Investment Exuberance under Cross Learning

Shiyang Huang               Yao Zeng
London School of Economics  Harvard University

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Abstract

We investigate how cross-learning by firms amplifies industry-wide investment exuberance. When investing, firms learn from each other’s stock prices to gain more knowledge about a common shock, generating higher investment sensitivity to the shock. Speculators respond by putting more weight on the shock when trading, making prices even more informative about it. This spiral produces higher investment (and price) comovements in investment exuberance. Meanwhile, cross-learning generates a new externality by making other firms’ prices less informative about their idiosyncratic shocks. This externality increases in the number of firms, suggesting that more competitive industries may exhibit more inefficient investment exuberance.

KEYWORDS: cross learning, investment exuberance, systematic risks, pecuniary externality, inefficiency, competition

JEL: D62, D83, G14, G31

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1 Introduction

Industry-wide investment exuberance has been commonly observed in history, especially after the arrival of major innovations that involve large uncertainty.\(^1\) Existing theories usually focus on boom-bust cycles, over-investments and price bubbles whereas ignore the second-moment implications of investment exuberance: high systematic risks. Specifically, in exuberant industries, a firm’s real investment and asset price comove greatly with those of other firms, as well as with the market (see Rhodes-Kropf, Robinson and Viswanathan, 2005, Pastor and Veronesi, 2009, Hoberg and Phillips, 2010, Bhattacharyya and Purnanandam, 2011, Greenwood and Hanson, 2014, for recent empirical evidence in different contexts, and the herding literature, for example, Scharfstein and Stein, 1990, for early rational theories). These joint comovements are hard to be explained by fundamental shocks. More surprisingly, more competitive industries exhibit more inefficient investment exuberance with higher systematic risks, in the sense that their investment exuberance is more predictable and their firms’ real and financial performances also comove more (Hoberg and Phillips, 2010, Greenwood and Hanson, 2014). Our paper provides a new rational amplification theory that helps unify these observations of industry-wide investment exuberance.

Our theory features a natural fact, firm cross learning, meaning that firms learn from other firms’ asset prices (in addition to their own asset prices) in making investment decisions.\(^2\) Although firm cross learning has been explicitly documented by recent empirical work, Foucault and Fresard (2014), Ozoguz and Rebello (2013), and more broadly Graham and Harvey (2001),\(^3\) it is overlooked in the theoretical literature. Our model attempts to fill this gap by providing a micro-foundation for firm cross learning and more importantly exploring its aggregate implications.

Our theory builds on the burgeoning literature that highlights the feedback from asset prices to investment decisions (see Bond, Edmans and Goldstein, 2012, for a survey). Specifically, since secondary market participants may have incremental information that is unavailable to firms or primary market participants, firms or their capital providers may learn from the asset prices in the secondary markets for making investment decisions, and this in turn affects the asset prices (see Luo, 2005, Chen, Goldstein, and Jiang, 2007, Bakke and Whited, 2010, Edmans, Goldstein, and Jiang, 2012, Foucault and Fresard, 2012, among many others, for the empirical evidence). The feedback literature, however, has not explored the multi-firm context and the cross-learning

\(^1\) The most typical examples include the “railway mania” of the UK in the 1840s, the rapid development of automobiles and radio in the 1920s, and the surge of the Internet in the 1990s, among many others. Other notable examples include major financial innovations like asset-backed securities (ABS) and credit default swaps (CDS), as well as the Mississippi Scheme and the South Sea Bubble, in which market structures experienced dramatic changes.

\(^2\) This definition of cross learning differs from the observational learning literature (for example, Banerjee, 1992, Bikhchandani, Hirshleifer, and Welch, 1992), which suggests that agents may directly observe other agents’ actions and then take similar actions. We do not deny the importance of direct observational learning and spill-over in the firm investment setting, but we put it aside in order to highlight our cross learning channel, which is less understood.

\(^3\) In the survey by Graham and Harvey (2001), CFOs of firms report that they tend to rely on other firms’ prices in making capital budgeting decisions, and this in turn affects CEOs’ investment decisions.
mechanism we emphasize which help generate and amplify industry-wide investment exuberance.

![Figure 1: Self-Feedback Benchmark](image)

To address industry-wide investment exuberance and cross learning, we construct a feedback model to admit multiple firms. We postulate that firms in an industry (or an economy) have correlated investment opportunities, so that they have incentives to learn from each other’s asset prices. To clarify the idea, Figure 1 depicts the existing feedback models without this feature, even if they can literally accommodate many firms. Although these firms can learn from their own asset prices for making better investment decisions, they are essentially separated in segmented economies, in turn making other firms’ asset prices irrelevant. This is the reason that the existing feedback models usually feature one representative firm or one single asset.\(^4\)

Instead, the novelty of our work is developing a tractable multi-firm model featuring two-way cross learning of firms from other firms’ asset prices (in addition to their own), and identifying a new pecuniary externality involved. Inspired by the two-factor feedback framework introduced by Goldstein and Yang (2014a,b), we explicitly model correlated investment opportunities in our multi-firm setting. Specifically, the fundamental of each firm’s asset is subject both to a common productivity shock (an industry shock)\(^5\) and an idiosyncratic shock (a firm-specific shock). As speculators trade firms’ assets, the prices aggregate their private information about the two types of shocks. Therefore, for each firm, other firms’ asset prices become noisy but informative signals about

\(^4\)One exception is Subrahmanyam and Titman (2013), in which a private firm learns from the stock price of another public firm to make investment decision. The private firm’s investment affects the profitability of the public firm through competition, which further generates interesting macroeconomic implications. But the public firm does not invest by itself and the private firm also does not have its own asset price. Hence, their model still features the standard feedback channel as shown in Figure 1. Their formal model also admits two private firms, which introduces an additional externality in terms of investment complementarity that amplifies their feedback effect. But as the authors have claimed, the introduction of two private firms is not essential for most of their results.

\(^5\)In a broader sense, our common shock can also be interpreted as a shock to the entire economy. Hence, our model speaks to not only industry-wide investment exuberance but also more broadly economy-wide investment exuberance.
the common shock. Consequently, the firm in question uses other firms’ asset prices (and its own) to gain more knowledge about the common shock and make investment decisions accordingly, while other firms do the same. Such cross learning makes firms’ investments more sensitive to the common shock, encouraging secondary market speculators to weight information about the common shock more. This in turn makes firms’ asset prices more informative about the common shock, resulting in an even higher investment sensitivity to the common shock. Eventually, a tiny shock can be amplified significantly.\(^6\) Moreover, when one firm learns from other firms’ asset prices, it does not internalize a negative pecuniary externality where the prices become less informative about other firms’ idiosyncratic shocks due to the speculators’ endogenous over-weight on the information about the common shock (and thus under-weight on the information about the idiosyncratic shocks). This externality leads to high investment inefficiency. Interestingly, the new pecuniary externality takes effect through the informativeness rather than the level of prices. Figure 2 illustrates the idea of cross learning and contrasts it to the existing feedback models. Empirically, firm cross learning has been documented by recent studies like Foucault and Fresard (2014)\(^7\) and Ozoguz and Rebello (2013), and the magnitude is shown to be considerable, serving as a strong support to our theory.

The predictions of our model are consistent with many empirical regularities. Compared with a benchmark in which firms cannot learn from others’ asset prices, cross learning results in a higher weight of speculators on the information of the common shock in trading and firms’ higher

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\(^6\)This mechanism resembles rational herding in financial markets (see Scharfstein and Stein, 1990 for an example) and more closely the market-based two-task corporate investment setting in Aghion and Stein (2008), but we explicitly model learning from prices and our cross-learning mechanism does not rely on any form of short-termism.

\(^7\)For the purpose of developing empirical hypotheses, Foucault and Fresard (2014) build a model, featuring one-way learning: one focal firm may learn from its peer firm’s price while not the other way around, and the peer firm does not invest. That model lays out a nice foundation for their empirical analysis. However, it does not generate inefficient multi-firm investment exuberance or comovements with the common productivity shock as emphasized in our paper. The setup and mechanisms of their model are also different from ours.
investment sensitivity to the common shock.\footnote{See Peng, Xiong, and Bollerslev (2007), and more broadly Rhodes-Kropf, Robinson and Viswanathan (2005), Hoberg and Phillips (2010), Bhattacharyya and Purnanandam (2011) for empirical evidence. Complementary to another related theoretical literature that highlights investors’ higher attention to the common shock (see Veldkamp, 2006, Peng and Xiong, 2006), our work further speaks to its endogenous origin from firm cross learning as well as its real consequences on firms’ investment decisions.} We further show that under cross learning (plus learning from own prices), a firm’s investment and price comove more greatly with 1) other firms’ investments and prices, and 2) with the common productivity shock. We interpret these patterns as amplification of systematic risks. Since the cross-learning mechanism relies on observable prices in public financial markets, it is further supported by Maksimovic, Phillips and Yang (2013)’s empirical findings that systematic risks are stronger among public firms than private ones. To the best of our knowledge, our work is the first to show that firms’ two-way cross learning has significant aggregate amplification effects on investment, prices, and systematic risks.

We investigate many circumstances in which varying economic conditions amplify systematic risks through the cross-learning mechanism. First, an increasing uncertainty on the common shock, most typically associated with the introduction of major innovations, leads to more weight on the information of the common shock and amplification of systematic risks. This is consistent with the messages in Brunnermeier and Nagel (2004) and Pastor and Veronesi (2006, 2009) that systematic risks are too high to be explained by fundamental shocks during investment exuberance like the so-called Nasdaq bubble. This amplification also fits in line with the broader evidence of investment exuberance in Rhodes-Kropf, Robinson and Viswanathan (2005), Hoberg and Phillips (2010) and Bhattacharyya and Purnanandam (2011) in various contexts. Second, an improvement of capital providers’ knowledge about the common shock leads to higher systematic risks. Last, lower market liquidity or higher variances of idiosyncratic non-fundamental trading activities also lead to higher systematic risks. These empirical regularities have frequently been ascribed to separate behavioral explanations in the previous literature, whereas our work provides a unified rational explanation.

Furthermore, our framework offers a novel perspective to look at the relationship between inefficient investment exuberance and industrial competition. This relationship is particularly helpful to separate aggregate effects specific to cross learning from other potential mechanisms. Due to the unaligned interests of firms and speculators in feedback and the new pecuniary externality associated with cross learning, the investment exuberance is inefficient. In particular, we show that, when the number of firms in an industry increases, that is, when the industry potentially becomes more competitive, the extent of cross learning becomes stronger, leading to more severe pecuniary externalities. This suggests a rationale for the puzzling facts identified in Hoberg and Phillips (2010) and Greenwood and Hanson (2014) that investment exuberance is more predictable and firms’ investment (and price) comovements are greater in more competitive industries. According to Hoberg and Phillips (2010), no single existing theory can accommodate their findings. Our cross-learning mechanism with the new pecuniary externality implies that firms and investors in
more competitive industries are more likely to overweight common shock whereas to underweight idiosyncratic shocks, leading to more inefficient investment exuberance with higher systematic risks. Such industry-level heterogeneity helps differentiate the cross-learning mechanism from other potential ones that may also generate investment exuberance. In this regard, Ozoguz and Rebello (2013) have identified that firms in more competitive industries have higher investment sensitivity to stock prices of their peers, which directly supports our predictions.

Fundamentally, the amplification effect of firm cross learning stems from two-way endogenous strategic complementarities across the primary and secondary markets, which are absent in the previous literature. Unlike the existing literature involving complementarities in financial markets (see Veldkamp, 2011, for an extensive review), our mechanism does not posit any exogenous complementarity in actions or externality but a well-documented natural fact that firms learn from their own and other firms’ asset prices.

**Related Literature.** Our framework contributes to the burgeoning feedback theories (see Bond, Edmans and Goldstein, 2012, for a survey). Most related to the present approach are Goldstein, Ozdenoren and Yuan (2013), Goldstein and Yang (2014a,b), and Sockin and Xiong (2014a,b), all of which highlight the feedback from secondary market speculators’ information aggregation to primary market capital providers’ scale-varying investment in various contexts. In particular, pioneering the two-factor feedback framework, Goldstein and Yang (2014a) uncover new relationship between market efficiency and real efficiency, and Goldstein and Yang (2014b) deliver new implications regarding the effectiveness of disclosure policies. Our contribution is to develop a two-way cross-learning framework with multiple firms, which generates the new pecuniary externality and various aggregate implications regarding inefficient investment exuberance and competition. Methodologically, our approach is highly tractable despite the complexity of the equilibrium, offering a framework for future research involving market feedback with multiple firms.

Our work also contributes to the literature of rational models on investment exuberance by exploring new implications. In particular, this literature has been paying increasing attention to the role of learning in financial markets. Other notable rational models on investment exuberance include DeMarzo, Kaniel, and Kremer (2007), featuring endogenous relative wealth concerns, Carlson, Fisher and Giammarino (2004), featuring investments as real options, and Ozdenoren and Yuan (2013), featuring contractual externalities.

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10The bubble literature is the first to provide an exploration for investment exuberance in the time series (see Brunnermeier and Oehmke, 2013, Xiong, 2013, for surveys of various models and evidence). These theories have focused on the over-investment or over-valuation of one single firm, and have often referred to behavioral aspects. The modern literature of macro-finance (see Brunnermeier, Eisenbach and Sannikov, 2013, for an extensive survey) also generates various forms of over-investment, over-borrowing, and over-lending in the time series, by highlighting agency or financial frictions. The focuses of those literatures are mostly on the time series and thus different from ours. Our model is not intended as a general dynamic theory of booms and busts.

Viswanathan (2004), which shares a similar signal extraction problem with ours, managers cannot differentiate between common misevaluation and possible idiosyncratic synergies, leading to herding and merger and acquisition exuberance. Pastor and Veronesi (2009) propose an explicit learning model, in which the uncertain productivity of a new technology is subject to learning. Learning and the ensuing technology adoption make the uncertainty from idiosyncratic to systematic, generating rational herding and investment exuberance. Our contribution to this literature is three-fold. First, our model features multiple firms and their cross learning explicitly. This allows us to study the second-moment implications of industry-wide investment exuberance, which is less understood. Second, we cast the microstructure of public asset markets explicitly, ensuring us to reflect the indispensable role of public financial markets as suggested by Maksimovic, Phillips and Yang (2013). Lastly, our model identifies a new externality regarding the use of information about common shocks and idiosyncratic shocks in making investment decisions.

Identifying the externality associated with cross learning contributes to the large pecuniary externality literature.\textsuperscript{12} The classical pecuniary externality takes effect through the level of prices: under various frictions agents do not internalize the impacts of their actions on equilibrium price levels, leading to a welfare loss. Instead in our multi-firm cross-learning framework, firms that make real investment decisions do not fully internalize the impact of cross learning on equilibrium price informativeness. This leads to a “tragedy of the commons” regarding the use of the price system as an information source. In this sense, our pecuniary externality is reminiscent of the learning externality in the early dynamic learning and herding literature (for example, Vives, 1997) that an agent, when responding to her private information, does not take into account the benefit of increased informativeness of public information in the future.\textsuperscript{13}

Our work is also related to the literature on the interaction across different asset markets, in particular the models that highlight learning. This literature has focused on speculator learning rather than firm cross learning as we model. Cespa and Foucault (2014) consider the contagion of illiquidity across segmented markets by introducing a concept of cross-asset learning among speculators. By cross-asset learning, speculators trading in one market can potentially learn from the asset price in another market, which in turn generates propagation. Goldstein and Yang (2014c) model an environment in which different speculators are informed of different fundamentals affecting one single asset. Trading on information about the two fundamentals exhibits complementarity, suggesting that greater diversity of information improves price informativeness. Our model complements those papers by focusing on the implications of firm cross learning on both real investments and asset prices, in contrast to their exchange economy setting that focuses on trading.


\textsuperscript{13}The recent study of Vives (2014) further combines the classical pecuniary externality (through the level of prices) and the learning externality associated with exogenous public information in an industrial competition context. This is different from our new pecuniary externality through endogenous price informativeness on the two shocks. Its focus is also on the strategic interaction in product markets instead of our endogenous cross learning in financial markets.
Finally, our framework is broadly related to a macroeconomic literature focusing on dispersed information and boom-bust cycles. Closely related are Angeletos, Lorenzoni and Pavan (2012) on the role of beauty contest and Amador and Weill (2010) on the crowding-out effect of exogenous public information provision to the use of private information. Compared with Angeletos, Lorenzoni and Pavan (2012), our cross-learning framework with more detailed financial market structures can generate the new second-moment implications of investment exuberance. Our externality is also different from theirs, which instead features beauty contest in signaling and higher-order uncertainties. Complementary to Amador and Weill (2010), the externality in our model derives from a different mechanism and suggests a new crowding-out effect: endogenous over-weight on the common shock crowds out the use of information about the idiosyncratic shocks. Relatedly, Liu (2014) has embedded a single-firm feedback and disclosure mechanism (in the spirit of Goldstein and Yang, 2014b,c) into a macroeconomic setting. None of the papers aforementioned have distinguished between common and idiosyncratic shocks or considered firm cross learning.

