth, and let \( U^n \) be the equilibrium utility of a household. For the remainder of this section, \( p^n, \mu^n, w^n \) and \( U^n \) always refers to this sequence of equilibrium variables. Let \( f : [a, b] \rightarrow \mathbb{R}_{++} \) be the density of the limit endowment.

For \( w > 0 \), \( \bar{L}^n(w) := \sup_{c \in B_{k-1}(p^n, w)} U^n(c) \) is the supremum over attainable utilities of a consumer with coarseness constrain \( k - 1 \) at the equilibrium price \( p^n \). In this section, we prove the following lemma:

**Lemma 13.** \( \lim inf_n [U^n - \bar{L}^n(w^n)] > 0. \)

#### A.4.1 Proof of Lemma 13

The proof proceeds by contradiction. We show that if (in the limit) moving from a \((k-1)\)-coarse plan to a \( k \)-coarse plan does not yield any utility gain, then the relative prices within each of the \( k - 1 \) cells are constant. This, in turn, implies that relaxing the coarseness constraint entirely does not yield any utility gain, which is an impossibility. In the course of this argument we extract a subsequence of economies three times (in expression (a) below, just before expression (c) below, and in Lemma 17 below); each time we pass to that subsequence. We end up obtaining a subsequence of economies for which (in the limit) the contradiction holds. The proof relies on some auxiliary lemmas, whose proofs appear in the next subsection.
First, notice that since \( B_k(p^n, w^n) \geq B_{k-1}(p^n, w^n) \) it follows that \( U_n^* - \bar{L}^n(w^n) \geq 0 \) for all \( n \); thus, \( \liminf_n [U_n^* - \bar{L}^n(w^n)] \geq 0 \). It remains to show strict positivity. Suppose that 

\[
\lim[\bar{U}^n - \bar{L}^n(w^n)] = 0 
\tag{a}
\]

along some subsequence. Pass to that subsequence.

Note that \( p^n_i \) may be zero for some \( n \) and, therefore, \( \bar{L}^n(w^n) \) may not be attained. Define \( \hat{p}^n \in \mathbb{R}^n_{+} \) such that

\[
\hat{p}^n_i = \begin{cases} 
p^n_i & \text{if } p^n_i > 0 \\
\pi^n_i / 2^n & \text{if } p^n_i = 0
\end{cases}
\]

and let \( L^n(w) := \max_{c \in B_{k-1}(\hat{p}^n, w)} U^n(c) \). Note that \( B_{k-1}(\hat{p}^n, w) \) is compact and, therefore, \( L^n(w) \) is well defined for all \( w > 0 \).

**Lemma 14.** (i) \( \hat{p}^n = p^n \) if \( \rho \leq 1 \); (ii) \( \lim(\bar{L}^n(w^n) - L^n(w^n)) = 0 \).

Lemma 14 and (a) implies that

\[
\lim[\bar{U}^n - L^n(w^n)] = 0; 
\tag{b}
\]

thus we can now work with \( \hat{p}^n \). For each \( n \), choose \( c^n \in B_{k-1}(\hat{p}^n, w^n) \) such that \( U^n(c^n) = L^n(w^n) \).

For each \( n \) define events \( A^n_1, \ldots, A^n_{k-1} \) as follows: let \( A^n_1 \) be the set of states with the lowest consumption in \( c^n \), \( A^n_2 \) be the set of states with the second lowest consumption in \( c^n \), etc. For any \( l \) define \( \beta^n_l := \sum_{A^n_l} \pi^n_i \) to be the probability of event \( A^n_l \) and \( \bar{p}^n_l := \sum_{A^n_l} p^n_i \) to be the sum of prices on \( A^n_l \).

For \( l \) such that \( A^n_l \neq \emptyset \) define \( \bar{s}^n_l := \frac{1}{\beta^n_l} \sum_{i \in A^n_l} \pi^n_i s^n_i \) to be the expected value of the endowment conditional on \( A^n_l \); define \( \bar{s}^n_l = 0 \) otherwise. Let \( A^n_{1l} := \{ i \in N : s^n_i \leq \bar{s}^n_l \} \cap A^n_l \) be the states in \( A^n_l \) with endowment below \( \bar{s}^n_l \) and \( A^n_{2l} := \{ i \in N : s^n_i > \bar{s}^n_l \} \cap A^n_l \) be the states in \( A^n_l \) with endowment above \( \bar{s}^n_l \). Define \( \beta^n_{1l} := \sum_{A^n_{1l}} \pi^n_i \) and \( \beta^n_{2l} := \sum_{A^n_{2l}} \pi^n_i \). Also, define \( \bar{p}^n_{1l} := \sum_{A^n_{1l}} p^n_i \) and \( \bar{p}^n_{2l} := \sum_{A^n_{2l}} p^n_i \).

Since the set \([u(a), u(b)] \times [0,1]^{4(k-1)}\) is compact there exists a subsequence such that \((L^n(w^n), (\beta^n_{1l}, \beta^n_{2l}, \bar{p}^n_{1l}, \bar{p}^n_{2l}))_{l=1}^{k-1}\) converges. Pass to that subsequence and let \((L, (\beta_{1l}, \beta_{2l}, \bar{p}_{1l}, \bar{p}_{2l}))_{l=1}^{k-1}\)
be its limit. Notice that (b) implies
\[
\lim U^n_* = L. 
\] (c)

The next lemma shows that on each fragment of the set \( A^n_i \) the limit relative price is the same.

**Lemma 15.** If \( \lim U^n_* = L \), then
\[
\frac{\bar{p}_1}{\beta_1} = \frac{\bar{p}_2}{\beta_2}.
\]
for all \( A_i \) with \( \beta_i > 0, \bar{p}_l > 0 \).

This in turn implies that relaxing the coarseness constraint entirely does not yield any gain in the limit.

**Lemma 16.** Suppose that \( \frac{\bar{p}_1}{\beta_1} = \frac{\bar{p}_2}{\beta_2} \) for all \( A_i \) with \( \beta_i > 0, \bar{p}_l > 0 \). Then \( L \geq \lim U^n(s^n) \).

Thus, (c) and Lemmas 15 and 16 imply that
\[
\lim U^n_* \geq \lim U^n(s^n). 
\] (d)

**Lemma 17.** \( \lim[U^n(s^n) - U^n_+] > 0 \) along some subsequence.

Lemma 17 is a contradiction with (d).

### A.4.2 Proofs of auxiliary lemmas

**Proof of Lemma 14.** To prove part (i) note that \( u \) is unbounded if \( \rho \leq 1 \). Since \( p^n \) is an equilibrium price it follows that \( p^n_i > 0 \) for all \( i \) and \( \hat{p}^n = p^n \) as desired. If \( \rho \leq 1 \) then part (i) implies part (ii). Thus, it remains to prove part (ii) for \( \rho > 1 \). In that case, \( u \leq 0 \).