Section 2 lays out the model, featuring correlated investment opportunities and cross learning. Section 3 characterizes the cross-learning equilibrium and benchmarks it to the self-feedback case. Section 4.2 investigates important implications of cross learning with a focus on systematic risks in investment exuberance. Section 5 explores the externality and the relationship between investment inefficiency and competition. Section 6 discusses some extensions of the model. All the proofs are delegated to Appendix unless otherwise noted.

2 The Model

2.1 The Economy

Inspired by Goldstein, Ozdenoren and Yuan (2013) and Goldstein and Yang (2014a,b), we build a new multi-firm feedback model to study industry-wide investment exuberance. To focus on the second moments and inefficiency of investment exuberance, which are less understood in the literature, we set aside a full dynamic investigation of the boom-bust cycles.

We consider a continuum of 1 of firms, $i \in [0, 1)$, each having an asset traded in a secondary market. Each firm $i$’s corresponding asset market is occupied by a mass 1 of exclusive informed risk-neutral speculators and uninformed noisy traders. We index speculators for firm $i$ by $(i, j)$, with $j \in [0, 1)$. Each firm also has an exclusive capital provider $i$ who decides how much capital to provide to the firm for investment purposes.

There are three dates, $t = 0, 1, 2$. At date 0, the speculators trade in their corresponding asset market based on their private information, and the asset price aggregates their information. At date 1, the capital providers observe the asset prices of both their own firm and all the other firms.

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14Since the speculators do not have a diversification motive, our results are unaffected if we assume that they can trade all assets. In other words, market segmentation regarding speculator trading plays no roles in our model.
Having observed all the prices and received their private information, the capital providers decide the amount of capital to provide to their corresponding firms and the firms undertake investment accordingly. All the cash flows are realized at date 2.

### 2.2 Capital Providers and Investment

All the firms in the economy have an identical linear production technology: \( Q(I_i) = AF_i I_i \), where \( I_i \) is the amount of capital provided by capital provider \( i \) to firm \( i \), and \( A \) and \( F_i \) are two stochastic productivity shocks. Specifically, shock \( A \) captures an industry-wide common productivity shock, and shock \( F_i \) captures an idiosyncratic productivity shock for firm \( i \) only. Denote by \( a \) and \( f_i \) the natural logs of these shocks, and assume that they are normal and mutually independent:

\[
a \sim N(0, 1/\tau_a), \quad \text{and} \quad f_i \sim N(0, 1/\tau_f),
\]

where \( \tau_a \) and \( \tau_f \) are positive and \( i \in [0, 1) \).

The introduction of multiple firms and of the two different productivity shocks plays an important role in necessitating firm cross learning. Specifically, if the investment opportunities are uncorrelated, cross learning does not make sense. On the other hand, if the investment opportunities are perfectly correlated, all assets become identical and thus it is less interesting to consider firm cross learning. To flesh the cross-learning mechanism out, we intentionally abstract away from any possible industrial structure of the firms’ product market.\(^{15}\)

At date 1, all the capital providers choose the amount of capital \( I_i \) simultaneously in their respective primary markets. Capital provider \( i \) captures a proportion \( \kappa \in (0, 1) \) of the output \( Q(I_i) \) by providing \( I_i \), which incurs a private quadratic adjustment cost, \( C(I_i) = \frac{1}{2} c I_i^2 \).\(^{16}\) Thus, capital provider \( i \)'s problem at \( t = 1 \) is

\[
\max_{I_i} \mathbb{E} \left[ \kappa A F_i I_i - \frac{1}{2} c I_i^2 | \Gamma_i \right],
\]

where \( \Gamma_i \) is the information set of capital provider \( i \) at \( t = 1 \). It consists of the price of firm \( i \)'s own asset, \( P_i \), and those of all the other firms’ assets, denoted by the set \( \{P_{-i}\} \) for brevity, formed at date 0, as well as their private signals about the (log) productivity shocks \( a \) and \( f_i \). Specifically, we assume that each capital provider \( i \) gets a private noisy and independent signal \( s_{a,i} \) about the (log) common productivity shock \( a \) with precision \( \tau_s \), and another private noisy and independent

\(^{15}\)However, in Session 5, we extend the baseline model to admit a finite number of \( n \) firms and use the number of firms to proxy the extent of competition. We discuss the plausibility of doing this in greater detail in Session 5.

\(^{16}\)The objective function of the capital providers is inherited from \textit{Goldstein, Ozdenoren and Yuan (2013)}, where the capital providers are interpreted as banks, private managers, or large shareholders of the firms. The specification helps ensure the tractability of the model.
signal $s_{f,i}$ about its own (log) idiosyncratic productivity shock $f_i$ with precision $\tau_f$:

$$s_{a,i} = a + \varepsilon_{a,i}, \quad \text{where } \varepsilon_{a,i} \sim N(0, 1/\tau_{sa}), \quad \text{and}$$

$$s_{f,i} = f_i + \varepsilon_{f,i}, \quad \text{where } \varepsilon_{f,i} \sim N(0, 1/\tau_{sf}).$$

Hence, for capital provider $i$, the information set is $\Gamma_i = \{P_i, \{P\}_{-i}, s_{a,i}, s_{f,i}\}$.

Different from existing literature, one major novelty of our setup is to allow capital providers to learn from other firms’ asset prices as well as own firms’ prices, which we formally call cross learning. As highlighted later, although the capital providers only care about their own firms, they use the prices of other firms’ assets for making better investment decisions.

2.3 Speculators and Secondary Market Trading

At date 0, the remaining cash flow $(1 - \kappa)Q(I_i)$, as an asset, is traded in a separate competitive secondary market for each firm $i$. For firm $i$, denote the price of this asset by $P_i$. To focus on capital providers’ cross learning, we do not consider any possible monetary transfers from the secondary market to the firm, but instead highlight the information revealed in the secondary market trading.\(^{17}\) In the asset market of firm $i$, each speculator $(i,j)$ has two private and independent signals about the common shock and the respective idiosyncratic shock. Specifically, the first signal is about the common shock:

$$x_{ij} = a + \varepsilon_{x,ij}, \quad \text{where } \varepsilon_{x,ij} \sim N(0, 1/\tau_x),$$

and the second signal is about the firm-specific idiosyncratic shock:

$$y_{ij} = f_i + \varepsilon_{y,ij}, \quad \text{where } \varepsilon_{y,ij} \sim N(0, 1/\tau_y).$$

In other words, the information set of speculator $(i,j)$ is $\Gamma_{ij} = \{x_{ij}, y_{ij}\}$. During trading, their private information will be aggregated into asset prices.

Based on their private information, the speculators submit market orders (so that their information set does not consist of the asset prices),\(^ {18}\) with an additional constraint that each speculator can buy or sell up to a unit of the asset.\(^ {19}\) Formally, the speculators maximize their expected

\(^{17}\)Hence, this asset can be interpreted as either equity of the firm or a derivative on the return from the firm’s investment.

\(^{18}\)The fact that the speculators submit market orders is not essential. For any firm $i$, even if its speculators can submit limit orders, all of our results are unaffected. See Goldstein and Yang (2014b) for another exposition of this point in a similar framework.

\(^{19}\)As discussed in the survey by Brunnermeier and Oehmke (2013), the specific size of this position limit is not essential for the results, as long as speculators cannot take unlimited positions, otherwise the prices would be fully revealing. This constraint can be easily justified by their capital or borrowing constraints.
trading profit given the market price. Their problems at \( t = 0 \) are

\[
\max_{d_{ij} \in [-1,1]} d_{ij} \mathbb{E}[(1 - \kappa) AF_i I_i - P_i | \Gamma_{ij}],
\]

where \( d_{ij} \) is speculator \((i, j)\)'s demand. The aggregate demand in market \( i \) is given by \( D_i = \int_0^1 d_{ij} d_j \).

We assume that the noisy supply in asset market \( i \) takes the following form:\(^{20}\)

\[
\Delta(\zeta, \xi_i, P_i) = 1 - 2\Phi(\zeta + \xi_i - \lambda \log P_i),
\]

where

\[
\zeta \sim N(0, \tau_{\zeta}^{-1}), \text{ and } \xi_i \sim N(0, \tau_{\xi}^{-1}).
\]

We elaborate the noisy supply. \( \Phi(\cdot) \) denotes the cumulative standard normal distribution function. The first term \( \zeta \) captures a common noisy supply shock that can be viewed as industry-wide sentiment or industry-wide fund flow. The second term \( \xi_i \) captures the idiosyncratic noisy supply shock in market \( i \) that can be viewed as market-specific liquidity shocks. The presence of a common noisy supply not only makes our framework more general, but more importantly prevents the aggregate price from fully revealing the common productivity shock. Both noisy supply shocks \( \zeta \) and \( \xi_i \) are independent and also independent of other shocks in the economy. Meanwhile, \( \lambda \) in the noisy supply function captures price elasticity and can be viewed as market liquidity. When \( \lambda \) is high, the demand from speculators can be easily absorbed and thus their aggregate demand has little impact on the asset prices.

Finally, in equilibrium, the prices clear each asset market by equalizing the aggregate speculator demand to the noisy supply in each asset market \( i \):

\[
D_i = \Delta(\zeta, \xi_i, P_i).
\]

\[ \text{(2.3)} \]

\subsection*{2.4 Discussion}

Before proceeding, we discuss several crucial differences of our settings from the existing feedback literature. First, to lay out a foundation for cross learning, our model features a continuum of many firms. Modeling multiple firms and two-way cross learning imposes new technical challenges in terms of finding closed form solutions. Despite this difficulty, our model provides a tractable approach not only suited for our purpose but potentially useful for future work in other directions.

Second, built upon the multiple-firm setup, our economy features two different productivity shocks. In most existing papers, there is only one productivity shock. Goldstein and Yang (2014a,b) first introduce two factors to the cash flow in feedback models. Their two factors differ in an

\[^{20}\text{This functional form resembles those used in Hellwig, Mukherji and Tsyvinski (2006) and Dasgupta (2007).} \]
informational sense that the capital providers perfectly observe one but not the other. Instead, in the multi-firm setting, our two shocks differ in a productivity-related sense that one is common to all firms while the other is firm-specific. The presence of such two different shocks helps deliver new aggregate implications regarding firm cross learning and investment.

Finally, in contrast to the theory for hypotheses development in Foucault and Fresard (2014), our framework fully features two-way cross learning instead of one-way learning by a focal firm from its peer firm. The one-way learning channel in Foucault and Fresard (2014) gives clear predictions on how the peer firm’s stock price may affect the focal firm’s investment, but the peer firm itself does not invest or learn. Our framework with two-way cross learning as well as more detailed real and financial market structures captures the new strategic complementarities and spiral toward the common shock. This new mechanism is crucial in generating industry-wide inefficient investment exuberance, which are consistent with the empirical regularities.

3 Cross-Learning Equilibrium

3.1 Equilibrium Definition

We formally introduce the equilibrium concept. We focus on symmetric linear equilibria that are commonly used in the literature. Specifically, the speculators in market $i$ long one share of the corresponding asset when $\phi_i x_{ij} + y_{ij} > \mu_i$, and short one share otherwise, where $\phi_i$ and $\mu_i$ are two constants to be determined in equilibrium. Since agents are risk neutral and firms are symmetric in our framework, symmetry further implies that $\phi_i = \phi$ and $\mu_i = \mu$, which mean that all the speculators use symmetric trading strategies in all asset markets, and the information contents of all asset prices are also symmetric.

**Definition 1.** A (symmetric) cross-learning equilibrium is defined as a combination of a price function for each firm $i$, $P_i(a, f_i, \zeta; \xi)$: $\mathbb{R}^4 \to \mathbb{R}$, an investment policy for each capital provider $i$, $I_i(s_{a,i}, s_{f,i}, P_i, \{P_{-i}\})$: $\mathbb{R}^2 \times \mathbb{R}^\infty \to \mathbb{R}$, and a linear monotone trading strategy for each speculator $(i,j)$, $d_{ij}(x_{ij}, y_{ij}) = 1(\phi_i x_{ij} + y_{ij} > \mu_i) - 1(\phi_i x_{ij} + y_{ij} \leq \mu_i)$, such that

i) each capital provider $i$’s investment policy $I_i(s_{a,i}, s_{f,i}, P_i, \{P_{-i}\})$ solves problem (2.1),

ii) each speculator $(i,j)$’s trading strategy $d_{ij}(x_{ij}, y_{ij})$ is identical and solves problem (2.2), and

iii) market clearing condition (2.3) is satisfied for each market $i$.

3.2 Equilibrium Characterization

We characterize the equilibrium, featuring the capital providers’ cross learning. We follow a step-by-step approach to streamline the presentation.

**Step 1.** First we solve for the price functions, which help characterize the information contents of prices from the capital providers’ perspective. We have the following lemma:
Lemma 1. The speculators’ trading leads to the following equilibrium price of each asset $i$:

$$P_i = \exp \left( \frac{\phi_i}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} a + \frac{1}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} f_i + \frac{\zeta + \xi_i}{\lambda} - \frac{\mu_i}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} \right). \tag{3.1}$$

Hence, from any capital provider $i$’s perspective, the price for its own firm $i$’s asset is equivalent to the following signal in predicting the common shock $a$:

$$z_a(P_i) = \frac{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}} \log P_i + \mu_i}{\phi_i} = a + \frac{1}{\phi_i} f_i + \frac{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}}{\phi_i} (\zeta + \xi_i), \tag{3.2}$$

and is equivalent to the following signal in predicting the corresponding idiosyncratic shock $f_i$:

$$z_f(P_i) = \lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}} \log P_i + \mu_i = f_i + \phi_i a + \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}} (\zeta + \xi_i). \tag{3.3}$$

Lemma 1 not only helps specify the information contents of a firm’s asset price to its own capital provider, but it also hints those to other firms’ capital providers. It suggests that capital providers would like to cross learn from each other when feasible. The next step formulates the idea.

Step 2. We then characterize the informational implications of cross learning. Specifically, we show that, when cross learning is feasible, that is, capital provider $i$’s information set includes both $P_i$ and $\{P_{-i}\}$, the capital provider will endogenously rely on an aggregate price as well as the target firm’s own asset price (in addition to the private signals) in inferring the two productivity shocks. To formally show the results, we impose the symmetry conditions $\phi_i = \phi$ and $\mu_i = \mu$ to conditions (3.1), (3.2) and (3.3) as we focus on symmetric equilibria, and we also define the aggregate price as

$$\bar{P} = \int_0^1 P_i \, di. \tag{21}$$

Lemma 2. For capital provider $i$, when her information set includes both $P_i$ and $\{P_{-i}\}$, these asset prices are informationally equivalent to the following two signals:

i) a signal based on the aggregate price $\bar{P}$:

$$z_a(\bar{P}) = a + \frac{\sqrt{\tau_x^{-1} \phi^2 + \tau_y^{-1}}}{\phi} \zeta \tag{3.4}$$

for predicting the common shock $a$, with the precision

$$\tau_{pa} = \frac{\tau_x \tau_y \tau_\zeta \phi^2}{\tau_x + \tau_y \phi^2}, \tag{3.5}$$

\footnote{The fact that the asset prices are equally weighted in calculating the aggregate price is inessential to their information contents. Our results carry through even if we choose arbitrarily positive weights.}
which is increasing in \( \phi \), and

ii) a signal based on the own asset price \( P_i \) as well as the aggregate price \( \overline{P} \):

\[
z_{f,i}(\overline{P}) = z_f(P_i) - \phi z_a(\overline{P}) = f_i + \sqrt{\tau_x^{-1} \phi^2 + \tau_y^{-1} \xi_i}
\]  

(3.6)

for predicting the corresponding idiosyncratic shock \( f_i \), with the precision

\[
\tau_{pf} = \frac{\tau_x \tau_y \tau_\xi}{\tau_x + \tau_y \phi^2},
\]  

(3.7)

which is decreasing in \( \phi \).

Along with Lemma 1, Lemma 2 implies that cross learning changes the feedback channel in which a capital provider uses asset prices to infer the two productivity shocks: she now uses the aggregate price \( \overline{P} \) to infer the common shock \( a \) and still uses her target firm’s own price \( P_i \) to infer the idiosyncratic shock \( f_i \). Intuitively, for capital provider \( i \), other firms’ asset prices \( \{P_{-i}\} \) are uninformative about the idiosyncratic shock \( f_i \) but informative about the common shock \( a \). Hence, when other firms’ asset prices are observable, which is natural in reality, the capital provider of the firm in question uses them to make better inference about the common shock. In particular, in a symmetric equilibrium, all prices are symmetric regarding their information contexts, so that the aggregate price is equally weighted. By the law of large numbers, \( \overline{P} \) only aggregates information about the common shock \( a \): all the information about idiosyncratic shocks and the idiosyncratic noisy supply shocks gets wiped out, while the presence of the common noisy supply shock still prevents the aggregate price from fully revealing the fundamental. This makes \( z_a(\overline{P}) \), as characterized in (3.4), the most informative signal about the common shock \( a \). Moreover, knowing \( z_a(\overline{P}) \), the capital provider also eliminates the information about the common shock and about the common noisy supply shock when she uses her own price \( P_i \) to infer the idiosyncratic shock \( f_i \), as characterized in (3.6).

**Step 3.** We then solve for the capital providers’ optimal investment policy under cross learning. This indicates the real consequences of cross learning. Lemma 2 implies that, under cross learning, capital provider \( i \) uses the new signal \( z_a(\overline{P}) \) and her private signal \( s_{a,i} \) to infer the common shock \( a \), and the new signal \( z_{f,i}(\overline{P}) \) and the private signal \( s_{f,i} \) to infer the idiosyncratic shock \( f_i \). Thus, we have the following lemma.

**Lemma 3.** Observing \( s_{a,i}, s_{f,i}, P_i \) and \( \{P_{-i}\} \), capital provider \( i \)'s optimal investment policy is

\[
I_i = \frac{\kappa}{c} \exp \left[ \frac{\tau_{sa} s_{a,i} + \tau_{sf} z_{f,i}(\overline{P})}{\tau_a + \tau_{sa} + \tau_{sf}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{sf})} + \frac{\tau_{sf} s_{f,i} + \tau_{pf} z_{f,i}(\overline{P})}{\tau_f + \tau_{sf} + \tau_{pf}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{pf})} \right].
\]  

(3.8)

The investment policy is intuitive. First, the optimal amount of investment is higher when the
share $\kappa$ of capital provider is higher while lower when the investment cost $c$ is higher. Moreover, the capital providers infer the two productivity shocks $a$ and $f_i$ independently but simultaneously in making investment decisions, reflected in the first and third terms in the parenthesis. In particular, the capital providers find it optimal to learn from both their target firms’ own asset prices as well as other firms’ prices, which are summarized in the two new signals $z_a(P)$ and $z_{f,i}(P)$. This fits quite in line with the recent empirical facts about firm and capital provider cross learning behavior (Foucault and Fresard, 2014, Ozoguz and Rebello, 2013).

According to Lemma 3, we propose the following intuitive concept of investment sensitivity to capture how the capital providers’ investment decision responds to the two productivity shocks under cross learning.

**Definition 2.** For capital providers, the investment sensitivity to the common productivity shock and the idiosyncratic productivity shock are defined as:

$$S_a(\tau_{pa}) = \frac{\tau_{sa} + \tau_{pa}}{\tau_a + \tau_{sa} + \tau_{pa}}, \text{ and } S_f(\tau_{pf}) = \frac{\tau_{sf} + \tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}},$$

respectively. We call $S_a$ the common investment sensitivity and $S_f$ the idiosyncratic investment sensitivity.

We highlight that the investment sensitivity depends not only on the capital providers’ private signals about the corresponding shock, but also on the new endogenous price signals coming from cross learning as characterized in Lemma 2. In particular, these two notions of investment sensitivity are increasing functions of $\tau_{pa}$ and $\tau_{pf}$, respectively, which are in turn affected by the speculators’ trading strategy. Hence, by Lemma 2, we have the following lemma that bridges the capital providers’ investment sensitivity and the speculators’ weight $\phi$ on the signal of the common productivity shock.

**Lemma 4.** The common investment sensitivity $S_a(\tau_{pa})$ is increasing in $\phi$ while the idiosyncratic investment sensitivity $S_f(\tau_{pf})$ is decreasing in $\phi$.

Lemma 4 is helpful because it offers an intuitive look at the real consequences of cross learning from asset prices in the economy with two productivity shocks. When the weight $\phi$ is higher, the speculators put more weight on the information about the common shock, and thus asset prices become more informative about the common shock while less informative about the idiosyncratic shocks. This in turn leads to a more sensitive investment policy in response to the common shock while less sensitive in response to the idiosyncratic shock.

**Step 4.** We finally close the model by solving for the speculators’ equilibrium trading strategy, characterized by the weight $\phi$ and the constant $\mu$. This also pins down other equilibrium outcomes since they are all functions of $\phi$. 

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For speculator \((i, j)\), her expected profit of trading given her available information is
\[
E \left[ (1 - \kappa)AF_i I_i - P_i | x_{ij}, y_{ij} \right], \tag{3.9}
\]
in which \(I_i\) and \(P_i\) have been characterized by conditions (3.8) and (3.1) respectively.

It is easy to show that speculators’ expected profit (3.9) of trading asset \(i\) can be expressed as
\[
E \left[ (1 - \kappa)AF_i I_i - P_i | x_{ij}, y_{ij} \right] = \frac{\kappa(1 - \kappa)}{c} \exp \left( \alpha_0 + \alpha_1 x_{ij} + \alpha_2 y_{ij} \right) - \exp \left( \gamma_0 + \gamma_1 x_{ij} + \gamma_2 y_{ij} \right),
\]
where \(\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \) and \(\gamma_2\) are all functions of \(\phi\):
\[
\begin{align*}
\alpha_1 &= (S_a + 1) \frac{\tau_x}{\tau_a + \tau_x}, \\
\alpha_2 &= (S_f + 1) \frac{\tau_y}{\tau_f + \tau_y}, \\
\gamma_1 &= \frac{\phi}{\lambda \sqrt{\tau_x + \tau_y \phi^2}} \frac{\tau_x}{\tau_a + \tau_x}, \\
\gamma_2 &= \frac{1}{\lambda \sqrt{\tau_x + \tau_y \phi^2}} \frac{\tau_y}{\tau_f + \tau_y}.
\end{align*}
\]

By definition, in a symmetric cross-learning equilibrium with cross learning, we have
\[
\phi = \frac{\alpha_1 - \gamma_1}{\alpha_2 - \gamma_2}.
\]
Plugging \(\alpha_1, \alpha_2, \gamma_1\) and \(\gamma_2\) yields
\[
\phi = \left( S_a + 1 - \frac{\phi}{\lambda \sqrt{\tau_x + \tau_y \phi^2}} \right) \frac{\tau_x}{\tau_a + \tau_x} - \left( S_f + 1 - \frac{1}{\lambda \sqrt{\tau_x + \tau_y \phi^2}} \right) \frac{\tau_y}{\tau_f + \tau_y}. \tag{3.10}
\]

Analyzing this equation by further plugging in \(S_a\) and \(S_f\), which are both functions of \(\phi\), we reach a unique cross-learning equilibrium, formally characterized by the following proposition.

**Proposition 1.** For a high enough noisy supply elasticity \(\lambda\), there exists a cross-learning equilibrium in which the speculators put a positive weight \(\phi > 0\) on the signal of the common productivity shock. For a high enough information precision \(\tau_y\) (of the speculators’ signal on the idiosyncratic shock), the equilibrium is unique.

Establishing the existence of a unique equilibrium is essential for our further analysis regarding investment exuberance, as it allows us to investigate that how changes in economic environment affect investments and prices through the cross-learning mechanism. When \(\phi\) is higher, the speculators put more weight on the information about the common shock in trading, encouraging all the
capital providers to respond to the common shock more sensitively through cross learning, which in turn leads to an even higher $\phi$. This new spiral gives rise to many implications in line with the empirical phenomena regarding industry-wide investment exuberance as we later explore.

The conditions to guarantee a unique cross-learning equilibrium are not only commonly seen in the feedback literature but also empirically plausible. A relatively high noisy supply elasticity $\lambda$ implies that markets are liquid enough. A relatively high information precision $\tau_y$ of the speculators’ signal on the idiosyncratic shock suggests that asset market participants understand their target firms better than the whole industry. These two conditions are appropriate in particular when we focus on the contexts leading to investment exuberance: relatively liquid markets and relatively more uncertain macroeconomic conditions.\footnote{We have numerically shown that these conditions are not restrictive. Even for reasonably small $\lambda$ and $\tau_y$, our model still features a unique cross-learning equilibrium. These numerical results are reported in Section 5. We have also explored other sufficient conditions that can guarantee a unique cross-learning equilibrium, which are less empirically irrelevant but the results are available upon request. The cross-learning mechanism itself is unaffected under other sets of sufficient conditions.}

### 3.3 Self-Feedback Benchmark

Having established the existence and uniqueness of a cross-learning equilibrium, we compare the cross-learning equilibrium to the corresponding self-feedback equilibrium in a comparable economy. This self-feedback benchmark helps understand how the presence of cross learning affects the capital providers’ investment policy and the speculators’ trading strategy, in contrast to the counterfactual where cross learning is absent. To show these effects, we again focus on the difference of the speculators’ weight $\phi$ on the signal of the common productivity shock in the two respective equilibria, as all equilibrium outcomes are functions of this weight. We still consider unique symmetric equilibrium and denote by $\phi'$ the speculators’ weight on the signal of the common productivity shock in the self-feedback benchmark.

Formally, the only difference of the benchmark economy is that each capital provider $i$ observes its own asset price $P_i$ but not other firms’ asset prices $\{P_{-i}\}$. In other words, capital provider $i$’s information set is $\Gamma_i = \{P_i, s_{a,i}, s_{f,i}\}$. We have

$$P_i = \exp \left( \frac{\phi'}{\lambda \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}}} a + \frac{1}{\lambda \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}}} f_i + \frac{\zeta + \xi_i}{\lambda \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}}} \right),$$

which is equivalent to the following two signals

$$z_a(P_i) = \frac{\lambda \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}} \log P_i + \mu_i}{\phi'} = a + \frac{1}{\phi'} f_i + \frac{\sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}}}{\phi'} (\zeta + \xi_i)$$

\footnote{We have numerically shown that these conditions are not restrictive. Even for reasonably small $\lambda$ and $\tau_y$, our model still features a unique cross-learning equilibrium. These numerical results are reported in Section 5. We have also explored other sufficient conditions that can guarantee a unique cross-learning equilibrium, which are less empirically irrelevant but the results are available upon request. The cross-learning mechanism itself is unaffected under other sets of sufficient conditions.}
in predicting the common shock \( a \) and

\[ z_f(P_i) = \lambda \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}} \log P_i + \mu_i = f_i + \phi' a \sqrt{\tau_x^{-1}(\phi')^2 + \tau_y^{-1}} (\zeta + \xi_i) \]

in predicting the corresponding idiosyncratic shock \( f_i \). The precisions of \( z_a(P_i) \) and \( z_f(P_i) \) are denoted as \( \tau_{pa} \) and \( \tau_{pf} \) where

\[ \tau_{pa} = \frac{1}{(\phi')^2 \tau_f^{-1} + \tau_x^{-1}(\phi')^2 + \tau_y^{-1}}(\zeta + \xi_i) \]

and

\[ \tau_{pf} = \frac{1}{(\phi')^2 \tau_a^{-1} + (\tau_x^{-1}(\phi')^2 + \tau_y^{-1})(\zeta + \xi_i)} \]

Following the same definition of investment sensitivity and the same analysis for the capital providers’ investment policy and the speculators’ trading strategy, we have

\[
S'_a = \frac{\tau_{sa} + \tau_{pa}}{\tau_a + \tau_{sa} + \tau_{pa}} + \frac{\tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}} \phi', \\
S'_f = \frac{\tau_{sf} + \tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}} + \frac{1}{\tau_a + \tau_{sa} + \tau_{pa}} \phi', \\
\alpha'_1 = \left( S'_a + 1 \right) \frac{\tau_x}{\tau_a + \tau_x}, \\
\alpha'_2 = \left( S'_f + 1 \right) \frac{\tau_y}{\tau_f + \tau_y}, \\
\gamma'_1 = \frac{\phi'}{\lambda \sqrt{\tau_x + \tau_y(\phi')^2}} \frac{\tau_a + \tau_x}{\tau_a + \tau_x}, \\
\gamma'_2 = \frac{1}{\lambda \sqrt{\tau_x + \tau_y(\phi')^2}} \frac{\tau_y}{\tau_f + \tau_y}.
\]

In the self-feedback equilibrium, we also have

\[ \phi' = \frac{\alpha'_1 - \gamma'_1}{\alpha'_2 - \gamma'_2} \]

to pin down the speculators’ weight on the information of the common shock. Plugging in \( \alpha'_1, \alpha'_2, \gamma'_1 \) and \( \gamma'_2 \) yields

\[ \phi' = \frac{\left( S'_a + 1 - \frac{\phi'}{\lambda \sqrt{\tau_x + \tau_y(\phi')^2}} \frac{\tau_a + \tau_x}{\tau_a + \tau_x} \right) \frac{\tau_x}{\tau_a + \tau_x}}{\left( S'_f + 1 - \frac{1}{\lambda \sqrt{\tau_x + \tau_y(\phi')^2}} \frac{\tau_y}{\tau_f + \tau_y} \right) \frac{\tau_y}{\tau_f + \tau_y}}. \tag{3.11}
\]

Therefore, we have the following proposition regarding the comparison between the cross-learning equilibrium and the corresponding self-feedback benchmark. We focus on comparable cases in which a self-feedback equilibrium and its corresponding cross-learning equilibrium are both unique.
Proposition 2. For a high enough noisy supply elasticity $\lambda$, a low enough idiosyncratic noisy supply shock precision $\tau_\xi$, and a high enough information precision $\tau_y$ (of the speculators’ signal on the idiosyncratic shock), there exists a unique self-feedback equilibrium in which speculators put a positive weight $\phi' > 0$ on the signal of the common productivity shock. In particular, $\phi' < \phi$, where $\phi$ is the speculators’ weight on the signal of the common productivity shock in the corresponding cross-learning equilibrium.

The comparison between a cross-learning equilibrium and its corresponding self-feedback equilibrium implies that, the presence of cross-learning may encourage the speculators to put a higher weight $\phi$ on the information about the common productivity shock.

The results in Proposition 2 uncover the informational and real consequences of cross learning in equilibrium. Intuitively, when the capital providers are able to cross learn from each other’s asset prices (in addition to their own firms’ prices), they indeed do so in equilibrium as other firms’ asset prices help them better infer the common shock. According to the results in Section 3.2, this makes firms’ investments relatively more correlated with the common shock. Thus, the speculators find it more profitable to put more weight on the information about the common shock. This further makes asset prices becoming relatively more informative about the common shock in guiding investment decisions, and thus the capital providers respond to the common shock even more sensitively in investing. This spiral is absent in existing feedback models, and it indeed plays an important role in amplifying industry-wide investment exuberance as we fully explore in the next section.

4 Systematic Risks

The most important implications of cross learning are on the systematic risks in industry-wide investment exuberance. This derives from the endogenous spiral between the capital providers’ investment sensitivity to the common shock and the speculators’ weight on the information about the common shock, as shown in Section 3. The new spiral and strategic complementarities help generate and amplify systematic risks in many relevant economic environments, which seem jointly puzzling otherwise. These implications also suggest differing extent of cross learning in industries with different economic conditions.

4.1 Impacts of Speculators’ Weight on Systematic Risks

It is instructive to first investigate the impact of the speculators’ weight $\phi$ (on the information of the common shock) on systematic risks. We also introduce our measures of systematic risks in investment exuberance along the way.
Definition 3. The correlation coefficients between the investments of two firms and between the asset prices of two firms are defined as:

\[ \beta_I = \frac{Cov(\log I_i, \log I_j)}{\sqrt{Var(\log I_i) \cdot Var(\log I_j)}}, \quad \text{and} \quad \beta_P = \frac{Cov(\log P_i, \log P_j)}{\sqrt{Var(\log P_i) \cdot Var(\log P_j)}}, \]

respectively. We call \( \beta_I \) the investment beta and \( \beta_P \) the price beta henceforth.

We take the investment beta \( \beta_I \) and the price beta \( \beta_P \) as two major measures of systematic risks in investment exuberance, with both real and financial respects. We have the following intuitive results on the impact of speculators’ weight \( \phi \) on the two betas. When the speculators put more weight on the information of the common shock, the capital providers’ investment sensitivity to the common shock increases, which makes their investments more correlated. This in turn encourages the speculators to put a higher weight on the common shock, resulting in a higher correlation between asset prices.