Choose any \( c^n \in B^n_{k-1}(p^n, w^n) \) such that \( U^n(c^n) \geq u(a) \); fix \( \epsilon > 0 \) and choose \( K \) such that \( u(K) = -\epsilon \). Define \( \hat{c}^n \) such that \( \hat{c}^n_i = \min\{c^n_i, K\} \) for all \( i \). Choose \( \bar{n} \) such that \( 2^{-\bar{n}} \cdot K \leq \epsilon \), then \( \left(\frac{w^n - \epsilon}{w^n}\right) \hat{c}^n \in B^n_{k-1}(\hat{p}^n, w^n) \) for \( n \geq \bar{n} \) and
\[
U^n\left(\left(\frac{w^n - \epsilon}{w^n}\right) \hat{c}^n\right) = \left(\frac{w^n - \epsilon}{w^n}\right)^\rho U^n(\hat{c}^n) \geq \left(\frac{w^n - \epsilon}{w^n}\right)^\rho (U^n(c^n) - \epsilon)
\]
Since \( 0 < a < w^n \) for all \( n \) and since \( 0 \geq U^n(c^n) \geq u(a) \) the result follows.
Lemma 18. (i) \( \beta_{11} > 0 \) if and only if \( \beta_{21} > 0 \); (ii) \( \beta_{1} > 0 \) and \( \frac{\beta_{11}}{\beta_{11}} = \frac{\beta_{12}}{\beta_{12}} \) imply

\[
\lim \sum_{A_{n}} |p^{n}_{i} - \pi^{n}_{i} \bar{p}^{n}_{i}/\beta^{n}_{i}| = 0.
\]

Proof of Lemma 18.

part (i) Let \( F^{n} \) be the cumulative distribution function of \( s^{n} \); let \( F \) be the limit cumulative distribution function. Since \( F \) is continuous, it follows that \( F^{n}(x) \to F(x) \) uniformly on \([a, b]\). Further, since \( F \) is continuous on \([a, b]\), it’s uniformly continuous, so for every \( \epsilon > 0 \) there is \( \delta > 0 \) (independent of \( x \)) such that \( F(x + \delta) - F(x) < \epsilon \). We assume \( \beta_{21} > 0 \) and show \( \beta_{11} > 0 \). The reverse implication follows from an analogous argument. Find \( \delta > 0 \) such that \( F(x + \delta) - F(x) < \beta_{21}/4 \) for all \( x \in [a, b] \); choose \( \bar{n} \) so that \( F^{n}(x + \delta) - F^{n}(x) < \beta_{21}/2 \) for all \( n > \bar{n} \) and for all \( x \in [a, b] \). Then, for \( n > \bar{n} \) and for any set \( E^{n} \subset N \) such that \( s^{n}_{i} > x \) for \( i \in E^{n} \) and \( \sum_{E^{n}} \pi^{n}_{i} s^{n}_{i} \geq \beta^{n}_{21} (x + \delta/2) \). To see that, notice that

\[
\sum_{E^{n}} \pi^{n}_{i} s^{n}_{i} = \sum_{\{i \in E^{n} : s^{n}_{i} \leq x + \delta\}} \pi^{n}_{i} s^{n}_{i} + \sum_{\{i \in E^{n} : s^{n}_{i} > x + \delta\}} \pi^{n}_{i} s^{n}_{i} \geq x \beta^{n}_{21} + \delta \sum_{\{i \in E^{n} : s^{n}_{i} > x + \delta\}} \pi^{n}_{i}
\]

Since \( \sum_{\{i \in E^{n} : s^{n}_{i} \leq x + \delta\}} \pi^{n}_{i} \leq F^{n}(x + \delta) - F^{n}(x) < \beta^{n}_{21}/2 \), it follows that \( \sum_{\{i \in E^{n} : s^{n}_{i} > x + \delta\}} \pi^{n}_{i} \geq \beta^{n}_{21}/2 \).

Setting \( x = \bar{s}_{l}^{n} \) and \( E^{n} = \{i \in A_{l}^{n} : s^{n}_{i} > \bar{s}_{l}^{n}\} \) we get \( \beta^{n}_{11} \bar{s}_{l}^{n} = \sum_{A_{l}^{n} \setminus E^{n}} \pi^{n}_{i} s^{n}_{i} + \sum_{E^{n}} \pi^{n}_{i} s^{n}_{i} \geq \beta^{n}_{11} a + \beta^{n}_{21} (\bar{s}_{l}^{n} + \delta/2) \). Since \( \beta^{n}_{21} \to \beta_{21} > 0 \) it follows that \( \beta^{n}_{11} \to \beta_{11} > 0 \).

part (ii) To prove part (ii) note that by part (i) \( \beta_{11} > 0 \) and \( \beta_{21} > 0 \). Define \( z = \frac{\bar{p}_{1}}{\bar{p}_{1}} = \frac{\bar{p}_{11}}{\bar{p}_{11}} = \frac{\bar{p}_{12}}{\bar{p}_{12}} \).

Fix \( \epsilon > 0 \). There is \( \bar{n} \) such that for \( n > \bar{n} \)

\[
|\frac{\bar{p}_{11}^{n}}{\bar{p}_{11}} - z| \leq \epsilon; |\frac{\bar{p}_{21}^{n}}{\bar{p}_{21}} - z| \leq \epsilon; |\frac{\bar{p}_{11}^{n}}{\bar{p}_{11}} - z| \leq \epsilon;
\]

(A)

Let \( n > \bar{n} \); let \( j_{1}^{n} \in N \) be such that \( s_{i}^{n} \leq j_{1}^{n} \) if and only if \( i \leq j_{1}^{n} \). Theorem 2 implies that \( \frac{p_{11}^{n}}{\pi_{i}^{n}} \) is non-increasing. Therefore, we can define \( j_{2}^{n} \in N \) such that \( p_{11}^{n} - \pi_{i}^{n} z > 0 \) if and only if \( i < j_{2}^{n} \).

Assume that \( j_{2}^{n} \leq j_{1}^{n} \). (The case where this inequality is reversed follows from an analogous
argument and is therefore omitted). Then, (a) implies
\[
\sum_{\{i \in A^n: i \geq j^n_2\}} |p^n_i - \pi_i^n z| = \beta_{2t}^n z - \bar{p}_{2t}^n \leq \beta_{2t}^n \epsilon \leq \epsilon. \quad (b)
\]
Since for all \(j^n_2 \leq i \leq j^n_1\) we have that \(0 \leq z - \frac{p^n_{j^n_2}}{\pi_i^n} \leq z - \frac{p^n_{j^n_1}}{\pi_i^n} \leq z - \frac{p^n_{j^n_1}}{\beta_{2t}^n}\), and
\[
\sum_{\{i \in A^n: j^n_1 \geq i \geq j^n_2\}} |p^n_i - \pi_i^n z| = \sum_{\{i \in A^n: j^n_1 \geq i \geq j^n_2\}} \pi_i^n z - p^n_i = \sum_{\{i \in A^n: j^n_1 \geq i \geq j^n_2\}} \pi_i^n \left(z - \frac{p^n_i}{\pi_i^n}\right) 
\leq \left(z - \frac{p^n_{j^n_1}}{\beta_{2t}^n}\right) \sum_{\{i \in A^n: j^n_1 \geq i \geq j^n_2\}} \pi_i^n \leq \epsilon \sum_{\{i \in A^n: j^n_1 \geq i \geq j^n_2\}} \pi_i^n \leq \epsilon \quad (c)
\]
where the second inequality follows from (a). Finally, note that (a) implies that
\[
\sum_{i \in A^n} p^n_i - \pi_i^n z = \bar{p}^n_i - \beta^n_i z \leq \epsilon. \quad (d)
\]
On the other hand, \(\sum_{i \in A^n} p^n_i - \pi_i^n z = \sum_{\{i \in A^n: i \leq j^n_2\}} |p^n_i - \pi_i^n z| - \sum_{\{i \in A^n: i > j^n_2\}} |p^n_i - \pi_i^n z|\); therefore
\[
\sum_{\{i \in A^n: i \leq j^n_2\}} |p^n_i - \pi_i^n z| \leq \sum_{i \in A^n} p^n_i - \pi_i^n z + \sum_{\{i \in A^n: i > j^n_2\}} |p^n_i - \pi_i^n z| \leq 3\epsilon, \quad (e)
\]
where the last inequality follows from (b), (c), and (d).

Inequalities (b), (c) and (e) imply that \(\sum_{\{i \in A^n\}} |p^n_i - \pi_i^n z| \leq 5\epsilon. \quad \square\)

**Lemma 19.** Let \(j^n \in A^n\) such that \(\beta_i > 0\). There exists \(\delta > 0\) such that \(c^n_{j^n} > \delta\) for \(n\) large.

**Proof.** If \(\rho \geq 1\), the result is immediate since \(\lim \inf c^n_{j^n} = 0\) implies \(\lim \inf U^n(\bar{c}^n) = -\infty\). So, assume \(\rho < 1\) and that passing to a subsequence \(\lim \inf U^n(\bar{c}^n) = 0\); recall that \(L = \lim U^n(\bar{c}^n)\). For \(\delta > 0\), define \(\delta c^n_j = (1 - \delta)^{\delta c^n_j}\) for all \(j \notin A^n\) and \(\delta c^n_j = a\delta\) for all \(j \in A^n\). Note that \(\delta c^n \in B_{k-1}(\bar{p}^n, w^n)\) since \(\sum_{A^n} p^n_i \leq 1\) and \(p^n = \bar{p}^n\) by Lemma 14 (i). Concavity of \(u\) implies
that
\[
\lim_n [U^n(\delta \hat{c}^n) - U^n(\bar{c}^n)] = \lim_n \sum_{j \in A^n} [u'(a\delta)(a\delta - \hat{c}^n)]\pi^n_i + \sum_{j \notin A^n} [u((1 - \delta)\hat{c}^n) - u(\bar{c}^n)]\pi^n_i \\
= a\delta u'(a\delta)\beta_t - \lim_n u'(a\delta)\bar{c}^n_t - [1 - (1 - \delta)^\rho] \lim_n \sum_{j \notin A^n} u(\bar{c}^n_j)\pi^n_i \\
\geq \beta_t \rho(a\delta) - [1 - (1 - \delta)^\rho]L
\]

Thus,
\[
\lim_n [U^n(\delta \hat{c}^n) - U^n(\bar{c}^n)] = (\beta_t \rho^\rho) \frac{1}{\delta^{1-\rho}} - L \frac{1 - (1 - \delta)^\rho}{\delta}
\]

Since \(\lim_{\delta \to 0} \frac{1 - (1 - \delta)^\rho}{\delta} = \rho\), we can choose \(\delta > 0\) so that \(\lim_n U^n(\delta \hat{c}^n) > \lim_n U^n(\bar{c}^n)\) contradicting the optimality of \(\bar{c}^n\). \(\square\)

**Proof of Lemma 15.** Suppose not; then \(\frac{\beta_{t2}^n}{\beta_{t1}^n} < \frac{\beta_{t1}^n}{\beta_{t2}^n}\) since, by Theorem 2, \(\frac{p_t}{\pi_t}\) is monotone. Let

\[
\tilde{V}(m_t) = \max_{\left\{(x,y) : \beta_{t1}x + \beta_{t2}y = m_t\right\}} \beta_{t1}u(x) + \beta_{t2}u(y)
\]

and let \((x_t, y_t)\) be a solution to the maximization problem. By a standard argument, \(x_t < y_t\) and \(\eta := \tilde{V}(m_t) - \beta_t u(m_t/\bar{p}_t) > 0\).

By Lemma 19, \(\bar{c}^n_i \geq \delta > 0\) for \(i \in A^n_t\). Let \(m^n_t = \bar{c}^n_i \bar{p}^n_t\) for any \(i \in A^n_t\). Note that \(m^n_t\) is bounded (clearly it is bounded from below by zero; to see the upper bound, note that \(\bar{c}^n_i = \frac{w^n_i}{\sum_{A^n_t} \bar{p}^n_t} \leq b/\bar{p}^n_t\)). Thus, \(m^n_t\) has a convergent subsequence. Let \(m_t\) be its limit and note that \(m_t > 0\).

Let \(\hat{c}^n\) be the following consumption plan:

\[
\hat{c}^n_i = \begin{cases} 
\bar{c}^n_i & \text{if } i \notin A^n_t \\
\frac{m^n_t}{\beta_{t1}x_t + \beta_{t2}y_t} x_t & \text{if } i \in A^n_t, s^n_i \leq \bar{s}^n_i \\
\frac{m^n_t}{\beta_{t1}x_t + \beta_{t2}y_t} y_t & \text{if } i \in A^n_{t2}, s^n_i > \bar{s}^n_i 
\end{cases}
\]

It follows that \(U^n(\hat{c}^n) - U^n(\bar{c}^n) \to \eta\). Since \(\hat{c}^n \in B_k(p^n, w^n)\) and \(U^n(\hat{c}^n) = L^n(w^n)\) this contradicts the hypothesis of the lemma. \(\square\)
Proof of Lemma 16. Let \( \hat{c}^n \) be the following consumption plan

\[
\hat{c}_i^n = \begin{cases} 
  s^n_i & \text{if } i \in A_i^n \text{ such that } \beta_l > 0 \\
  \min_{j \in A_i^n} s^n_j & \text{if } i \in A_i^n \text{ such that } \beta_l = 0 
\end{cases}
\]

Since \( s^n \) is bounded below, concavity of \( u \) implies that \( \lim \inf [U^n(\hat{c}^n) - U^n(s^n)] \geq 0 \). Thus, for any sequence \( \gamma^n < 1 \) with \( \lim \gamma^n = 1 \) we have \( \lim \inf [U^n(\gamma^n \hat{c}^n) - U^n(s^n)] \geq 0 \).

To conclude the proof, it remains to show that \( \gamma^n \hat{c}^n \in B_{k-1}(\hat{p}^n, w^n) \) for some sequence \( \gamma^n < 1 \) with \( \lim \gamma^n = 1 \). To this end, we show that \( \lim \sum_{A_i^n} p^n_i (\hat{c}_i^n - s^n_i) \leq 0 \) for all \( l \). This is immediate if \( \beta_l = 0 \) or if \( \hat{p}_l = 0 \). Hence, assume \( \beta_l > 0 \) and \( \hat{p}_l > 0 \).

First, note that \( \lim \sum_{A_i^n} p^n_i (\hat{c}_i^n - s^n_i) = \lim \sum_{A_i^n} \hat{p}^n_i (\hat{c}_i^n - s^n_i) \). To see that, notice that

\[
\lim \sum_{A_i^n} \hat{p}^n_i (\hat{c}_i^n - s^n_i) = \lim \sum_{\{i \in A_i^n: p^n_i > 0\}} \hat{p}^n_i (\hat{c}_i^n - s^n_i) + \sum_{\{i \in A_i^n: p^n_i = 0\}} \hat{p}^n_i (\hat{c}_i^n - s^n_i)
\]

\[
= \lim \sum_{\{i \in A_i^n: p^n_i > 0\}} \frac{\pi^n_i}{2^n} (\hat{c}_i^n - s^n_i) + \sum_{\{i \in A_i^n: p^n_i > 0\}} \hat{p}^n_i (\hat{c}_i^n - s^n_i)
\]

\[
= 0 + \sum_{\{i \in A_i^n: p^n_i > 0\}} \hat{p}^n_i (\hat{c}_i^n - s^n_i)
\]

\[
= \lim \sum_{\{i \in A_i^n: p^n_i = 0\}} p^n_i (\hat{c}_i^n - s^n_i) + \sum_{\{i \in A_i^n: p^n_i > 0\}} p^n_i (\hat{c}_i^n - s^n_i)
\]

\[
= \lim \sum_{A_i^n} p^n_i (\hat{c}_i^n - s^n_i)
\]

Second, note that

\[
\sum_{A_i^n} p^n_i (\hat{c}_i^n - s^n_i) = \sum_{A_i^n} \left( \frac{\hat{p}^n_i}{\beta^n_i} n_i s_i - p_i s_i \right) \leq b \sum_{A_i^n} \left| p^n_i - n_i s_i \right| \beta^n_i \hat{p}^n_i.
\]

Lemma 18 implies that last expression converges to zero.