Lemma 5. Both the investment beta \( \beta_I \) and the price beta \( \beta_P \) are increasing in \( \phi \) when \( \phi > 0 \).

Similarly, we also look at the correlations between investment and the two productivity shocks, respectively. As a complement to Definition 2 of investment sensitivity and the associated Lemma 4, the following definition shoots a closer look at the equilibrium investments’ correlation with the two shocks.

Definition 4. The correlation coefficient between investment and the common productivity shock, and between investment and the idiosyncratic productivity shock are defined as:

\[ \beta_A = \frac{Cov(\log I_i, \log A)}{\sqrt{Var(\log I_i) \cdot Var(\log A)}}, \quad \text{and} \quad \beta_F = \frac{Cov(\log I_i, \log F_i)}{\sqrt{Var(\log I_i) \cdot Var(\log F_i)}}, \]

respectively. We call \( \beta_A \) the common investment correlation and \( \beta_F \) the idiosyncratic investment correlation.

Intuitively, when the speculators put more weight on the information of the common shock, both investments and prices become more correlated with the common productivity shock instead of the idiosyncratic shocks. This is because the asset prices become more informative in predicting the common shock but less informative in predicting the idiosyncratic shocks.

Lemma 6. The common investment correlation \( \beta_A \) is increasing in \( \phi \) while the idiosyncratic investment correlation \( \beta_F \) is decreasing in \( \phi \) when \( \phi > 0 \).

In what follows, we focus on the investment beta \( \beta_I \) and the price beta \( \beta_P \) in exploring the comparative statics, highlighting the speculators’ endogenous weight and equilibrium systematic risks under cross learning. The investigation on the common investment correlation \( \beta_A \) and the idiosyncratic investment correlation \( \beta_F \) yields the same insights.
4.2 Endogenous Cross Learning and Systematic Risks

Having established the impacts of the speculators’ weight $\phi$ (on the information about the common shock) on systematic risks, we turn to one of the most interesting parts of the paper, which investigates how the changes of economic environments affect equilibrium systematic risks through the cross-learning mechanism. This unifies several observations regarding shock amplifications that are otherwise hard to reconcile with fundamental shocks only. Mathematically, we perform formal comparative statics of the equilibrium betas with respect to exogenous parameters. We elaborate the first comparative statics (with respect to the common uncertainty) in more detail to explore the underlying mechanism, while the other comparative statics follow the same intuition.

4.2.1 Common Uncertainty

We first focus on the effects of common uncertainty, which is captured by the prior precision $\tau_a$ of the common productivity shock. We pay greater attention to common uncertainty, because many cases of industry-wide and economy-wide investment exuberance are associated with an increasing common uncertainty in the first place. The most typical driver for an increasing common uncertainty is the arrival of all-purpose technological or financial innovations, as discussed in Brunnermeier and Nagel (2004), Pastor and Veronesi (2006, 2009), and more broadly the literature of bubbles. Highlighting the cross-learning mechanism, our predictions help deliver a new perspective to look at the impacts of innovations and the accompanying increasing common uncertainty on the systematic risks in investment exuberance.

**Lemma 7.** Increasing the common uncertainty leads to a higher weight of the speculators on the information about the common shock. Specifically, the speculators’ weight $\phi$ is decreasing in $\tau_a$.

Lemma 7 implies that an increase in the common uncertainty leads to a stronger cross-learning spiral towards the common shock. The following proposition further establishes the overall impact on the equilibrium systematic risks.

**Proposition 3.** Increasing the common uncertainty leads to both a higher investment beta and a higher price beta in equilibrium. Specifically, $\beta_I$ and $\beta_P$ are both decreasing in $\tau_a$. We further decompose the effects into two negative components (the same for $\beta_I$ and $\beta_P$):

$$\frac{d\beta(\tau_a, \phi)}{d\tau_a} = \left(\frac{\partial \beta(\tau_a, \phi)}{\partial \tau_a}\right) + \left(\frac{\partial \beta(\tau_a, \phi)}{\partial \phi}\right) \left(\frac{\partial \phi}{\partial \tau_a}\right) < 0.$$  

Proposition 3 indicates two effects contributing to the higher systematic risks associated with an increasing common uncertainty. The first is a mechanical effect that does not depend on the endogenous interaction between the capital providers and the speculators under cross learning.
Intuitively, when the common uncertainty increases, the firms’ fundamentals become more exposed to the common shock, and thus their capital providers’ investment sensitivity to the common shock increases mechanically. This immediately results in a higher comovement among firms’ investments and prices. Figure 3 illustrates this mechanical effect in a two-firm example. As pointed out in the literature, however, investment (and price) comovements during investment exuberance are hard to reconcile with fundamental shock comovements only, so a further amplification channel is needed.

Figure 3: Mechanical Effect on Systematic Risks

The second effect, the cross-learning effect, provides a new perspective to understand the amplification from fundamental shocks to large investment (and price) comovements. This effect is novel and only at play in our multi-firm cross-learning framework with two types of shocks. It reflects the new spiral between the capital providers’ investment sensitivity to the common shock and the speculators’ weight on the signal of the common shock.

Figure 4 illustrates this cross-learning effect in a two-firm example. Interestingly, this effect takes place even when only some (not all) firms in the economy perceive the increasing common uncertainty. Suppose, without loss of generality, firm 1’s capital provider perceives the increasing common uncertainty but firm 2’s does not. As seen in the upper-left panel, firm 1’s investment sensitivity to the common shock $S_{a1}$ first increases (along with a decreasing investment sensitivity to its idiosyncratic shock), leading to a higher weight $\phi_1$ on the information of the common shock by its speculators. Then, as seen in the upper-right panel, a higher $\phi_1$ results in an even higher $S_{a1}$ since firm 1 learns from its own price. More importantly, because of cross learning, firm 2’s investment sensitivity to the common shock $S_{a2}$ also increases, since firm 2 finds firm 1’s price more informative about the common shock and thus better understands the common shock. It then naturally leads to a higher weight $\phi_2$ on the information of the common shock by firm 2’s speculators, as seen in the lower-left panel. Finally, the increase of $\phi_2$ results in even higher $S_{a1}$ and $S_{a2}$ by cross learning, as seen in the lower-right panel. This spiral suggests two new strategic complementarities only under cross learning: the first is among speculators’ weight on the information about the common...
shock in each market, and the second is among different firms’ relative investment sensitivity to the common shock. With the two strategic complementarities, the spiral reinforces itself and eventually pushes the economy to a new equilibrium with much higher systematic risks.

Our predictions on systematic risks after an increasing common uncertainty are consistent with the literature (Brunnermeier and Nagel, 2004, Pastor and Veronesi, 2006, 2009) that documents the increasing systematic risks after major technological innovations, as these innovations often come with industry-wide uncertain market prospects. In particular, the cross-learning amplification effect helps understand the huge magnitude of systematic risks in these cases of investment exuberance, which is hard to be explained by fundamental shocks. These predictions also shed light on the empirical regularities on systematic risks in papers such as Rhodes-Kropf, Robinson and Viswanathan (2005) on merger and acquisitions, Hoberg and Phillips (2010) on investment waves, and Bhattacharyya and Purnanandam (2011) on financial institutions’ collective risk-taking activities.
4.2.2 Capital Providers’ Access to Information

We then turn to the capital providers’ access to private information, captured by the two precisions \( \tau_{sa} \) and \( \tau_{sf} \) regarding the two productivity shocks, respectively. Again, we have the following lemma pertaining to the speculators’ endogenous weight.

**Lemma 8.** Increasing the capital providers’ information precision on the common shock leads to a higher speculator weight on the information about the common shock, while increasing the capital providers’ information precision on the idiosyncratic shock leads to a lower weight. Specifically, the speculators’ weight \( \phi \) is increasing in \( \tau_{sa} \) while decreasing in \( \tau_{sf} \).

Lemma 8 prescribes that when the capital providers have better information on the common shock, they use that information more in investing, which in turn encourages the speculators to put more weight on the common shock. Hence, the equilibrium cross-learning spiral towards the common shock is stronger. In contrast, better information on the idiosyncratic shock pushes the cross-learning spiral towards the idiosyncratic shocks.

Moreover, similar to Proposition 3, when the overall impact of the mechanical effect and the cross-learning effect is considered, we have the following proposition.

**Proposition 4.** For the capital providers’ access to private information, we have the following results.

i) Increasing the precision on the common shock leads to a higher investment beta when the precision is not small, and always a higher price beta; specifically, \( \beta_I \) is increasing in \( \tau_{sa} \) when \( \tau_{sa} > \tau_a + \tau_x \tau_{\zeta} \) and \( \beta_P \) is always increasing in \( \tau_{sa} \).

ii) Increasing the precision on the idiosyncratic shock leads to both a lower investment beta and a lower price beta; specifically, \( \beta_I \) and \( \beta_P \) are always decreasing in \( \tau_{sf} \).

4.2.3 Liquidity Trading

We also investigate the effects of liquidity trading, captured by the market liquidity \( \lambda \) and the two precisions of noisy supplies \( \tau_{\zeta} \) and \( \tau_{\xi} \). Similarly, we have the following intuitive lemma on the speculators’ endogenous weight.

**Lemma 9.** For liquidity trading, a higher weight of the speculators on the information about the common shock results from a lower market liquidity, a lower variance of common noisy supply, or a higher variance of idiosyncratic noisy supply. Specifically, the speculators’ weight \( \phi \) is decreasing in \( \lambda \), increasing in \( \tau_{\zeta} \), and decreasing in \( \tau_{\xi} \).

The predictions along the three dimensions are all intuitive. When the market liquidity is higher, it is easier for the noisy traders to absorb speculators’ demand, making asset prices harder to aggregate information so that the cross-learning spiral towards the common shock becomes
weaker. When the variance of the common noisy supply is lower, speculators are more likely to trade upon the common productivity shock. In contrast, when the variance of the idiosyncratic noisy supply is lower, speculators are less likely to trade upon the common shock, which results in a weaker spiral towards the common shock.

These predictions are further reflected in the following proposition, speaking to the overall effects of liquidity trading on investment exuberance. Again, similar to Proposition 3, we have the mechanical effect and the cross-learning effect in the same direction.

**Proposition 5.** For liquidity trading, we have the following results.

i) A higher investment beta $\beta_I$ results from a lower market liquidity, a lower variance of common noisy supply, or a higher variance of idiosyncratic noisy supply. Specifically, $\beta_I$ is decreasing in $\lambda$, increasing in $\tau_\zeta$, and decreasing in $\tau_\xi$.

ii) A higher price beta $\beta_P$ results from a lower market liquidity, or a higher variance of idiosyncratic noisy supply. Specifically, $\beta_P$ is decreasing in $\lambda$ and decreasing in $\tau_\xi$. The effect of the variance of common noisy supply $\tau_\zeta$ on $\beta_P$ is uncertain.

5 Investment Inefficiency and Competition

An important question is how firm cross learning affects real investment efficiency. On the positive side, cross learning allows capital providers to take advantage of more information that would not be available if they were not able to observe their own and other firms’ asset prices. However, the interests between capital providers and speculators in learning the two types of shocks are not perfectly aligned. More importantly, each firm’s cross learning further creates a new pecuniary externality on other firms. These frictions result in investment inefficiency. In particular, the cross learning effect and the pecuniary externality associated with cross learning both increase in the number of firms, suggesting that more competitive industries may exhibit stronger and more inefficient investment exuberance. This industry-level heterogeneity is particularly helpful to separate aggregate effects specific to cross learning from other potential mechanisms which may also generate investment exuberance.

We proceed by two steps in evaluating how cross learning affects investment efficiency. First, we evaluate the overall investment efficiency and show that any cross-learning equilibrium always features investment inefficiency. Then we characterize the new pecuniary externality associated with cross learning for better understanding the origin of such inefficiency. By doing this, we in particular underscore the implications of competition on inefficient investment exuberance through the new pecuniary externality.
5.1 Overall Investment Efficiency

We follow Goldstein, Ozdenoren and Yuan (2013) to define investment efficiency by the ex-ante expected net benefit of the total investments by all of the firms, given that capital providers may learn from all publicly available asset prices:

**Definition 5.** The investment efficiency of the economy is defined as

\[ R = \int_0^1 R_i \, di, \]

where

\[ R_i = \mathbb{E}\left[ \mathbb{E}\left[ AF_i I_i - \frac{c}{2} I_i^2 | \Gamma_i \right] \right] \]

denotes each firm i’s ex-ante expected net benefit of investment, given its capital provider’s information set under cross learning: \( \Gamma_i = \{ P_i, \{ P_{-i}\}, s_{a,i}, s_{f,i} \} \).

We have the following proposition indicating the universal presence of investment inefficiency in a cross-learning equilibrium. We focus on unique cross-learning equilibria by assuming that the noisy supply elasticity \( \lambda \) and the information precision \( \tau_y \) (of speculators’ signal on the idiosyncratic shock) are high enough.

**Proposition 6.** There always exists a unique optimal weight \( \phi^* \geq 0 \) of the speculators on the signal of the common shock that maximizes investment efficiency. In particular, for a high enough noisy supply elasticity \( \lambda \) and a high enough information precision \( \tau_y \) (of the speculators’ signal on the idiosyncratic shock), the optimal weight is always smaller than that in the corresponding cross-learning equilibrium, i.e., \( \phi^* < \phi \).

Proposition 6 indicates that in a cross-learning equilibrium, the speculators tend to put an inefficient high weight on the signal about the common shock. This makes capital providers respond to the common shock too sensitively, leading to inefficient investment exuberance. This particular inefficiency fits quite in line with what we have observed in typical investment exuberance (for example, Rhodes-Kropf, Robinson and Viswanathan, 2005, Peng, Xiong, and Bollerslev, 2007, Hoberg and Phillips, 2010, Bhattacharyya and Purnanandam, 2011) that both primary and secondary market investors inefficiently pay too much attention to common shocks or noisy macroeconomics news while ignoring informative idiosyncratic news.

To better understand the impacts of cross learning on investment efficiency and potentially shed light on corrective policies, we perform comparative statistics of investment efficiency with respect to several economic parameters. Again, we focus on the cases in which a unique cross-learning equilibrium is guaranteed.

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23 Consistent with the literature, the private adjustment costs of the capital providers are included in the first-best welfare calculation, and they are also the only type of investment cost in the present framework. Thus, although the capital providers do not maximize firms’ total expected values, this fact does not imply any inefficiency per se.
Proposition 7. In a cross-learning equilibrium, investment efficiency is higher when the market liquidity is higher, the precision of idiosyncratic noisy supply is higher, or the capital providers’ information precision on the idiosyncratic productivity shock is higher. Specifically, \( R \) is increasing in \( \lambda, \tau_\xi, \) and \( \tau_{sf} \).

The comparative statics regarding the investment efficiency are intuitive. First, a higher market liquidity has a corrective effect on the investment efficiency. In other words, in a deeper asset market, the speculators’ trading positions can more easily be absorbed. Specifically, when an asset market is more liquid or deeper, it becomes harder for the same amount of informed trading to impact the asset price. This is beneficial in particular when cross learning is strong after the arrival of major innovations or other common news involving high uncertainty, because the inefficient impact from speculators’ overuse of information about the common shock can be better absorbed.

Importantly, this corrective effect on real investment efficiency helps justify recent regulatory concerns and practices by the SEC in limiting informed speculators’ trading positions but at the same time encouraging liquidity provision by less informed retail investors. These two are hard to be explained as approaches to correct investors’ irrationality or to sidestep limits to arbitrage. In this sense, our cross-learning mechanism does a better job in delivering policy implications than typical models featuring bubbles.

Second, increasing investment efficiency in an economy with cross learning calls for a better use of information about the idiosyncratic shocks in the economy. Any policies on financial disclosure or government communication failing to keep this point in mind may end up crowding out the idiosyncratic news and resulting in investment inefficiency. This policy implication fits broadly in line with the recent studies that speak to the dark side of financial disclosures or central bank communications (Di Maggio and Pagano, 2013, Kurlat and Veldkamp, 2013, Goldstein and Yang, 2014b). Theoretically, the endogenous overuse of information on the common shock due to multi-firm cross learning results in an inefficient crowding-out effect on the use of information on the idiosyncratic shock. Thus, it also complements the idea on the crowding-out effect of public information provision on the use of private information (see Amador and Weill, 2010).

5.2 Competition and Cross Learning

To better understand the origin of investment inefficiency, we further identify a new pecuniary externality induced by cross learning. In doing so, we extend our baseline model to admit finite number of firms. This not only allows us to underscore the efficiency change associated with different extent of cross learning, but to investigate the relationship between competition and inefficient investment exuberance which has been a well documented puzzle in recent empirical literature (see Hoberg and Phillips, 2010, Greenwood and Hanson, 2014, among others). To flesh out the
cross-learning channel, we use the number of firms as a proxy for competition\footnote{To use the number of firms to proxy competition is common in the literature, especially when the model focuses on information \citep[see][]{Vives2010}, for a survey.} and intentionally abstract away from any product market competition as that in \citet{Peress2010}.\footnote{\citet{Peress2010} offers an interesting analysis on the impacts of monopolistic competition in product markets on stock market efficiency, but does not consider feedback to real investments or cross learning as we do. He does not consider the implications on investment exuberance as well. To incorporate product market competition into our framework is possible and could make our model richer, and we leave it for future research.} Doing this also helps empirically identify the aggregate effects specific to cross learning because it predicts industry-level heterogeneity, that is, differing extent of cross learning in different industries.

We first outline the extended cross-learning framework. A major challenge in identifying the externality associated with cross learning is to deal with an information endowment effect. Specifically, when the number of firms increase, the total amount of information in the economy also increases, leading to an efficiency gain to each firm. This information endowment effect confounds the identification of externalities and thus needs to be controlled properly. To achieve this goal, our extended cross-learning framework still features a continuum of 1 of firms being able to learn from all asset prices. However, we assume that the speculators believe that each firm only observes and learns from as many as $n \geq 1$ asset prices, including its own price. This setting delivers an equilibrium weight $\phi$ (of the speculators on the information about the common shock) identical to that in a corresponding economy with $n$ finite firms operating and the speculators fully internalizing their cross learning, while keeping the total amount of information endowment invariant with $n$. Hence, we are able to distinguish the externality associated with cross learning as the number of firms increases.

We rigorously formulate the idea above as follows. We divide all the firms into $n \geq 1$ groups, a continuum of $1/n$ of firms in each group. The firms still observe and learn from all the asset prices as in the baseline model, regardless of the grouping. However, the speculators do not fully internalize firms’ cross learning. Specifically, let $i \in [0, 1/n)$ denote one firm in the first group. The speculators believe that for any $i$, the $n$ firms in the set $\{i + k/n | 0 \leq k \leq n - 1, k \in \mathbb{Z}\}$ only learn from the asset prices of each other but not from the asset prices of other firms outside the set. Figure 5 offers an illustration of the case when $n = 3$, in which the speculators believe that the three red firms ($i$, $i + 1/3$, and $i + 2/3$) cross learn only from each other and the three blue firms ($i'$, $i' + 1/3$, and $i' + 2/3$) cross learn only from each other, similar for other firm triples.

This setting has several advantages, both economically and technically. First, it casts industry competition in a straightforward way. Since the speculators are risk neutral, it looks to them as if there are exactly $n$ firms operating in the economy. Thus, the speculators’ weight in equilibrium is identical to that in a corresponding economy with exactly $n$ firms and the speculators fully internalizing their cross learning. Second, it helps identify the pecuniary externality induced by cross learning while keeping the total information endowment fixed. Especially, the efficiency change
associated with cross learning takes place only through the speculators’ endogenous weight over the two types of shocks, making it possible to distinguish that from the change of firms’ actual information endowment. Last, this setting offers a smooth transition between the baseline model with full cross learning (as \( n \) goes to infinity) and the self-feedback benchmark (as \( n \) equals to 1). This not only makes our analysis analytically tractable but also helps unify all of the results and intuitions.

We again acknowledge that we are abstracting away from any possible industrial organization of the firms’ product markets. Rather, we make use of the number of firms as a proxy for competition, which we believe is the most straightforward and relevant measure. This also allows us to underscore the cross-learning mechanism by rendering it as the only form of interaction among firms. In this sense, our model serves as a benchmark for further research that may take into account more respects of industrial competition along with firm cross learning.

We proceed to characterize the equilibrium in the extended framework and the corresponding investment efficiency. We still consider symmetric equilibria, and denote by \( \phi_n \) the speculators’ weight on the signal of the common shock when the speculators believe that each firm only learns from as many as \( n \) asset prices, including its own. For convenience, we call the associated equilibrium an \( n \)-learning equilibrium.

Formally, each capital provider \( i \) still observes its own asset price \( P_i \) and all other firms’ asset prices \( \{P_{-i}\} \). Same as before, the information content of any asset price is characterized by

\[
P_i = \exp \left( \frac{\phi_n}{\lambda \sqrt{\tau_x^{-1} \phi_n^2 + \tau_y^{-1}}} a + \frac{1}{\lambda \sqrt{\tau_x^{-1} \phi_n^2 + \tau_y^{-1}}} f_i + \frac{\zeta + \xi_i}{\lambda} - \frac{\mu_i}{\lambda \sqrt{\tau_x^{-1} \phi_n^2 + \tau_y^{-1}}} \right),
\]

equivalent to a signal

\[
z_n(P_i) = \phi_n a + f_i + \sqrt{\tau_x^{-1} \phi_n^2 + \tau_y^{-1}} (\zeta + \xi_i).
\]

However, in an \( n \)-learning equilibrium, the speculators believe that each capital provider only learns from its own price as well as the other \( n - 1 \) firms’ asset prices. Specifically, from the speculators’ perspective, due to symmetry, each capital provider \( i \) has four signals: her own private signals \( s_{a,i} \) and \( s_{f,i} \), the signal \( z_n(P_i) \) from its own asset price, and another signal \( z_n(P_{-i}) \) coming
from the other \( n - 1 \) asset prices:\footnote{When \( n = 1 \), only the first three signals are relevant and the \( n \)-learning equilibrium degenerates to a self-feedback equilibrium.}

\[
  z_n(P_{-i}) = \phi_n a + \frac{\sum_{l \neq i} f_l}{n - 1} + \sqrt{\tau_x^{-1} \phi_n^2 + \tau_y^{-1}} \left( \zeta + \frac{\sum_{l \neq i} \xi_l}{n - 1} \right).
\]

From the speculators’ perspective, capital provider \( i \) uses these four signals to infer the sum of the two (log) productivity shocks, \( a + f_i \), in making investment decisions. Concretely, the speculators believe that capital provider \( i \) updates her belief as

\[
  \mathbb{E}[a + f_i | \Gamma_i] = z' \text{Var}(z)^{-1} \text{Cov}(a + f_i, z), \tag{5.12}
\]

where \( z = [s_{a,i}, s_{f,i}, z_n(P), z_n(P_{-i})]' \). As a consequence, the speculators’ perceived investment sensitivity \( S_{an} \) to the common shock and \( S_{fn} \) to the idiosyncratic shocks are read off from the conditional expectation (5.12). Following the same approach as before in solving for the speculators’ optimal weight in trading, we finally get

\[
  \phi_n = \frac{S_{an} + 1 - \frac{\phi_n}{\lambda \sqrt{\tau_x + \tau_y \phi_n^2}}}{S_{fn} + 1 - \frac{1}{\lambda \sqrt{\tau_x + \tau_y \phi_n^2}}} \frac{\tau_y}{
\]

Clearly, the equilibrium condition (5.13), with the investment sensitivity prescribed by condition (5.12), is equivalent to that in a corresponding economy with \( n \) firms operating and speculators fully internalizing their cross learning, so is the equilibrium weight \( \phi_n \), while the expression of investment efficiency can be shown to be the same as that in Definition 5. Therefore, we have the following proposition regarding the equilibrium weight \( \phi_n \) and investment efficiency \( R_n \) in an \( n \)-learning equilibrium. We still focus on comparable cases in which all \( n \)-learning equilibria are unique.

**Proposition 8.** For a high enough noisy supply elasticity \( \lambda \), a low enough idiosyncratic noisy supply shock precision \( \tau_x \), and a high enough information precision \( \tau_y \) (of the speculators’ signal on the idiosyncratic shock),

i) for all \( n \geq 1 \), there exists a unique \( n \)-learning equilibrium in which the speculators put a positive weight \( \phi_n > 0 \) on the signal of the common productivity shock,

ii) for all \( n > 1 \), \( \phi^* < \phi' (= \phi_1) < \phi_n < \phi \), in particular, \( \phi_n \) is increasing in \( n \), where \( \phi \), \( \phi' \) are the equilibrium weights in the baseline cross-learning equilibrium and in the self-feedback equilibrium, respectively, and \( \phi^* \) is the optimal weight that maximizes investment efficiency, and

iii) for all \( n \geq 1 \), \( R < R_n < R^* \), in particular, \( R_n \) is decreasing in \( n \), where \( R \) is the investment efficiency in the baseline cross-learning equilibrium and \( R^* \) is the optimal investment efficiency.
Proposition 8 offers a clear identification of the externality and efficiency loss associated with cross learning. Under the parameters guaranteeing a unique equilibrium, when the number of firms increases, cross learning makes the speculators put an increasing weight $\phi_n$ on the signal of the common shock. Along with the established results in Section 4.2, this implies investment exuberance with higher systematic risks. Moreover, since the information endowment is controlled, an increase in $n$ leads to decreasing investment efficiency, associated with an increasing extent of cross learning.

The key to understand this is a new externality through the speculators’ weight over the two types of shocks in response to the capital providers’ cross learning. When each capital provider learns from other firms’ asset prices, she only cares about her own investment decision and wants to use other firms’ asset prices for better inferring the common shock. This makes her investment more sensitive to the common shock, which in turn encourages the speculators to put more weight on the signal of the common shock. However, she does not internalize the endogenous cost on other firms’ investment decisions because her cross learning makes asset prices endogenously less informative on other firms’ idiosyncratic productivity shocks, through the speculators’ endogenous response in terms of allocating weight over the two shocks. When there are more firms in the economy, the speculators respond more heavily to the capital providers’ cross learning and each asset price is also used by more firms, which implies a stronger externality not being internalized by each capital provider.

Along with the results in Section 4.2, the new externality associated with cross learning offers a new perspective to investigate the puzzling fact that more competitive industries exhibit more inefficient investment exuberance with higher systematic risks. This fact has recently been documented in Hoberg and Phillips (2010) and shown to be robust after many relevant controls. As they suggest, however, no single theory in the literature can accommodate their findings. More recently, Greenwood and Hanson (2014) find a similar pattern in the cargo ship industry that also applies to other industries. They estimate a behavioral theory in which firms over-extrapolate exogenous demand shocks and partially neglect the endogenous investment responses of their competitors. Our fully rational cross-learning framework helps reconcile these facts by explicitly identifying the externality associated with competition and its impacts on real investment efficiency. Relatedly, Ozoguz and Rebello (2013) have empirically identified that firms in more competitive industries adapt their investment more sensitively to stock prices of their peers, which supports our theory.

We highlight the externality that we have identified as a new pecuniary externality, taking effect through the informativeness of prices instead of price levels. In the classical pecuniary externality literature (see Stiglitz, 1982, Greenwald and Stiglitz, 1986, and Geanakoplos and Polemarchakis, 1985), agents do not internalize the impacts of their actions on equilibrium price levels, leading to a welfare loss under various frictions. In particular, the classical pecuniary externality generates welfare transfers across agents through the levels of prices. Instead in our framework, the capital
providers do not fully internalize the impacts of cross learning on equilibrium price informativeness. This leads to a typical “tragedy of the commons” regarding the use of the price system as an information source under multi-firm cross learning. This tragedy-of-the-commons observation is absent in existing single-firm feedback models. In this sense, our pecuniary externality is also reminiscent of the notion of learning externality in the earlier dynamic learning and herding literature (for example, Vives, 1997) that an agent, when responding to private information, does not take into account the benefit of increased informativeness of public information in the future. This literature, however, does not explicitly consider the role of financial markets and in particular the feedback from market prices to investments as we do.

The new pecuniary externality associated with cross learning resembles and generalizes the “mismatch channel” of feedback identified in Goldstein and Yang (2014a). Due to cross learning, the capital providers already get a relatively better understanding about the common shock, so that they will want more information about the idiosyncratic shocks and want to see their own prices more informative about them. But this goes against the speculators’ incentives in trading (and information aggregation), because the speculators still find it profitable to put considerable weight on the signal of the common shock in trading. This conflict of interest between capital providers (or firms) and speculators is generally present in the feedback literature in different forms (see the survey by Bond, Edmans and Goldstein, 2012), and Goldstein and Yang (2014a) formally identify it as the mismatch channel of feedback. Specifically, to uncover this channel, Goldstein and Yang (2014a) assume that capital providers can perfectly observe one shock but not the other and they do not consider the possibility of multi-firm cross learning. Thus, the contribution of our work is to endogenize capital providers’ information advantage on the common shock through multi-firm cross learning, and then to show that the underlying conflict of interest becomes more severe as the number of firms get larger. Therefore, our cross-learning framework nicely resembles the classical notion of pecuniary externalities, in the sense that many capital providers endogenously ignore the full impacts of their cross learning actions on equilibrium prices.

Although our framework allows for an analytical characterization, we also offer numerical examples to help illustrate the pecuniary externality and efficiency loss associated with different extent of cross learning. We set $\tau_a = \tau_f = \tau_{sa} = \tau_{sf} = \tau_x = \tau_\zeta = 1$, $\tau_y = 10$, $\tau_\xi = 0.1$, $\lambda = 2$, $\kappa = 1$, and $c = 0.5$. The left panel of Figure 6 depicts the equilibrium weight $\phi_n$ as $n$ increases as the blue solid line. When $n$ becomes larger, the weight gradually approaches that in the baseline cross-learning equilibrium, as depicted by the red dashed line. The right panel of Figure 6 depicts the log of the efficiency loss due to cross learning, measured by $\log(R^*/R_n)$. As shown in the solid blue line, the efficiency loss associated with cross learning increases in $n$, suggesting a more severe pecuniary externality as competition becomes stronger. In the baseline cross-learning equilibrium, the pecuniary externality is the strongest, as depicted by the dashed red line. These
results are robust to a very wide range of parameters when $\lambda$ and $\tau_y$ are relatively large while $\tau_\xi$ is relatively small (and thus a unique equilibrium is guaranteed), which are empirically relevant as aforementioned.

![Graph](image)

Figure 6: Competition on Cross Learning and Efficiency Loss

Admittedly, our identification of the pecuniary externality associated with cross learning does not attempt to offer a comprehensive evaluation of the merits of competition. Relatedly, the investment efficiency $R_n$ in an $n$-learning equilibrium cannot be interpreted as a direct measure of the investment efficiency in an actual competitive industry with $n$ firms. Our point is focused, however, to suggest a new perspective to look at the relationship between inefficient investment exuberance and competition, a puzzling fact recently documented and hard to be reconciled with existing theories. We admit that, despite the fact that competition increases the extent of cross learning, with new adverse implications for investment efficiency, it may well remain desirable when all other social benefits and costs of competition are taken into account.

6 Discussion and Extension

Our cross-learning framework focuses on the second moments and inefficiency of industry-wide investment exuberance, which are less understood in the literature. It is convenient to further extend the present model to generate more implications and to shed light on some other commonly observed phenomena, which offer additional empirical predictions to help differentiate cross learning from other well-understood mechanisms.

6.1 Growth Opportunities and Systematic Risks under Cross Learning

In a structural asset pricing framework, Berk, Green, and Naik (1999) first argues that firms with more growth opportunities are systematically more risky and their returns also comove more.
Papanikolaou (2011) and Kogan and Papanikolaou (2014) further propose a concept of investment-specific technology shocks and argue that growth firms are more risky because they are more exposed to such shocks. The cross-learning framework can provide an alternative to look at the systematic risks associated with growth firms or industries. In the same spirit of the relationship between competition and cross learning, the theory predicts that industries with more growth opportunities may exhibit stronger extent of cross learning and higher systematic risks.

We could adapt our baseline model by incorporating both assets in place and growth opportunities while keeping all other settings. Specifically, each firm has an asset in place and a growth option, both subject to a common shock and an idiosyncratic shock. However, the capital provider can only invest in the growth option but not in the asset in place. In this setting, a larger growth option (relative to the asset in place) implies a stronger extent of cross learning and a higher weight of the speculators on the common shock. Consequently, the common beta becomes higher, implying higher systematic risks. Intuitively, as the size of growth option becomes higher, the capital providers tend to cross learn more, making the speculators over-value information about the common shock in trading. This in turn reinforce capital providers’ cross learning, leading to a higher investment sensitivity to the common shock even if the shock to growth opportunities does not change per se.