Proof of Lemma 17. For \( l = 0, \ldots, k + 1 \), let \( x_i := a + (b - a) \frac{t}{k+1} \) and define \( N_l^n := \{ i \in N | s^n_i \in [x_l, x_{l+1}] \} \) for \( l < k \) and \( N_k^n := \{ i \in N | s^n_i \in [x_k, b] \} \). Let \( \delta^n_l := \sum_{N_l^n} n_i s_i^n \) and \( \delta_l := \int_{x_l}^{x_{l+1}} f(z)dz = \lim \delta^n_l \) and note that for \( n \) large enough, \( \delta^n_l > 0 \) for all \( l \). Define \( A_i^n \subset C_i^n \).
inductively: \( A^n_0 \) is the set of \( k \)-coarse plans that are constant on \( N^n_0 \); \( A^n_j \) is the set of \( k \)-coarse plans that are constant on \( N^n_l \) but are not in \( A^n_j \) for any \( j < l \). Define \( \alpha^n_i := \mu^n(A^n_i) \). Then, since any monotone \( k \)-coarse plan belongs to exactly one \( A^n_i \), we have \( \sum_{i=0}^{k} \alpha^n_i = 1 \). For all \( l \) with \( \alpha^n_l > 0 \) let \( i_l = N^n_l \); also let \( y^n_l = \sum_{c \in A^n_l} \mu^n(c)c_l / \alpha^n_l \); that is, \( y^n_l \) is the average consumption of plans in \( A^n_l \cap K(\mu^n) \) conditional on the state being in \( N_l \). Note that \( U^n(c) = U^n_* \) for all \( c \in K(\mu^n) \) and hence \( \sum_{l \in N^n} \mu^n(c)U^n(c) = U^n_* \).

Define the function \( R^n_l \) as follows: \( R^n_l(x) = \alpha^n_l u(y^n_l) + (1 - \alpha^n_l)u\left(\frac{x - \alpha^n_l y^n_l}{1 - \alpha^n_l}\right) \) if \( \alpha^n_l < 1 \) and \( R^n_l(x) = u(y^n_l) \) otherwise. Note that \( R^n_l(x) \) is the maximal average utility of agents in \( E^n \) given that they consume \( x \) on average and the fraction \( \alpha^n_l \) of them consumes \( y_l \). Feasibility and concavity of \( u \) imply that \( \sum_{c \in C_k} \mu^n(c)u(c_i) \leq \sum_{c \in C_k} R^n_l(s^n_i) \) for every \( i \in N^n_l \) and thus

\[
U^n_* = \sum_{c \in C^n_k} \mu^n(c)U^n(c) = \sum_{c \in C^n_k} \mu^n(c) \left( \sum_{i \in N^n_l} \pi^n_i u(c_i) + \sum_{i \in N^n_l} \pi^n_i u(c_i) \right) \\
\leq \sum_{i \notin N^n_l} \pi^n_i \sum_{c \in C_k} \mu^n(c)u(c_i) + \sum_{i \in N^n_l} \pi^n_i \sum_{c \in C_k} \mu^n(c)u(c_i) \\
\leq \sum_{i \notin N^n_l} \pi^n_i u(s^n_i) + \sum_{i \in N^n_l} \pi^n_i R^n_l(s^n_i).
\]

The concavity of \( u \) ensures

\[
\sum_{i \in N^n_l} \pi^n_i [u(s^n_i) - R^n_l(s^n_i)] \geq 0
\]

Thus, to prove the Lemma, it suffices to show that the above expression cannot converge to zero for all \( l \). Note that \( \alpha^n_l \geq 1/(k + 1) \) for some \( l \) and hence we may choose \( l \) such that \( \lim \alpha^n_l = \alpha > 0 \) along some subsequence; pass to that subsequence. Feasibility requires that \( y^n_l \) stays bounded; hence, let \( \lim y^n_l = y_l \) along some subsequence; pass to that subsequence.

If \( \alpha_l = 1 \), then feasibility ensures that \( y_l \leq x_l \). Also, \( \lim_{\gamma \to 1}(1 - \gamma)u(z/(1 - \gamma)) = 0 \) and therefore \( R^n_l(z) \to u(y_l) \) uniformly over \([x_l, x_{l+1}]\); thus, since \( f > 0 \),

\[
\lim \sum_{i \in N^n_l} \pi^n_i [u(s^n_i) - R^n_l(s^n_i)] \geq \int_{x_l}^{x_{l+1}} \left[ u(z) - u(x_l) \right] f(z)dz > 0.
\]

If \( 0 < \alpha_l < 1 \), then \( R^n_l(z) \to \alpha_l u(y_l) + (1 - \alpha_l)u(\frac{x - \alpha_l y_l}{1 - \alpha_l}) =: R_l(z) \) uniformly over \([x_l, x_{l+1}]\); thus,
since \( f > 0 \),

\[
\lim_{i \in N_1} \pi_i^n [u(s_i) - R^n_i(s^n_i)] = \int_{x_i}^{x_{i+1}} [u(z) - R_i(z)] f(z)dz > 0.
\]

where the last inequality follows from the strict concavity of \( u \). In the above inequalities we can pass to the limit, since whenever a sequence of probability measures \( P_n \) converges weakly to \( P \) and a sequence of functions \( f^n \) converges uniformly to a bounded and continuous function \( f \), we have \( \int f^n dP_n \to \int fdP \).

\[ \square \]

### A.5 Proof of Theorem 3

Let \( \{E^n\} \) be a convergent sequence of economies where \( E^n = \{(u, k, \pi^n, s^n)\} \). Let \( (p^n, \mu^n) \) be a CCE of \( E^n \) and let \( w^n = p^n \cdot s^n \) be the consumer’s equilibrium wealth and let \( U^n_* \) be the equilibrium utility of a household. For the remainder of this section, \( p^n, \mu^n, w^n \) and \( U^n_* \) always refers to this sequence of equilibrium variables. Let \( f : [a, b] \to \mathbb{R}^+ \) be the density of the limit endowment.

**Lemma 20.** If \( \sigma_i(\mu^n) < s_i \) then \( p^n_i = 0 \).

**Proof.** If \( c \in C_k \) then \( \gamma c \in C_k \) for all \( \gamma > 0 \). Since \( U^n \) is strictly increasing it follows that \( \sum_{i \in N} p^n_i c_i = \sum_{i \in N} p^n_i s^n_i \) for any \( c \in B^n_k(p^n, w^n) \) that maximizes utility. Therefore, \( \sum_{i \in N} p^n_i s^n_i = \sum_{c \in K(\mu^n)} \mu^n(c) \sum_{i \in N} p^n_i c_i = \sum_{i \in N} p^n_i \sigma_i(\mu^n) \). Since \( s^n_i \geq \sigma_i(\mu^n) \) for all \( i \), the lemma follows. \[ \square \]

**Lemma 21.** There is \( \delta > 0 \) such that \( 1 - \delta > p^n_i \geq \delta \) for all \( n \)

**Proof.** If \( p^n_1 \to 1 \), then \( \sum_{i > 1} p^n_i \to 0 \) and, since \( \pi^n_1 \to 0 \), \( U^n_* > u(b) \) for \( n \) large. This violates feasibility since \( s^n \leq b \). Thus, we have shown that \( p^n_1 \leq 1 - \delta \) for some \( \delta > 0 \).