Importantly, the cross-learning framework provides a new micro-foundation for growth firms’ higher exposure to investment-specific shocks, without assuming that such shocks are larger for growth firms per se. Consistent with the thesis of Papanikolaou (2011) and Kogan and Papanikolaou (2014), as firms grow, the uncertainty of their assets in place decreases relative to growth opportunities, and investment-specific technology shocks become more positively correlated with their average $q$, so that they become systematically more risky. In this sense, our theory helps generate additional testable predictions regarding the aggregate effects specific to cross learning, highlighting the cross-sectional heterogeneity among industries with differing growth opportunities.

6.2 Over-investment under Cross Learning

Investment exuberance usually exhibits both high systematic risks (second moment) and over-investment (first moment). Although the latter has been well addressed in the literature, our framework is easily adaptable to generate so.

We could keep all the settings in our baseline model except for introducing two different investment technologies. Specifically, each firm now has two mutually exclusive projects, one only subject to a common shock while the other only subject to an idiosyncratic shock. We call the former common project and the latter idiosyncratic project. This adapted setting is in the similar spirit of Dow, Goldstein and Guembel (2011) but enriches it with both cross learning and the firm’s debate between the common project and the idiosyncratic project.
Following the same equilibrium concept as our baseline model, a cross-learning equilibrium in this adapted setting features over-investment in the common project while under-investment in the idiosyncratic project, compared to the first best. The intuition is the same as before. When the capital providers are able to cross learn, the speculators again find it more profitable to put a higher weight on the information about the common shock. This makes the prices more informative about the common shock and thus encourages the capital providers to invest more on the common project while less on their idiosyncratic projects.

Complementary to the existing literature about over-investment, our cross-learning mechanism has two new implications. First, over-investment is more likely to happen in technologies or industries that are more sensitive to common shocks, which is reflected by the common project in our stylized model. This fits quite in line with the recent episodes of over-investment in the IT industry and housing markets. Second, which is perhaps more subtle and interesting, over-investment in the common project is always accompanied by under-investment in the idiosyncratic projects at the same time. This suggests that over-investment does not necessarily imply an inefficiently large economy scale but rather an inefficient composition of various economic activities.

6.3 Industry Momentum under Cross Learning

The contemporaneous study by Sockin and Xiong (2014b) uses a feedback model to generate return momentum in a housing cycle context. Although our model does not aim to provide a general dynamic account for boom-bust cycles, the introduction of multiple firms and the two types of shocks also help shed light on the understanding of momentum by further establishing a channel between cross learning and industry momentum.

Industry momentum, first identified by Moskowitz and Grinblatt (1999), suggests that industry portfolios also exhibit considerable short-term momentum, and it even accounts for much of the individual stock momentum. As discussed by Moskowitz and Grinblatt (1999), individual stock momentum may be explained by a number of behavioral theories focusing on investors’ information barrier or risk appetite. But there have been no formal theories that directly speak to the existence and the magnitude of industry momentum. Our framework can potentially offer a consistent rational theory for both individual stock momentum and industry momentum, instead highlighting firms’ investment activities and their cross learning.

In our benchmark three-period model, the standard definition of overall individual momentum is

\[ M_t = \text{Cov} (\log(AF_i I_i) - \log P_i, \log P_i) \, , \]
and industry momentum can be defined as

\[ \mathcal{M} = \text{Cov} \left( \log \int_0^1 AF_i I_i \, di - \log P, \log P \right). \]

It is straightforward to show that both individual stock momentum and industry momentum exist in equilibrium, and their magnitudes increase in the speculators’ weight \( \phi \) on the information about the common shock. Specifically, \( M_i \) and \( \mathcal{M} \) are always positive when the noisy supply elasticity \( \lambda \) and the information precision \( \tau_y \) (of speculators’ signal on the idiosyncratic shock) are large enough, and they increase in \( \phi \). Intuitively, when the speculators put a higher weight on the common shock, the asset prices become more informative about the common shock and thus firm investment also becomes more sensitive to the common shock. Therefore, the common shock plays a more important role in determining both the asset prices and eventual cash flow of the firms, implying both a stronger individual stock momentum and a stronger industry momentum. Moreover, according to the results regarding the relationship between cross learning and competition in Section 5, both individual stock momentum and industry momentum may be stronger in more competitive industries, also due to a stronger cross learning effect.

It is worth highlighting that our mechanism to generate individual momentum and industry momentum is fundamentally different from the prevailing explanations that highlight overconfidence (Daniel, Hirshleifer, and Subrahmanyam, 1998), sentiment (Barberis, Shleifer, and Vishny, 1998), or slow information diffusion (Hong and Stein, 1999). In those models, investors generally ignore some information contents revealed by asset prices. In contrast, in our model, the capital providers’ rational cross learning from all available asset prices plays a central role.

7 Conclusion

Firm cross learning behavior is not only empirically important but also theoretically relevant for commonly observed investment exuberance. We have developed a tractable model to admit cross learning and delivered a series of predictions regarding the second moments of investment exuberance. We have illustrated that the high systematic risks associated with investment exuberance come from a new spiral that coordinate capital providers’ investment sensitivity and speculators’ weight in trading towards the common productivity shock. Moreover, cross learning may lead to higher investment inefficiency, because capital providers do not internalize the new externality that other firms’ asset prices become less informative on their idiosyncratic productivity shocks. In more competitive industries, cross learning tends to be stronger, potentially leading to more inefficient investment exuberance with higher systematic risks. Hence, appropriate policy interventions are called for to correct the inefficiency in industry-wide investment exuberance.
8 Appendix

This appendix provides all proofs omitted above with auxiliary results.

Proof of Lemma 1. According to the equilibrium definition, any speculator \((i, j)\) longs one share of asset \(i\) when \(\phi_i x_{ij} + y_{ij} > \mu_i\) and shorts one share otherwise. Equivalently, speculator \((i, j)\) longs one share of asset \(i\) when

\[
\frac{\phi_i x_{ij} + y_{ij}}{\sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} > \frac{\mu_i - (\phi_i a + f_i)}{\sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}},
\]

and shorts one share otherwise. Thus, in asset market \(i\), all speculators’ aggregate demand is

\[
D_i = 1 - 2\Phi\left(\frac{\mu_i - (\phi_i a + f_i)}{\sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}}\right).
\]

Hence, in equilibrium, market clearing implies

\[
1 - 2\Phi\left(\frac{\mu_i - (\phi_i a + f_i)}{\sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}}\right) = 1 - 2\Phi(\zeta + \xi_i - \lambda \log P_i),
\]

which further implies that the equilibrium price in asset market \(i\) is

\[
P_i = \exp\left(\frac{\phi_i}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} a + \frac{1}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} f_i + \frac{\zeta + \xi_i}{\lambda} - \frac{\mu_i}{\lambda \sqrt{\tau_x^{-1} \phi_i^2 + \tau_y^{-1}}} \right).
\]

This concludes the proof. \(\square\)

Proof of Lemma 2. In a symmetric equilibrium, the capital providers put a same weight \(\phi\) on the information of the common shock in any asset market \(i\). Thus, by Lemma 1, for asset price \(P_i\), its equivalent signal in predicting the common shock \(a\) becomes

\[
z_a(P_i) = a + \frac{1}{\phi} f_i + \sqrt{\frac{\tau_x^{-1} \phi^2 + \tau_y^{-1}}{\phi}} (\zeta + \xi_i).
\]

Since \(f_i\) and \(\xi_i\) are both i.i.d. and have zero means, the aggregate price \(\bar{P}\) is equivalent to the following signal

\[
z_a(\bar{P}) = \int_0^1 \left(a + \frac{1}{\phi} f_i + \sqrt{\frac{\tau_x^{-1} \phi^2 + \tau_y^{-1}}{\phi}} (\zeta + \xi_i)\right) di = a + \sqrt{\frac{\tau_x^{-1} \phi^2 + \tau_y^{-1}}{\phi}} \zeta
\]

in predicting the common shock \(a\). It immediately follows the construction of the other signal \(z_{f,i}(\bar{P})\) in predicting the idiosyncratic shock.

Finally, it is easy to verify that any combination of the asset prices \(\{P_i, i \in [0, 1]\}\) cannot be more informative in predicting the two productivity shocks. This concludes the proof. \(\square\)
**Proof of Lemma 3.** From the capital providers’ problem (2.1), the first order condition is

\[
I_i = \frac{\kappa}{c} \mathbb{E}[\exp(a + f_i) | \Gamma_i] \\
= \frac{\kappa}{c} \exp \left( \mathbb{E}[a + f_i | \Gamma_i] + \frac{1}{2} \text{Var}[a + f_i | \Gamma_i] \right).
\]

By Lemma 2, we know that \( s_{a,i} \) and \( z_{a}(P) \) are only informative about the common shock \( a \) and \( s_{f,i} \) and \( z_{f,i}(P) \) are only informative about the idiosyncratic shock \( f_i \). Applying Bayesian updating immediately leads to the following optimal investment policy

\[
I_i = \frac{\kappa}{c} \exp \left[ \frac{\tau_{sa}s_{a,i} + \tau_{P}z_{a}(P)}{\tau_a + \tau_{sa} + \tau_{P}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{P})} + \frac{\tau_{sf,i} + \tau_{P}z_{f,i}(P)}{\tau_f + \tau_{sf} + \tau_{P}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{P})} \right].
\]

This concludes the proof. \( \square \)

**Proof of Lemma 4.** This is a direct application of Lemma 2 to Definition 2. \( \square \)

**Proof of Proposition 1.** We proceed step by step.

**Step 1:** Proof of the existence of the solution.

Following the equilibrium condition (3.10) for the cross-learning case, let

\[
g(\phi) = \phi - \frac{\alpha_1 - \gamma_1}{\alpha_2 - \gamma_2} = \phi - \frac{\tau_{sa}s_{a,i} + \tau_{P}z_{a}(P)}{\tau_a + \tau_{sa} + \tau_{P}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{P})} + \frac{\tau_{sf,i} + \tau_{P}z_{f,i}(P)}{\tau_f + \tau_{sf} + \tau_{P}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{P})},
\]

where \( \tau_{P} \) is given by (3.5) and \( \tau_{P} \) is given by (3.7), both being function of \( \phi \). It is easy to check that \( \lim_{\phi \to -\infty} g(\phi) < 0 \) and \( \lim_{\phi \to +\infty} g(\phi) > 0 \) by the following two equations:

\[
\lim_{\phi \to -\infty} \phi - \frac{\tau_{sa}s_{a,i} + \tau_{P}z_{a}(P)}{\tau_a + \tau_{sa} + \tau_{P}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{P})} + \frac{\tau_{sf,i} + \tau_{P}z_{f,i}(P)}{\tau_f + \tau_{sf} + \tau_{P}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{P})} = \frac{\tau_a + \tau_{sa} + \tau_{P}}{\tau_a + \tau_{sa} + \tau_{P}} \frac{\tau_f + \tau_{sf} + \tau_{P}}{\tau_f + \tau_{sf} + \tau_{P}} + 1 - \frac{\sqrt{\tau_x}}{\lambda},
\]

and

\[
\lim_{\phi \to +\infty} \phi - \frac{\tau_{sa}s_{a,i} + \tau_{P}z_{a}(P)}{\tau_a + \tau_{sa} + \tau_{P}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{P})} + \frac{\tau_{sf,i} + \tau_{P}z_{f,i}(P)}{\tau_f + \tau_{sf} + \tau_{P}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{P})} = \frac{\tau_a + \tau_{sa} + \tau_{P}}{\tau_a + \tau_{sa} + \tau_{P}} \frac{\tau_f + \tau_{sf} + \tau_{P}}{\tau_f + \tau_{sf} + \tau_{P}} + 1 - \frac{\sqrt{\tau_x}}{\lambda}.
\]

The analysis above indicates that there always exists a solution of \( \phi \) to the equilibrium condition (3.10), i.e., \( g(\phi) = 0 \), by the intermediate value theorem. Especially, when \( \lambda > 1/\sqrt{\tau_x} \), we know that

\[
f(0) = -\frac{\tau_a + \tau_{sa} + \tau_{P}}{\tau_a + \tau_{sa} + \tau_{P}} \frac{\tau_f + \tau_{sf} + \tau_{P}}{\tau_f + \tau_{sf} + \tau_{P}} + 1 - \frac{\tau_a + \tau_{sa} + \tau_{P}}{\tau_a + \tau_{sa} + \tau_{P}} \frac{\tau_f + \tau_{sf} + \tau_{P}}{\tau_f + \tau_{sf} + \tau_{P}} < 0.
\]

We conclude that there always exists a positive solution \( \phi > 0 \) as long as \( \lambda \) is large enough.

**Step 2:** Proof of the uniqueness of the solution when \( \tau_f \) is large enough.
By simple algebra, the equilibrium condition (3.10) is re-expressed as
\[
\left(\frac{\tau_{sa} + \tau_{\text{pr}}}{\tau_a + \tau_{sa} + \tau_{\text{pr}}} + 1 - \frac{\phi}{\lambda \sqrt{\tau_x^{-1} \phi^2 + \tau_y^{-1}}} \right) \frac{\tau_x}{\tau_a + \tau_x} = \phi \left(\frac{\tau_{sf} + \tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}} + 1 - \frac{1}{\lambda \sqrt{\tau_x^{-1} \phi^2 + \tau_y^{-1}}} \right) \frac{\tau_y}{\tau_f + \tau_y}.
\] (8.1)

Applying Taylor expansion to the terms in equation (8.1) with respect to \(\tau_y^{-1}\) yields:
\[
\frac{\tau_{sa} + \tau_{\text{pr}}}{\tau_a + \tau_{sa} + \tau_{\text{pr}}} = -\frac{\tau_a}{\tau_a + \tau_{sa} + \tau_{\text{pr}}} - \frac{\tau_a}{(\tau_a + \tau_{sa} + \tau_{\text{pr}})^2} \tau_{\text{c}} \phi^2 \tau_y^{-1} + o(\tau_y^{-1}),
\]
\[
\frac{\tau_{sf} + \tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}} = 1 - \frac{\tau_f}{\tau_f + \tau_{sf} + \tau_{pf}} - \frac{\tau_f}{(\tau_f + \tau_{sf} + \tau_{pf})^2} \tau_{\text{c}} \phi^2 \tau_y^{-1} + o(\tau_y^{-1}),
\]
\[
\frac{\phi}{\lambda \sqrt{\tau_x^{-1} \phi^2 + \tau_y^{-1}}} = \frac{1}{\lambda \sqrt{\tau_x}} - \frac{1}{2 \lambda \tau_x^2 \phi^2} \tau_y^{-1} + o(\tau_y^{-1}),
\]
and
\[
\frac{1}{\lambda \phi \sqrt{\tau_x^{-1}}} = \frac{1}{2 \lambda \tau_x^2 \phi^3} \tau_y^{-1} + o(\tau_y^{-1}).
\]

Plugging them back into equation (8.1), we have:
\[
\frac{\tau_x}{\tau_a + \tau_x + \tau_y} \left[ 2 - \frac{\tau_a}{\tau_a + \tau_{sa} + \tau_{\text{pr}}} - \frac{\tau_a}{(\tau_a + \tau_{sa} + \tau_{\text{pr}})^2} \tau_{\text{c}} \phi^2 \tau_y^{-1} - \frac{1}{\lambda \sqrt{\tau_x}} + \frac{1}{2 \lambda \tau_x^2 \phi^2} \tau_y^{-1} + o(\tau_y^{-1}) \right]
\]
\[
= \phi \left[ 2 - \frac{\tau_f}{\tau_f + \tau_{sf} + \tau_{pf}} - \frac{\tau_f}{(\tau_f + \tau_{sf} + \tau_{pf})^2} \tau_{\text{c}} \phi^2 \tau_y^{-1} - \frac{1}{\lambda \phi \sqrt{\tau_x}} + \frac{1}{2 \lambda \tau_x^2 \phi^3} \tau_y^{-1} + o(\tau_y^{-1}) \right],
\]
which becomes a cubic equation of \(\phi\) when \(\tau_y\) goes to infinity:
\[
\left(\frac{\tau_{sa} + \tau_x \tau_{\text{c}}}{\tau_a + \tau_{sa} + \tau_x \tau_{\text{c}}} + 1 - \frac{\sqrt{\tau_x}}{\lambda} \right) \frac{\tau_x}{\tau_a + \tau_x} = \frac{\tau_{sf} \phi^3 + \tau_{pf} \phi^2 + \tau_{\xi} \phi}{\tau_f \phi^2 + \tau_{sf} \phi^2 + \tau_{\xi} \phi} + \phi - \frac{\sqrt{\tau_x}}{\lambda}.
\] (8.2)

Note that, the left hand side of equation (8.2) does not depends on \(\phi\). Denote by \(h(\phi)\) the right hand side of (8.2), and its first order derivative with respect to \(\phi\) is given by
\[
\frac{\partial h(\phi)}{\partial \phi} = 1 - \frac{\tau_f \phi^2}{\tau_f \phi^2 + \tau_{sf} \phi^2 + \tau_{\xi} \phi} + 1 - \frac{2 \tau_f \tau_{\xi} \tau_{\phi^2}}{(\tau_f \phi^2 + \tau_{sf} \phi^2 + \tau_{\xi} \phi)^2} > 0,
\]
which indicates that the right hand side of equation (8.2) is increasing in \(\phi\) and thus we have a unique solution to equation (8.2). Therefore, since \(g(\phi)\) is a continuous function of \(\tau_y\), there always exists one unique solution to \(g(\phi) = 0\), i.e., equation (8.1), when \(\tau_y\) is large enough. This concludes the proof.

**Proof of Proposition 2.** This proof is similar to the proof of Proposition 1. In the benchmark
with respect to \( \tau \) is smaller than the left hand side of (8.4) (for the case with cross learning). By the continuity are the same, while the left hand side of (8.3) (for the benchmark case without cross learning).

Thus, as \( \xi \) goes to infinity, \( \phi \) decreases to zero and \( \lambda > \sqrt{\tau_x} \). Since the right hand side of equation (8.3) is increasing if \( \phi \), we know that there must exist one unique solution to equation (8.3).

On the other hand, when \( \tau_y \) goes to zero (and when \( \tau_y \) goes to infinity and \( \lambda > \sqrt{\tau_x} \)), equation (8.2) in the case with cross learning becomes

\[
\left( \frac{\tau_{sa} + \tau_{pa}}{\tau_a + \tau_{sa} + \tau_{pa}} + \frac{\tau_{pf}}{\tau_f + \tau_{sf} + \tau_{pf}} \phi' + 1 - \frac{\phi}{\lambda \sqrt{\tau_x - 1}} \right) \frac{\tau_x}{\tau_a + \tau_x} = 0,
\]

which further reduces to

\[
\left( \frac{\tau_{sa} + \tau_{sa}}{\tau_a + \tau_{sa} + \tau_{sa}} + 1 - \frac{\sqrt{\tau_x}}{\lambda} \right) \frac{\tau_x}{\tau_a + \tau_x} = \phi \left( \frac{\tau_{sf} + \tau_{sf}}{\tau_f + \tau_{sf} + \tau_{sf}} + 1 - \frac{\sqrt{\tau_x}}{\lambda \phi} \right), \tag{8.3}
\]

when \( \tau_y \) goes to infinity, \( \tau_y \) goes to zero and \( \lambda > \sqrt{\tau_x} \). Since the right hand side of equation (8.3) is increasing inf \( \phi \), we know that there must exist one unique solution to equation (8.3).

Proof of Lemma 5. We first consider the investment beta \( \beta_f \). Recall the investment policy (3.8), we have

\[
\log I_i = \frac{\tau_{sa} s_{a,i} + \tau_{par} z_{a,i}(P)}{\tau_a + \tau_{sa} + \tau_{par}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{par})} + \frac{\tau_{f} s_{f,i} + \tau_{par} z_{f,i}(P)}{\tau_f + \tau_{sf} + \tau_{par}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{par})}.
\]

Following the definition of \( \beta_f \) and after some algebra, we reach

\[
\beta_f = \frac{S_a/\tau_a - \tau_{sa} + \tau_{par}}{S_a/\tau_a + S_f/\tau_f}.
\]

To simplify the analysis, let \( g_1 = S_a/\tau_a, g_2 = \tau_{sa} / (\tau_a + \tau_{sa} + \tau_{par})^2, \) and \( g_3 = S_f/\tau_f \). By Lemma 4, it is straightforward that \( g_1 \) is increasing in \( \phi \) and both \( g_2 \) and \( g_3 \) are decreasing in \( \phi \). Thus, as \( \phi > 0 \), we also have that \( g_1 \) is increasing in \( \phi^2 \) and both \( g_2 \) and \( g_3 \) are decreasing in \( \phi^2 \).
Furthermore, we have
\[
\frac{\partial \beta_I}{\partial \phi^2} = \frac{g'_1 - g'_2 - (g_1 - g_2)(g'_1 + g'_3)}{g_1 + g_3} - \frac{(g_1 + g_3)^2}{[(g'_1 - g'_2)g_1 - (g_1 - g_2)g'_1] + (g'_1 - g'_2)g_3 - (g_1 - g_2)g'_3},
\]
where \(g'_1, g'_2\) and \(g'_3\) stands for the first order derivative with respect to \(\phi^2\).
Since we know that \(g'_1 - g'_2 > g'_1\) (due to the fact that \(g_2\) is decreasing in \(\phi^2\)) and \(g_1 > g_1 - g_2\), we have \((g'_1 - g'_2)g_1 - (g_1 - g_2)g'_1 > 0\). Meanwhile, we have \(g'_1 > 0\), \(g'_2 < 0\), and \(g_3 > 0\), so that \((g'_1 - g'_2)g_3 > 0\). Lastly, since \(g_1 > g_2\) and \(g_3 < 0\), we also know that \((g_1 - g_2)g'_3 < 0\). Therefore, we conclude that \(\frac{\partial \beta_I}{\partial \phi^2} > 0\), i.e., \(\beta_I\) is an increasing function of \(\phi\) when \(\phi > 0\).

We then consider the price beta \(\beta_P\). Recall the pricing function (3.1), we have
\[
\log P_i = \phi \frac{\phi^2}{\lambda\sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} a + \frac{1}{\lambda\sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} f_i + \frac{\zeta + \xi_i}{\lambda \sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}}.
\]
Following the definition of \(\beta_P\) and after some algebra, we reach that
\[
\beta_P = \frac{\frac{\phi^2}{\lambda\sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} a + \frac{1}{\lambda \sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}}}{\frac{\phi^2}{\lambda\sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} + \frac{1}{\lambda \sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} \frac{\zeta + \xi_i}{\lambda \sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} + \frac{1}{\lambda \sqrt{\tau_x^{-1}\phi^2 + \tau_y^{-1}}} \frac{1}{\tau_a}}.
\]
To simplify, let
\[
h_1 = \frac{1}{\tau_x^{-1}\phi^2 + \tau_y^{-1}} \frac{1}{\tau_a} + \frac{1}{\tau_c} \tag{8.5}
\]
and
\[
h_2 = \frac{1}{\tau_x^{-1}\phi^2 + \tau_y^{-1}} \frac{1}{\tau_f} + \frac{1}{\tau_c} \tag{8.6}
\]
It is straightforward that \(h_1\) is increasing in \(\phi\) and \(h_2\) is decreasing in \(\phi\). Hence, we have
\[
\frac{\partial \beta_P}{\partial \phi^2} = \frac{-g_1 \frac{\partial g_2}{\partial \phi^2}}{(g_1 + g_2)^2} > 0,
\]
which indicates that \(\beta_P\) is an increasing function of \(\phi\) when \(\phi > 0\).

**Proof of Lemma 6.** We first consider common investment correlation \(\beta_A\). Recall the investment policy (3.8) and the definition of \(\beta_A\), we have
\[
\beta_A = \frac{\text{Cov}(\log I_i, \log A)}{\sqrt{\text{Var}(\log I)} \sqrt{\text{Var}(\log A)}} = \frac{\tau_{aa} + \tau_{af} + \tau_{af}}{\tau_a + \tau_{aa} + \tau_{af}} \frac{\tau_f + \tau_{ff} + \tau_{af}}{\tau_f + \tau_{ff} + \tau_{af}} = \frac{S_a \sqrt{\tau_a S_a / \tau_a + S_f / \tau_f}}{\sqrt{\tau_a S_a / \tau_a + S_f / \tau_f}}.
\]

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It is convenient for us to consider instead
\[
\frac{1}{\tau_a \beta_A^2} = \frac{S_a/\tau_a + S_f/\tau_f}{S_a^2/\tau_a + S_f^2/\tau_f}.
\]

By Lemma 4, since \( S_a \) is increasing in \( \phi^2 \) and \( S_f \) is decreasing in \( \phi^2 \) when \( \phi > 0 \), it is straightforward that \( 1/\tau_a \beta_A^2 \) is decreasing in \( \phi^2 \). This indicates that \( \beta_A \) is an increasing function of \( \phi \) when \( \phi > 0 \).

We then consider the idiosyncratic investment correlation \( \beta_F \). Again, recall the investment policy (3.8) and the definition of \( \beta_F \), we have
\[
\beta_F = \frac{\text{Cov}(\log I_i, \log F_i)}{\sqrt{\text{Var}(\log I_i)} \sqrt{\text{Var}(\log F_i)}} = \frac{S_f}{\tau_f \sqrt{S_a/\tau_a + S_f/\tau_f}}.
\]

Similarly, it is convenient for us to consider instead
\[
\frac{1}{\tau_f \beta_F^2} = \frac{S_a/\tau_a + S_f/\tau_f}{S_f^2/\tau_f} = \frac{S_a}{\tau_a S_f} + \frac{1}{\tau_f S_f},
\]
which is decreasing in \( \phi^2 \), again by Lemma 4. Hence, we conclude that \( \beta_F \) is an increasing function of \( \phi \) as well, when \( \phi > 0 \).

**Proof of Lemma 7.** We focus on the case when \( \tau_y \) and \( \lambda \) are large enough so that a unique solution of \( \phi \) is guaranteed. Recall that the reduced equilibrium condition (8.2) (in the proof of Proposition 1):
\[
\left( \frac{\tau_{sa} + \tau_x \tau_{\xi}}{\tau_a + \tau_{sa} + \tau_x \tau_{\xi}} + 1 - \frac{\sqrt{\tau_x}}{\lambda} \right) \frac{\tau_x}{\tau_a + \tau_x} = \frac{\tau_{sf} \phi^3 + \tau_x \tau_{\xi} \phi}{\tau_f \phi^2 + \tau_{sf} \phi^2 + \tau_x \tau_{\xi}} + \phi - \frac{\sqrt{\tau_x}}{\lambda}.
\]

It is clear that the left hand side of (8.2) is decreasing in \( \tau_a \). Hence, when \( \tau_a \) increase, the right hand side of (8.2) decreases as well. On the other hand, we have already known that the right hand side of (8.2) is increasing in \( \phi \). Hence, in equilibrium, \( \phi \) decreases. This indicates that \( \phi \) is a decreasing function of \( \tau_a \).

**Proof of Proposition 3.** We still focus on the case when \( \tau_y \) and \( \lambda \) are large enough so that a unique solution of \( \phi \) is guaranteed.

We first consider the investment beta
\[
\beta_I = \frac{S_a - g}{S_a + \tau_a S_f/\tau_f},
\]
where
\[
g = \left( \frac{\tau_{sa}}{\tau_a + \tau_{sa} + \tau_{pr}} \right)^2 \frac{\tau_a}{\tau_{sa}}.
\]
It is instructive for us to decompose the total effects of the changing of $\tau_a$ on $\beta_I$ into two parts: the mechanical effect and the cross-learning effect:

$$\frac{d\beta_I(\tau_a, \phi)}{d\tau_a} = \frac{\partial \beta_I(\tau_a, \phi)}{\partial \tau_a} \frac{\partial \psi}{\partial \phi} + \frac{\partial \beta_I(\tau_a, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_a}.$$ 

The sign of the second term, the cross-learning effect, is straightforward by Lemma 5 and Lemma 7. Specifically, Lemma 5 indicates that $\partial \beta_I(\tau_a, \phi)/\partial \phi > 0$ and Lemma 7 indicates that $\partial \phi/\partial \tau_a < 0$, so that the cross-learning effect is negative in this case.

For the first term, the mechanical effect, by the facts that $\tau_{ym} = \tau_x \tau_\xi$ when $\tau_y$ goes to infinity and that $S_f$ is independent of $\tau_a$, we know that $\partial \beta_I(\tau_a, \phi)/\partial \tau_a < 0$, i.e., the mechanical effect is negative as well.

Taking the two effects together, we know that the total effect is also negative, i.e., $d\beta_I(\tau_a, \phi)/d\tau_a < 0$.

We then consider the price beta

$$\beta_P = \frac{\phi^2}{\tau_a \phi^2 + \tau_y} \frac{1}{\tau_\xi} + \frac{1}{\tau_a} + \frac{1}{\tau_x \phi^2 + \tau_y} \frac{1}{\tau_\xi} + \frac{1}{\tau_\xi} = \frac{h_1}{h_1 + h_2},$$

where $h_1$ and $h_2$ are already defined in (8.5) and (8.6) (in the proof of Lemma 5).

Again, we decompose the total effects of the changing of $\tau_a$ on $\beta_P$ into two parts: the mechanical effect and the cross-learning effect:

$$\frac{d\beta_P(\tau_a, \phi)}{d\tau_a} = \frac{\partial \beta_P(\tau_a, \phi)}{\partial \tau_a} \frac{\partial \psi}{\partial \phi} + \frac{\partial \beta_P(\tau_a, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_a}.$$ 

First, it is straightforward to see that the mechanical effect $\partial \beta_P(\tau_a, \phi)/\partial \tau_a$ is negative, because $\beta_P$ is an increasing function of $h_1$ that is in turn decreasing in $\tau_a$ at the same time. Furthermore, Lemma 5 indicates that $\partial \beta_P(\tau_a, \phi)/\partial \phi > 0$ and Lemma 7 indicates that $\partial \phi/\partial \tau_a < 0$, which together imply that the cross-learning effect is negative as well. Therefore, we conclude that the total effect is also negative, i.e., $d\beta_P(\tau_a, \phi)/d\tau_a < 0$. 

**Proof of Lemma 8.** The proof is similar to the proof of Lemma 7. We again focus on the case when $\tau_y$ and $\lambda$ are large enough so that a unique solution of $\phi$ is guaranteed. In this case, we recall the reduced equilibrium condition (8.2) (in the proof of Proposition 1):

$$\left(\frac{\tau_{sa} + \tau_x \tau_\xi}{\tau_a + \tau_{sa} + \tau_x \tau_\xi} + 1 - \frac{\sqrt{\tau_x}}{\lambda}\right) \frac{\tau_x}{\tau_a + \tau_x} = \frac{\tau_{sf} \phi + \tau_x \tau_\xi \phi}{\tau_f \phi^2 + \tau_{sf} \phi^2 + \tau_x \tau_\xi} + \phi - \sqrt{\tau_x}/\lambda.$$ 

It is clear that the left hand side of (8.2) is increasing in $\tau_{sa}$. Hence, when $\tau_{sa}$ increase, the right hand side of (8.2) increases as well. On the other hand, we have already known that the right hand side of (8.2) is increasing in $\phi$. Hence, in equilibrium, $\phi$ increases. This indicates that $\phi$ is an increasing function of $\tau_{sa}$.

The analysis is similar for $\tau_{sf}$. The right hand side of (8.2) is increasing in $\tau_{sf}$, while the left hand side is independent of $\tau_{sf}$. Thus, when $\tau_{sf}$ increase, $\phi$ decreases to ensure a constant right
Proof of Proposition 4. We first consider the comparative statics with respect to $\tau_{sa}$. For the investment beta $\beta_I$, we have

$$\beta_I = 1 - \frac{S_f/\tau_f + g/\tau_a}{S_a/\tau_a + S_f/\tau_f},$$

where $g$ is already defined in (8.7) (in the proof of Proposition 3).

We also decompose the total effects of the changing of $\tau_{sa}$ on $\beta_I$ into two parts: the mechanical effect and the cross-learning effect:

$$\frac{d\beta_I(\tau_{sa}, \phi)}{d\tau_{sa}} = \frac{\partial \beta_I(\tau_{sa}, \phi)}{\partial \tau_{sa}} + \frac{\partial \beta_I(\tau_{sa}, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_{sa}}.$$

Clearly, Lemma 5 and Lemma 8 indicate that the second term, the cross-learning effect, is positive. For the first term, the mechanical effect, we first know that $S_a$ is increasing in $\tau_{sa}$, given $\phi$ fixed. Also, it is easy to show that $g/\tau_a$ is increasing in $\tau_{sa}$ (given $\phi$ fixed) when $\tau_{sa} < \tau_a + \tau_x \tau_c$, and decreasing in $\tau_{sa}$ (also given $\phi$ fixed) when $\tau_{sa} > \tau_a + \tau_x \tau_c$. Hence, we conclude that when $\tau_{sa} > \tau_a + \tau_x \tau_c$, the mechanical effect is positive, and thus the total effect $d\beta_I(\tau_{sa}, \phi)/d\tau_{sa}$ is positive as well. When $\tau_{sa} < \tau_a + \tau_x \tau_c$, the mechanical effect is ambiguous and so is the total effect.