Next, we show that for all \( n \) there is \( c \in K(\mu^n) \) such that \( c^n_1 < c^n_2 \). If not then \( \sigma_1(\mu^n) = \sigma_2(\mu^n) \leq s^n_1 < s^n_2 \) and, by Lemma 20, \( p^n_2 = 0 \). Then, by Theorem 2, \( p^n_i = 0 \) for all \( i \geq 2 \) contradicting the fact that \( p^n_1 \leq 1 - \delta \) for some \( \delta > 0 \).
If $c^n \in K(\mu^n)$ then $c^n_2 < b/\delta$. To prove this assertion, note that $b \geq w^n \geq \sum_{i>1} c^n_i p^n_i$ since $s^n \leq b$ and $\sum_N p^n_i = 1$. Since $c$ is monotone, $\sum_{i>1} c^n_i p^n_i \geq \sum_{i>1} c^n_{i} \delta$ and, therefore, $b/\delta \geq c^n_2$.

To complete the proof of the lemma, let $c \in K(\mu^n)$ with $c^n_1 < c^n_2$. Let $\hat{c} \in C_{k-1}$ be the following consumption plan:

$$\hat{c}^n_i = \begin{cases} c^n_2 & \text{if } i = 1, 2 \\ c^n_i & \text{otherwise.} \end{cases}$$

Note that $U^n(\hat{c}^n) \geq U^n(c^n) = U^*_n$. Define $\gamma^n = w^n + \frac{w^n}{w^n + p^n_1 \frac{b}{\delta}}$ and note that $\gamma^n \hat{c}^n \in B_{k-1}^n(p^n, w^n)$. If $p^n_i \to 0$ then $\gamma^n \to 1$ and therefore $U^n(\hat{c}^n) - U^n(\gamma^n \hat{c}^n) \to 0$. Since $U^*-\tilde{L}^n(w^n) \leq U^n - U^n(\gamma^n \hat{c}^n)$ this, in turn, contradicts Lemma 13.

Lemma 22. If $\rho > 1$ there is $\bar{n}, \epsilon > 0$ such that such that $n \geq \bar{n}$ and $\sum_{j \geq i} \bar{\pi}^n_j < \epsilon$ implies $p^n_{i+1} = 0$.

Proof. Define $i^n := \max \{ i \in N : c^n_i < c^n_{i+1} \text{ for some } c \in K(\mu^n) \}$. Below, we show that there is $\epsilon > 0$ such that $\sum_{j \geq i^n} \bar{\pi}^n_j \geq \epsilon$ for $n$ sufficiently large. It follows that for any $j^n$ such that $\sum_{j \geq j^n} \bar{\pi}^n_j < \epsilon$ and any $c^n \in K(\mu^n)$, $c^n_j = c^n_{j^n+1}$. By Lemma 20 it follows that $p^n_{j^n+1} = 0$ and, by Theorem 2, $p^n_j = 0$ for all $j \geq j^n + 1$, establishing the lemma.

Thus, it remains to show that $\sum_{j \geq i^n} \bar{\pi}^n_j \geq \epsilon$ for some $\epsilon > 0$. First, fix $c^n \in K(\mu^n)$. Note that there exists $\delta' > 0$ such that $c^n_i \geq \delta'$ for large $n$. This is true, since, by monotonicity, $\lim c^n_i = 0$ implies $\lim c^n_i = 0$ for all $i < i^n$ and $U^n(c^n) = -\infty$ (since $\rho > 1$). Define the consumption plan $\hat{c}^n$ such that

$$\hat{c}^n_j = \begin{cases} c^n_i & \text{if } j \leq i^n \\ c^n_{i^n} & \text{if } j \geq i^n \end{cases}$$

Note that $\hat{c}^n \in B_{k-1}(p^n, w^n)$ and, therefore, Lemma 13 implies that $U(c^n) - U(\hat{c}^n) \geq \delta$ for $n$ large. Hence,

$$\delta \leq U^n(c^n) - U^n(\hat{c}^n) = \sum_{j \geq i^n} \bar{\pi}^n_j (u(c^n_j) - u(\hat{c}^n_j)) \leq \sum_{j \geq i^n} \bar{\pi}^n_j (u(c^n_j) - u(\delta')) \leq -u(\delta') \sum_{j \geq i^n} \bar{\pi}^n_j$$

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and let $\epsilon = -\delta / u(\delta')$. \hfill \Box

**Lemma 23.** If $\{i^n\}$ satisfies $\lim s^n_{i^n} = b$ then $\lim (p^n_i / \pi^n_i) = 0$

*Proof.* If $\rho > 1$ then Lemma 22 implies the result. Thus, assume $\rho \leq 1$ and note that since $\lim_{z \to \infty} u(z) = \infty$ it follows that $p^n_i > 0$ for all $i$. First, we prove the following claim.

**Claim:** If $\{i^n\}$ satisfies $\lim s^n_{i^n} = b$ then $\lim \sum_{i \geq i^n} p^n_i / \sum_{i \geq i^n} \pi^n_i = 0$.

*Proof of Claim:* By Lemma 20 there is $c^n \in K(\mu^n)$ such that $c^n_{i^n} - 1 < c^n_{i^n}$. Let $c^n$ be such a consumption plan and let $A^n = \{i^n, \ldots, j^n\}$ be such that $c^n_i = c^n_{i^n}$ if and only if $i \in A^n$. Let $\beta^n := \sum_{i \in A^n} \pi^n_i$ and let $\bar{p}^n := \sum_{i \in A^n} p^n_i$. Let $\hat{c}^n \in B_{k-1}^n(p^n, w^n)$ be the following consumption plan:

$$
\hat{c}^n_i = \begin{cases} 
  c^n_{i^n} - 1 & \text{if } i \in A^n \\
  c^n_i & \text{otherwise}.
\end{cases}
$$

Note that there is $\delta' > 0$ such that $c^n_{i^n - 1} \geq \delta' > 0$ for large $n$. Otherwise, by monotonicity, $c^n_i \to 0$ for all $i \leq i^n - 1$ along some subsequence. Since $\sum_{i \leq i^n - 1} \pi^n_i \to 1$ this contradicts optimality of $c^n$.

By Lemma 13 there is $\delta > 0$ such that $U^n(c^n) - U^n(\hat{c}^n) \geq \delta$ for large $n$. This, in turn, implies $\sum_{i \in A^n} \pi^n_i (u(c^n_i) - u(\delta')) \geq \sum_{i \in A^n} \pi^n_i (u(c^n_i) - u(\hat{c}^n_i)) \geq \delta$. Note that $c^n_j \leq w^n / \bar{p}^n$; hence

$$
\sum_{j \in A^n} \pi^n_j \left( u \left( \frac{w^n}{\bar{p}^n} \right) - u(\delta') \right) \geq \delta
$$

Since $w^n \leq b$ we have

$$
\bar{p}^n \leq \frac{b}{u^{-1} \left( \frac{\delta + \beta^n u(x)}{\beta^n} \right)}
$$

Substituting for $u$ and setting $\delta^n := \delta + \beta^n u(x)$ we obtain

$$
\bar{p}^n \leq \begin{cases} 
  \frac{b}{((1 - \rho) (\delta^n / \beta^n))^{1-\rho}} & \text{if } \rho \in (0, 1) \\
  \frac{b}{\beta^{\rho}} & \text{if } \rho = 1
\end{cases}
$$

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Since \( \lim \beta^n = 0 \) and \( \lim \delta^n = \delta > 0 \), it follows that \( \lim \bar{p}^n/\beta^n = 0 \). Since prices are monotone (Theorem 2), the claim follows.

Suppose, contrary to the assertion of the lemma, there is \( \bar{v} \) satisfying the hypothesis of the Lemma such that \( \bar{p}^n/\pi^n \geq m > 0 \) for all \( n \). Let \( \beta^n := \sum_{i \geq m} \pi^n_i \) and let \( j^n := \max\{ j \in \mathbb{N} : \sum_{i \geq j} \pi^n_i \geq 2\beta^n \} \). Note that \( \sum_{i \geq j} \pi^n_i < 2\beta^n + \pi^n_j \). Since \( \pi^n_j \to 0 \), it follows that \( \sum_{i \geq j} \pi^n_i \to 0 \). Therefore, \( \{j^n\} \) satisfies the conditions of claim above and \( \lim \sum_{i \geq j} p^n_i/\sum_{i \geq j} \pi^n_i = 0 \). But \( p^n_i/\pi^n_i \geq m \) for all \( j^n \leq i \leq \bar{v} \) and therefore

\[
\lim \frac{\sum_{i \geq j} p^n_i}{\sum_{i \geq j} \pi^n_i} \geq \frac{m\beta^n}{2\beta^n}
\]

yielding the desired contradiction. \( \square \)

**Lemma 24.** Assume \( \rho \geq 1 \) and let \( \{v^n\} \) be such that \( \lim s^n_{v^n} = a \) then \( \lim \frac{p^n_{v^n}}{\pi^n_{v^n}} = \infty \).

**Proof.** Assume \( \{v^n\} \) satisfies the hypothesis of the lemma. If \( p^n_{v^n+1} = 0 \) along any subsequence then by monotonicity \( p^n_j = 0 \) for all \( j \geq v^n + 1 \). Since \( \lim \sum_{j \leq v^n} p^n_j = 0 \) and \( u \) is unbounded, it follows that \( U^n_i > u(b) \) violating feasibility. Thus, \( p^n_{v^n+1} > 0 \) for all \( n \) sufficiently large. By Lemma 20 this, in turn, implies that there is \( c^n \in K(\mu^n) \) such that \( c^n_{v^n} < c^n_{v^n+1} \).

For the remainder of this proof, fix \( c^n \in K(\mu^n) \) for each \( n \) with \( c^n_{v^n} < c^n_{v^n+1} \). From Lemma 13 it follows that \( c^n \) has exactly \( k \) distinct consumption levels. Let \( (x^n_1, \ldots, x^n_k) \) be those \( k \) consumption levels in increasing order \( (x^n_1 < x^n_2 < \ldots < x^n_k) \); let \( A^n_i = \{ i \in \mathbb{N} : c^n_i = x^n_i \} \).

Let \( \bar{p}^n_i = \sum_{l \in A^n_i} p^n_l \) and let \( \beta^n_i := \sum_{l \in A^n_i} \pi^n_l \). Since \( c^n \) is monotone, there are \( v^n_l, l = 0, \ldots, k \) with \( v^n_k = n, v^n_0 = 0 \) such that \( A^n_i = \{ i \in \mathbb{N} : v^n_l < i \leq v^n_{l+1} \} \). By construction, \( v^n = v^n_l \) for some \( l = 1, \ldots, k - 1 \). Let \( l^n \) be this value. Then, \( c^n_{v^n} = x^n_{l^n} \) for all \( i \in A^n_{l^n} \).

By construction, \( \beta^n_{l^n} \to 0 \). Adapting the argument given in the proof of Lemma 21 (with \( v^n_{l^n} \) taking the place of \( i = 1 \) and \( \bar{p}^n_{l^n} \) replacing \( \bar{p}^n_1 \)) it follows that there exists \( \delta > 0 \) such that \( \bar{p}^n_{l^n} \geq \delta \) for \( n \) large enough. Optimality of \( x^n_{l^n} \) implies that for \( \theta = 1/\rho \)

\[
x^n_{l^n} = \left( \frac{\beta^n_{l^n}}{\bar{p}^n_{l^n}} \right)^\theta \frac{u^n}{\left( \sum_{l=1}^k (\beta^n_l)^\theta (\bar{p}^n_l)^{1-\theta} \right)}
\]
If $\beta^n_l \overline{p}_l^n$ converges to zero for all $l$ then $U^n_\pi > u(b)$ for large $n$. Thus, it follows that there is $l$ and $\delta' > 0$ such that $\beta^n_l \overline{p}_l^n > \delta'$ for $n$ large and, therefore, $x^n_l$ is bounded from zero, so $\sum_{l=1}^{K} \frac{w^n}{(\overline{p}_l^n)^{1-\theta}}$ is bounded. The above stated optimality condition then implies that $x^n_l \to 0$.

Assume that $\frac{p^n_i}{\pi^n_i} \leq K < \infty$ for all $n$. To prove the lemma, we will show that $c^n$ cannot be an optimal consumption plan for $n$ sufficiently large. Let $\ell$ be as in the paragraph above. The budget constraint implies that $x^n_i$ stays bounded away from infinity. Let $\hat{c}_i^n$ be the following consumption plan:

$$
\hat{c}_i^n = \begin{cases} 
c_i^n & \text{if } i \notin A^n_l \cup \{i^n\} \\
\frac{x^n_i \overline{p}_l^n}{\overline{p}_l^n + K \pi^n_i} & \text{if } i \in A^n_l \cup \{i^n\}
\end{cases}
$$

By construction, $\hat{c}_i^n \in B_k(p^n, w^n)$. Consider a subsequence such that $x^n_i \to x_i > 0$, $\overline{p}_l^n \to \overline{p}_l > 0$, and $\overline{p}_l^n \to \overline{p}_l > 0$. Then,

$$
\lim \frac{U^n(\hat{c}_i^n) - U^n(c^n)}{\pi^n_i} = \lim [u(x^n_i) - u(x^n_i)] + \lim \frac{\beta^n_l}{\pi^n_i} \left[ u \left( \frac{x^n_i \overline{p}_l^n}{\overline{p}_l^n + K \pi^n_i} \right) - u(x^n_i) \right] \tag{13}
$$

It is straightforward to verify that the second term on right hand side of (13) converges to $-\frac{\beta_i}{\overline{p}_l} x_i u'(x_i) K$. Note that $\lim [u(x^n_i) - u(x^n_i)] = u(x_i) - \lim u(x^n_i) = +\infty$ since $\rho \leq 1$ and therefore the expression on the right hand side of 13 converges to $+\infty$. Hence $U^n(\hat{c}_i^n) > U^n(c^n)$ for large $n$, as desired.

**Proof of Theorem 3:** Lemma 21 proves that $\lim p^n_l > 0$. Next, we show that $\lim \Pr(\kappa^n_i > K) > 0$ for all $K$ if $\rho \geq 1$. Without loss of generality, choose $K > 1$ and let $j^n := \min \{ j \in N | \kappa^n_j \leq K \}$. Note that $j^n$ exists because $\sum_{j=1}^{N} p^n_j = \sum_{j=1}^{N} \pi^n_j$. Lemma 24 implies that $j^n > 1$ for $n$ sufficiently large. Then, by Theorem 2, $\{ i \in N : \kappa^n_i > K \} = \{ 1, \ldots, j^n - 1 \}$. From Lemma 24 it follows that $\lim s^n_{j^n} > a$ and, therefore, $\lim s^n_{j^n - 1} > a$. Since the limit endowment has a strictly positive density on $[a, b]$ the result follows.

Next, we show that $\lim \Pr(\kappa^n < \epsilon) > 0$ for all $\epsilon > 0$. Without loss of generality, choose $\epsilon < 1$ and let $j^n := \max \{ j \in N | \kappa^n_j \geq \epsilon \}$. Note that $j^n$ exists because $\sum_{j=1}^{N} p^n_j = \sum_{j=1}^{N} \pi^n_j$. Lemma 23 implies that $j^n < n$ for large $n$. Theorem 2 then implies that $\{ i \in N : \kappa^n_i < \epsilon \} = \{ j^n + 1, \ldots, n \}$. From Lemma 23 it follows that $\lim \sup s^n_{j^n} < b$ and, therefore, $\lim \sup s^n_{j^n+1} < b$. Since the limit
endowment has a strictly positive density on \([a, b]\) the result follows. Lemma 22 proves the final part of the Theorem. 