For the price beta $\beta_P$, there is only cross-learning effect but no mechanical effect. Hence, by Lemma 5 and Lemma 8 we have that

$$\frac{d\beta_P(\tau_{sa}, \phi)}{d\tau_{sa}} = \frac{\partial \beta_P(\tau_{sa}, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_{sa}} > 0.$$

We then consider the comparative statics with respect to $\tau_{sf}$ in a similar manner. For the investment beta $\beta_I$, we have

$$\beta_I = \frac{S_a/\tau_a - g/\tau_a}{S_a/\tau_a + S_f/\tau_f}.$$

By the similar decomposition and again by Lemma 5 and Lemma 8, we know that both the mechanical effect and the cross-learning effect in this case are negative. So that the total effect is also negative:

$$\frac{d\beta_I(\tau_{sf}, \phi)}{d\tau_{sf}} = \frac{\partial \beta_I(\tau_{sf}, \phi)}{\partial \tau_{sf}} + \frac{\partial \beta_I(\tau_{sf}, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_{sf}} < 0.$$

For the price $\beta_P$, again, there is only cross-learning effect but no mechanical effect. Hence, by Lemma 5 and Lemma 8 we have that

$$\frac{d\beta_P(\tau_{sf}, \phi)}{d\tau_{sf}} = \frac{\partial \beta_P(\tau_{sf}, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_{sf}} < 0.$$

This concludes the proof.

Proof of Lemma 9. We focus on the case when $\tau_y$ and $\lambda$ are large enough so that a unique solution of $\phi$ is guaranteed. We first consider the effect of $\lambda$. Re-express the reduced equilibrium
condition (8.2) (in the proof of Proposition 1):

\[
\left( \frac{\tau_{sa} + \tau_x\tau_\zeta}{\tau_a + \tau_{sa} + \tau_x\tau_\zeta} + 1 \right) \frac{\tau_x}{\tau_a + \tau_x} + \frac{\sqrt{\tau_x}}{\tau_a + \tau_x} = \frac{\tau_s f \phi^\beta + \tau_x\tau_\xi \phi}{\tau_f \phi^2 + \tau_x \phi^2 + \tau_x \tau_\xi} + \phi. \tag{8.8}
\]

It is clear that the left hand side of (8.8) is decreasing in \(\lambda\) while the right hand side is independent of \(\lambda\). Hence, when \(\lambda\) increases, \(\phi\) decreases in equilibrium.

We then consider the effects of \(\tau_\zeta\) and \(\tau_\xi\). Again recall the reduced equilibrium condition (8.2):

\[
\left( \frac{\tau_{sa} + \tau_x\tau_\zeta}{\tau_a + \tau_{sa} + \tau_x\tau_\zeta} + 1 - \frac{\sqrt{\tau_x}}{\lambda} \right) \frac{\tau_x}{\tau_a + \tau_x} = \frac{\tau_s f \phi^\beta + \tau_x\tau_\xi \phi}{\tau_f \phi^2 + \tau_x \phi^2 + \tau_x \tau_\xi} + \phi - \sqrt{\tau_x} \frac{\lambda}{\lambda}.
\]

On the one hand, the left hand side of (8.2) is increasing in \(\tau_\zeta\), while the right hand side is independent of \(\tau_\zeta\), so that \(\phi\) is increasing in \(\tau_\zeta\) in equilibrium. On the other hand, the right hand side of (8.2) is increasing in \(\tau_\xi\) while the left hand side is independent of \(\tau_\xi\), so that \(\phi\) is decreasing in \(\tau_\xi\) in equilibrium.

\[\Box\]

**Proof of Proposition 5.** We first consider the comparative statics with respect to \(\lambda\). Since \(\lambda\) has no mechanical effect on ether \(\beta_I\) or \(\beta_P\), we focus on the cross-learning effect along. By Lemma 5 and Lemma 9, we know that both \(\beta_P\) and \(\beta_I\) are decreasing in \(\lambda\).

We then consider the comparative statics with respect to \(\tau_\zeta\). For the investment beta \(\beta_I\), we have

\[
\beta_I = \frac{S_a - g}{S_a + \tau_a S_f / \tau_f},
\]

where \(g\) is defined in (8.7) (in the proof of Proposition 3).

Again, we decompose the total effects of the changing of \(\tau_\zeta\) on \(\beta_I\) into two parts: the mechanical effect and the cross-learning effect:

\[
\frac{d\beta_I(\tau_\zeta, \phi)}{d\tau_\zeta} = \frac{\partial \beta_I(\tau_\zeta, \phi)}{\partial \tau_\zeta} + \frac{\partial \beta_I(\tau_\zeta, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_\zeta}.
\]

By Lemma 5 and Lemma 9, we know that the cross-learning effect is positive. For the mechanical effect, when \(\phi\) is fixed, it is easy to show that \(\partial \beta_I(\tau_\zeta, \phi)/\partial \tau_P > 0\). Since we know that \(\partial \tau_P/\partial \tau_\zeta > 0\), we get that the mechanical effect is also positive. Hence, the total effect \(d\beta_I(\tau_\zeta, \phi)/d\tau_\zeta\) is positive.

However, the total effect on the price beta \(\beta_P\) is ambiguous in this case. We have

\[
\beta_P = \frac{h_1}{h_1 + h_2},
\]

where \(h_1\) and \(h_2\) are already defined in (8.5) and (8.6) (in the proof of Lemma 5). Decomposition gives

\[
\frac{d\beta_P(\tau_\zeta, \phi)}{d\tau_\zeta} = \frac{\partial \beta_P(\tau_\zeta, \phi)}{\partial \tau_\zeta} + \frac{\partial \beta_P(\tau_\zeta, \phi)}{\partial \phi} \frac{\partial \phi}{\partial \tau_\zeta}.
\]

Lemma 5 and Lemma 9 give a positive cross-learning effect, i.e., the second term. However, it is easy to show that the first term, the mechanical effect, is negative. Hence, the total effect is ambiguous and is determined by other model parameters.
We finally consider the comparative statics with respect to $\tau\xi$. Similarly, we follow the decomposition above. For the investment beta $\beta_I$, by Lemma 5 and Lemma 9, we know that the cross-learning effect is negative. For the mechanical effect, when $\phi$ is fixed, it is easy to show that $\frac{\partial \beta_I(\tau\xi, \phi)}{\partial \tau_pf}$ > 0. Since we know that $\frac{\partial \tau_pf}{\partial \tau\xi}$ > 0, we get that the mechanical effect is also negative. Hence, the total effect $d\beta_I(\tau\xi, \phi)/d\tau\xi$ is negative. Following similar arguments and again by Lemma 5 and Lemma 9, we know that both the mechanical effect and the cross-learning effect on the price beta $\beta_P$ are also negative, so that the total effect on $\beta_P$ is negative as well.

**Proof of Proposition 6.** Following the definition of real investment efficiency, we know that

$$R = \int_0^1 R_idi,$$

where

$$R_i = \mathbb{E}\left[AF_i I_i - \frac{c}{2} I_i^2\right] = \frac{\kappa(2 - \kappa)}{2c} \mathbb{E}\left[AF_i \mathbb{E}[AF_i|\Gamma_i]\right],$$

and

$$\mathbb{E}[AF_i|\Gamma_i] = \exp\left[\frac{\tau_{sa}s_{a,i} + \tau_{pm}z_{a}P}{\tau_a + \tau_{sa} + \tau_{pm}} + \frac{1}{2(\tau_a + \tau_{sa} + \tau_{pm})} + \frac{\tau_{f} s_{f,i} + \tau_{pf} z_{f,i} P}{\tau_f + \tau_{sf} + \tau_{pf}} + \frac{1}{2(\tau_f + \tau_{sf} + \tau_{pf})}\right].$$

Since $\kappa$ and $c$ are constant, without loss of generality, we set $\kappa = 1$ and $c = 0.5$ to ease the exposition. After some tedious algebra, the investment efficiency $R$ is re-expressed in a much simpler and more intuitive form:

$$R = \exp\left(\frac{1 + S_a}{\tau_a} + \frac{1 + S_f}{\tau_f}\right). \tag{8.9}$$

We solve for the socially optimal $\phi^*$ that maximizes $R$. Taking the first order condition gives

$$\frac{\partial \log R}{\partial (\phi^2)} = \frac{\tau_x^2 \tau_y \tau_{\xi}}{(\tau_a + \tau_{sa} + \tau_{pm})^2 (\tau_x + \tau_{y}\phi^2)^2} - \frac{\tau_x \tau_y^2 \tau_{\xi}}{(\tau_f + \tau_{sf} + \tau_{pf})^2 (\tau_x + \tau_{y}\phi^2)^2} = 0, \tag{8.10}$$

which reduces to

$$\frac{(\tau_a + \tau_{sa} + \tau_{pm})^2}{(\tau_f + \tau_{sf} + \tau_{pf})^2} = \frac{\tau_x \tau_{y}}{\tau_y \tau_{\xi}}. \tag{8.11}$$

Since $\tau_{pm}$ is increasing in $\phi^2$ while $\tau_{pf}$ is decreasing in $\phi^2$, we know that the left hand side of (8.11) is increasing in $\phi^2$. Therefore, there is a unique non-negative solution of $\phi^*$.

We further compare between the socially optimal weight $\phi^*$ and the weight $\phi$ in the cross-learning equilibrium, focusing on the case in which $\tau_y$ and $\lambda$ are large enough so that there is always a unique positive solution of $\phi$. We re-express the first order condition (8.10) as

$$\frac{\partial \log R}{\partial (\phi^2)} = \frac{\tau_x^2 \tau_y}{(\tau_a + \tau_{sa} + \tau_{pm})^2} \left[\tau_x \tau_{\xi} - \frac{(\tau_a + \tau_{sa} + \tau_{pm})^2 \tau_{\xi}}{(\tau_f + \tau_{sf} + \tau_{pf})^2}\right] = 0. \tag{8.12}$$
When $\tau_y$ goes to infinity, $\tau_x\tau_\zeta/\tau_y$ goes to 0, and we also have

$$\left(\frac{\tau_a + \tau_{sa} + \tau_{sa}}{\tau_f + \tau_{sf} + \tau_{sf}}\right)^2 = \left(\frac{\tau_a + \tau_{sa} + \tau_x\tau_\zeta}{\tau_f + \tau_{sf} + \tau_x\tau_\zeta/\phi^2}\right)^2 > 0.$$ 

Hence, when $\tau_y$ and $\lambda$ are large enough, the left hand side of (8.12) is always negative. Therefore, we conclude that the cross-learning equilibrium $\phi$ is always larger than the socially optimal $\phi^*$ when $\tau_y$ and $\lambda$ are large enough.

\[\Box\]

**Proof of Proposition 7.** Recall the expression of investment efficiency (8.9):

$$R = \exp\left(\frac{1 + S_a}{\tau_a} + \frac{1 + S_f}{\tau_f}\right).$$

We again focus on the case when $\tau_y$ and $\lambda$ are large enough so that a unique solution of $\phi$ is guaranteed. We first consider the comparative statics with respect to $\lambda$. Lemma 4 implies that $\partial S_f/\partial \phi < 0$ and Lemma 9 implies that $\partial \phi/\partial \lambda < 0$. Since there is no direct effect of $\lambda$ on $S_f$, we know that $S_f$ is increasing in $\lambda$ in equilibrium. Moreover, because the effect of $\phi$ on $S_a$ is negligible when $\tau_y$ is sufficiently large, we eventually know that that $R$ is an increasing function of $\lambda$.

We then consider $\tau_\xi$. Similarly, Lemma 4 implies that $\partial S_f/\partial \phi < 0$ and Lemma 9 implies that $\partial \phi/\partial \tau_\xi < 0$, so that $S_f$, and thus $R$ is increasing in $\tau_\xi$ in equilibrium.

We finally consider $\tau_{sf}$. It is clear that $\partial S_f/\partial \tau_{sf} > 0$, i.e., the mechanical effect is positive. For the cross-learning effect, Lemma 4 implies that $\partial S_f/\partial \phi < 0$ and Lemma 8 implies that $\partial \phi/\partial \tau_\xi < 0$. Hence, the total effect is positive as well, i.e., $S_f$ is increasing in $\tau_{sf}$ in equilibrium. Since the effect of $\phi$ on $S_a$ is negligible when $\tau_y$ is sufficiently large, we eventually know that that $R$ is an increasing function of $\tau_{sf}$.

\[\Box\]

**Proof of Proposition 8.** Part i) is straightforward following the proofs of Proposition 1 and Proposition 2. For part ii), we make use of the conditional expectation (5.12). Specifically, we have

$$\text{Var}(z) = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{bmatrix},$$

where

$$\sigma_{11} = \tau_a^{-1} + \tau_{sa}^{-1},$$
$$\sigma_{12} = 0,$$
$$\sigma_{13} = \phi_n\tau_a^{-1},$$
$$\sigma_{14} = \phi_n\tau_a^{-1},$$
$$\sigma_{22} = \tau_f^{-1} + \tau_{sf}^{-1},$$
$$\sigma_{23} = \tau_f^{-1},$$
$$\sigma_{24} = 0.$$
\[
\sigma_{33} = \tau_f^{-1} + \phi_n^2 \tau_a^{-1} + (\tau_x^{-1} \phi_n^2 + \tau_y^{-1})(\tau_c^{-1} + \tau_x^{-1}) ,
\]
\[
\sigma_{34} = \phi_n^2 \tau_a^{-1} + (\tau_x^{-1} \phi_n^2 + \tau_y^{-1})\tau_c^{-1} ,
\]
\[
\sigma_{44} = \frac{\tau_f^{-1} + \phi_n^2 \tau_a^{-1} + (\tau_x^{-1} \phi_n^2 + \tau_y^{-1})\tau_c^{-1}}{n - 1} ,
\]
and
\[
\text{Cov}(z, a + f_1) = [\tau_a^{-1}, \tau_f^{-1}, \phi_n \tau_a^{-1} + \tau_f^{-1}, \phi_n \tau_a^{-1}] .
\]

By condition (5.12), we get the expressions of \( S_{an} \) and \( S_{fn} \) after some tedious algebra and plug them into the equilibrium condition. Denote by \( RHS \) the right hand side of the equilibrium condition (5.13) and we get
\[
\frac{\partial \lim_{n \to \infty} RHS(\phi_n, n)}{\partial n} = C_1C_2C_3C_4C_5(C_6 + C_7 + C_8)^2 ,
\]
where
\[
\begin{align*}
C_1 &= \phi_n^2 (\tau_f + \tau_x)\tau_x \tau_c \tau_c - \phi_n (\tau_x + \tau_x \tau_c \tau_c) ,
C_2 &= \phi_n^2 \tau_a (\tau_f + \tau_x)\tau_x + 2\tau_a \tau_x \tau_c \tau_c + \phi_n \tau_f (\tau_a + \tau_x \tau_x) ,
C_3 &= \phi_n^2 \tau_f + \tau_x \tau_c ,
C_4 &= \phi_n^2 \tau_f \tau_x \tau_c \tau_c ,
C_5 &= \tau_a + \tau_x ,
C_6 &= \phi_n^2 \tau_x \tau_c \tau_c \tau_c + \phi_n \tau_f \tau_x \tau_x \tau_c \tau_c + 2(\tau_a + \tau_x \tau_x \tau_c \tau_c) ,
C_7 &= \phi_n^4 \tau_f (\tau_f + 2\tau_x) \tau_x + (\tau_a + \tau_x \tau_x \tau_a) ,
C_8 &= \phi_n^2 \tau_x \tau_c \tau_c \tau_c \tau_c + (2\tau_a + \tau_x \tau_x \tau_a) ,
\end{align*}
\]

Note that, only the first term \( C_1 \) has a negative component. However, when \( \tau_x \) is small enough, \( C_1 \) is always strictly positive, so is the entire derivative (8.13). It implies that when \( \tau_f \) is large enough and \( \tau_x \) is small enough, the equilibrium \( \phi_n \) is increasing in \( n \). Also, the proof of Proposition 6 directly implies that \( \phi' (= \phi) > \phi^* \), so that \( \phi_n > \phi^* \) for all \( n \geq 1 \).

Finally, for part iii), by the proof of Proposition 6, in particular condition (8.9), we know that
\[
R_n = \exp \left( \frac{1 + S_a(\phi_n)}{\tau_a} + \frac{1 + S_f(\phi_n)}{\tau_f} \right) ,
\]
where \( S_a \) and \( S_f \) are the capital providers’ investment sensitivities in the baseline cross-learning case pinned down by however the equilibrium weight in the corresponding n-learning equilibrium. By Lemma 4, it follows that \( R < R_n < R^* \) for all \( n \geq 1 \). This concludes the proof.

References