A.6 Limit Price

Fix a convergent sequence of economies \(E^n = (u, k, \pi^n, s^n)\). Let \(\mu^n\) be a CCE allocation for \(E^n\), \(K(\mu^n)\) be the set of consumption plans in its support. Let \(P^n\) a sequence of CCE cumulative price of \(E^n\). For any \(x \in [0, 1]\), let \(\Pi^n(x) = \sum_{i \leq x} \pi^n_i\) and let \(I(x) = \max\{0, \min\{x, 1\}\}\).

**Lemma 25.** \(E^n\) is convergent implies \(\lim \Pi^n = I\).

**Proof.**Straightforward and omitted. 

**Definition 9.** If \(H, H^n \in \mathcal{H}\) and \(H(x) = \lim H^n(x)\) at every continuity point \(x\) of \(H\), we say that \(H\) is a weak limit of \(H^n\).

**Lemma 26.** \(H\) is a weak limit of \(H^n\), \(\lim x_n = x\) and \(H\) is continuous at \(x\) implies \(\lim H^n(x_n) = H(x)\).

**Proof.** For every \(\epsilon > 0\), there exists \(\delta > 0\) such that \(H(x - \delta) > H(x) - \epsilon\). Since \(H\) is nondecreasing, we can choose \(\delta\) so that \(H\) is continuous at \(x - \delta\). Hence, there exists \(\bar{n}\) such that \(H^n(x - \delta) > H(x - \delta) - \epsilon\) and \(x_n \geq x - \delta\) for all \(n > \bar{n}\). It follows that \(H^n(x_n) > H(x) - 2\epsilon\) for all \(n > \bar{n}\). Together with a symmetric argument, this yields \(\hat{n}\) such that \(H(x) + 2\epsilon > H^n(x_n) > H(x) - 2\epsilon\) for all \(n > \hat{n}\). 

**Lemma 27.** \(P^n\) has a subsequence that has a weak limit. The restriction of any such weak limit to \([0, 1]\) is continuous and concave.

**Proof.** Since \(\mathcal{H}\) is a tight family, Helly selection theorem (Theorem 25.9 of Billingsley, 1995) implies that there exists \(P \in \mathcal{H}\) and a subsequence \(P^{nm}\) such that \(P\) is the weak limit of \(P^{nm}\). To simplify the notation, we assume that this sequence is \(P^n\) itself. Hence, the remainder of this proof, we assume \(P^n\) is the cumulative of some CCE price \(p^n\) for the economy \(E^n\) and \(P = \lim P^n\). To conclude the proof, we will show that \(P\) has all the desired properties.
First, we prove that the restriction of $P$ to the unit interval is concave: fix $0 \leq x_1 < x_2 \leq 1$ and $\lambda \in (0,1)$; let $x_3 := \lambda x_1 + (1-\lambda)x_2$. First, assume $x_i$, for $i = 1, 2, 3$ are all continuity points of $P$. Let $z^n_j = \Pi^n(x_j)$ for $j = 1, 2, 3$ so that $f^n_j = f^n_{x_j}$. Since $p^n / \pi^n$ is monotone,

$$\frac{P^n(z^n_3) - P^n(z^n_1)}{z^n_3 - z^n_1} \geq \frac{P^n(z^n_2) - P^n(z^n_3)}{z^n_2 - z^n_3}$$

and therefore,

$$P^n(z^n_3) \geq \frac{z^n_2 - z^n_3}{z^n_2 - z^n_1} P^n(z^n_1) + \frac{z^n_3 - z^n_1}{z^n_2 - z^n_1} P^n(z^n_2).$$

Since $E^n$ is convergent $z^n_i$ converges to $x_i$ (Lemma 25) and hence $\frac{z^n_i}{z^n_{i+1}}$ converges to $\lambda$; by Lemma 26, $P^n(z^n_j)$ converges to $P(x_j)$ for $j = 1, 2, 3$. Therefore, $P(x_3) = \lim P^n(z^n_3) \geq \lambda \lim P^n(z^n_1) + (1-\lambda) \lim P^n(z^n_2) = \lambda P(x_1) + (1-\lambda)P(x_2)$ as desired.

Since $P$ is continuous except possibly on a countable set and right-continuous, if $P(x_3) < \lambda P(x_1) + (1-\lambda)P(x_2)$ for any $x_1, x_2, x_3 = \lambda x_1 + (1-\lambda)x_2$, we can find $y_j > x_j$ close to $x_j$ such that $y_j$’s are continuity points of $P$, $y_3 = \lambda y_1 + (1-\lambda)y_2$, and $P(y_3) < \lambda P(y_1) + (1-\lambda)P(y_2)$, contradicting the above argument.

Since the restriction of $P$ to the unit interval is concave, it is continuous on $(0,1)$. Since $P$ is right-continuous, it is continuous at 0. Since $P$ is nondecreasing, $P(x) \leq P(1)$ for all $x \in [0,1)$; thus, $\lim_{x \uparrow 1} P(x) \leq 1$. On the other hand, by concavity, $P(x) \geq (1-x)P(0) + xP(1) \geq x$, so $\lim_{x \uparrow 1} P(x) \geq 1$ proving the continuity of $P$ at 1. Hence $P$ is continuous on $[0,1]$. □

**Proof of Lemma 2** By Lemma 27, $P^n$ has a weak limit $P$ whose restriction to $[0,1]$ is continuous and concave. Thus, $P$ is continuous on $(0,1]$ and $\lim P^n(x) = P(x)$ for all $x \in (0,1]$. Hence, $P$ is a limit CCE price $E^n$. Thus we have shown that every convergent $E^n$ has a limit price $P$ and that the restriction of $P$ to the unit interval is concave. □

**Proof of Theorem 4:** That $0 < P(0)$ follows from Lemma 21. Next, we prove that $P'(1) = 0$. For each $r = 1, 2, \ldots$, let $x(r) = \frac{r+1}{r}$. By Lemmas 25 and 27 we can choose $n(r)$ so that for $A(r) := \{ j^{n(r)}_{x(r)} + 1, \ldots, n(r) \}$

$$\frac{1 - P(x(r))}{1 - x(r)} - \frac{1 - P^{n(r)}(x(r))}{1 - \Pi^{n(r)}(x(r))} = \left| r(1 - P(x(r))) - \sum_{A(r)} \frac{P^{n(r)}_i}{\pi^{n(r)}_i} \right| \leq \frac{1}{r} \quad (14)$$

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The sequence $\{j_{x(r)}^n\}, r = 1, 2, \ldots$ satisfies the hypothesis of Lemma 23, so by Theorem 2,

$$\lim_r \frac{\sum A(r) p_{i(r)}^n}{\sum A(r) \pi_{i(r)}^n} = 0$$

Inequality (14) then implies that $\lim_r r(1 - P\left(\frac{r-1}{r}\right)) = 0$ and, since $P$ is monotone, $P'(1) = 0$.

Next, we prove that $P'(0) = \infty$ if $\rho \geq 1$. For $r = 1, 2, \ldots$, let $y(r) = \frac{2}{r}, z(r) = \frac{1}{r}$. By Lemmas 25 and 27 we can choose $n(r)$ so that for $A(r) = \left\{j_{z(r)}^n, j_{y(r)}^n, \ldots, j_{z(r)}^n, j_{y(r)}^n\right\}$

$$\left| r(P(y(r)) - P(z(r))) - \frac{\sum A(r) p_{i(r)}^n}{\sum A(r) \pi_{i(r)}^n} \right| \leq 1/r$$

(15)

Since $P$ is concave, $\frac{P(\epsilon) - P(0)}{\epsilon} \geq r(P(z(r)) - P(y(r)))$ for $\epsilon \leq 1/r$ and hence the result follows if

$$\lim_r \frac{\sum A(r) p_{i(r)}^n}{\sum A(r) \pi_{i(r)}^n} = \infty.$$ 

The sequence $\{j_{y(r)}^n\}$ satisfies the hypothesis of Lemma 24, so by Theorem 2 the argument is complete. Finally, note that Lemma 22 implies that, for $\rho > 1$, $P(1 - \epsilon) = 1$ for some $\epsilon > 0$. □

A.7 Dynamic Economy

Let $E = (u, k, \pi, s)$ and let $E^* = (u, \beta, k, \pi, s, \Phi)$. Let $W^*(\nu) = \sum_d V(d) \cdot \nu(d)$ and let $W_k^* = \sup_{M^*(D_k)} W(\nu)$. We call $\nu \in M^*(D_k)$ a solution to the planner’s problem in the dynamic economy if $W^*(\nu) = W_k^*$. Let $W_k = \max_{M(\mathcal{C}_h)} W(\mu)$ be the value of the planner’s problem for the static economy.

Consider the static economy $E^t = (u, \pi^t, s^t)$ where $\pi_i^t = \lambda_h, s_i^t = s_{j(h)}$ for all $h \in N^t$. Let $W^t$ be the planner’s value function for this economy. Note that $d \in \mathbb{R}^n \times \mathbb{R}^{n^2} \times \ldots$. For any $\nu \in M^*(D_k)$ let $\nu_t$ be the marginal of $\nu$ on $\mathbb{R}^{n^t}$.

**Lemma 28.** The allocation $\nu$ is a solution to the planner’s problem in the dynamic economy if and only if $W^t(\nu_t) = W_k$ for all $t$.  

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Proof. If $\nu$ is a feasible allocation for the dynamic economy, then $\nu_t$ is a feasible allocation for $E'$. By definition, $W^*(\nu) = \sum_{t \geq 1} (1 - \beta)^{t-1} W_t(\nu_t)$. Then, since $F_{\nu_t} = F_\nu$, Lemma 6 implies that for any feasible $\nu$, $W^t(\nu_t) \leq W_k$ for all $t$. Therefore, to conclude the proof, it suffices to show that $W^*_k \geq W_k$. Let $\mu$ be a solution to the planner’s problem in the stationary economy, then the stationary allocation $T_3(\mu)$ is feasible for the dynamic economy and $W^*(T_3(\mu)) = W^t((T_3(\mu))_t) = W_k$ for all $t$.

**Lemma 29.** Every solution to the planner’s problem in the dynamic economy is stationary.

Proof. Suppose $\nu$ is a nonstationary solution to the planner’s problem. By Lemma 28, $W^t(\nu_t) = W_k$ and since $F_{\nu_t} = F_\nu$, we conclude that $\nu_t$ solves the planner’s problem for $E'$. If follows that $\nu_t$ is measurable and in particular, $\nu(d) > 0$ implies $d_h = d_{h'}$ for all $h, h' \in H_i$ and $i, t$. Then, since $\nu$ is not stationary, there must be $t, t', h, h', d, i$ such that $h \in N^t$ and $h' \in N^{t'}$ and $j(h) = j(h') = i$, $d_h \neq d_{h'}$ and $\nu(d) > 0$. Consider the economy $\hat{E} = (u, \hat{\pi}, \hat{s})$ where $\hat{N} = N^t \cup N^{t'}$, $s_{\hat{h}} = s_j(h)$ for all $\hat{h} \in \hat{N}$ let $\hat{\pi}_{\hat{h}} = .5\lambda_{\hat{h}}$. Define the consumption plan $\hat{c}$ for $\hat{E}$ as follows $\hat{c}_{\hat{h}} = d_{\hat{h}}$ for all $\hat{h} \in \hat{N}$. Our choice of $t, t', d$ ensures that $\hat{c}$ fails measurability.

Let $\hat{W}$ be the planner’s value function for the economy $\hat{E}$ and note that since $F_{\hat{s}} = F_{\nu}$, by Lemma 6, $\hat{W} = W_k$. Let $\nu^2$ be the marginal of $\nu$ on $N^t \times N^{t'}$ and note that $\nu^2$ is an allocation for $\hat{E}$. Then, by Lemma 28, $\hat{W}(\nu^2) = .5W^t(\nu_t) + .5W^t(\nu_{t'}) = W_k$. Therefore $\nu^2$ solves the planner’s problem in $\hat{E}$. But $\nu^2$ fails measurability since $\nu^2(\hat{c}) > 0$, contradicting Lemma 1.

**Proof of Theorem 5** Assume $(p, \mu)$ is a CCE of $E$ but $(T_2(p), T_3(\mu))$ is not a CCE of $E^*$. Hence, there exists $d \in B^*(T_2(p))$ such that $V(d) > W(\mu) = U^n(c)$ for $c \in D(p)$. Let $X = \{d_h : h \in H\}$ and for all $i \in N$, $x \in X$, let $H^i(x) = \{h \in H^i : d_h = x\}$ and

$$\chi_{ix} = \sum_{t \geq 1} \sum_{h \in H^i(x)} (1 - \beta)^{t-1} \lambda_h.$$ 

Let $\hat{N} = N \times X$, $\hat{\pi}_{ix} = \chi_{ix}$ for all $i \in N$ and $x \in X$, $\hat{s}_{ix} = s_i$ for all $i$. Define $g(ix) = i$ and note it is a $[\hat{E}, E]$-homomorphism. Therefore, by Lemma 10, $\theta_1(c) \in \hat{D}(\theta_2(p))$ whenever $c \in D(p)$. Define $\hat{c}_{ix} = x$ for all $ix \in \hat{N}$ and note that $U^n(\hat{c}) = V(d) > U(c) = U(\theta_1(c))$ for any $c \in D(p)$, contradicting the fact that $\theta_1(c) \in D(\theta_2(p))$.

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Finally, let \((q, \nu)\) be any CCE for \(E^*\). Lemma 28 implies that there is a fair solution to the planner’s problem. Standard arguments ensure that \(\nu\) must be Pareto-efficient; since all households have the same endowment \(\nu\) must also be fair. It follows that \(\nu\) must solve the planner’s problem. Then, Lemma 29 establishes that \(\nu\) is stationary. If \(T^{-1}_3(\nu)\) is not a CCE allocation for \(E\), then there exists an allocation \(\mu\) for \(E\) such that \(W(\mu) > W(T^{-1}_3(\nu))\), which implies that \(W^*(T_3(\mu)) > W(\nu)\), so \(\nu\) is not a solution to the planner’s problem, a contradiction.

**Proof of Theorem 6:** Since \(z^n\) is bounded, it follows that \(\sum_N p^n_i z^n_i \in [\gamma_1, \gamma_2]\). Theorem 6 (i) then follows from Theorem 3(i) and Theorem 6 (ii) follows from Theorem 3(ii).

**Proof of Theorem 7:** Since \(\pi^n_i \to 0\), it follows that \(e^n_{i\epsilon} = 0\) for \(n\) sufficiently large. Substituting for \(r^n\), this implies that

\[
\frac{r^n_i(e^n)}{r^n_i(e^{n\epsilon})} \geq \frac{\sum_{i \in N} p^n_i}{\sum_{i > 1} p^n_i}
\]

for \(n\) sufficiently large. Theorem 3(i) now implies the result.

**References**


