Note that we use the metric convention \((-+++)\).

1. The Klein-Gordon field has mode expansion

\[
\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right).
\]

In the interaction picture explain what is meant by the time ordered product; the normal ordered product; the Feynman propagator.

A time ordered product

\[
T\{\phi(x)\phi(y)\} = \begin{cases} 
\phi(x)\phi(y), & x^0 > y^0 \\
\phi(y)\phi(x), & x^0 < y^0.
\end{cases}
\]

A normal ordered product always has annihilation operators to the right of the creation operators, i.e.,

\[
: a^\dagger a : = : a a^\dagger : = a^\dagger a,
\]

such that the vacuum will always be annihilated by the operator.

The Feynman propagator is the vacuum expectation value of the time ordered product, i.e.,

\[
\langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle.
\]

The difference between the Feynman, retarded, and advanced propagators lies in the prescriptions of handling the poles in the \(p^0\) integral. For the Feynman one, we pick up a pole at \(p^0 = E_p > 0\) when \(x^0 > y^0\), and at \(p^0 = -E_p < 0\) when \(x^0 < y^0\), so that positive energy modes flow forward in time.

Derive the Feynman rules from the path-integral for the particle case.

Notice that we can write

\[
\int_{-\infty}^{\infty} dq \ e^{-\frac{i}{2} m^2 q^2 + J q} q^{4n} = \left( \frac{d}{dJ} \right)^4 \int_{-\infty}^{\infty} dq \ e^{-\frac{i}{2} m^2 q^2 + J q}.
\]
Hence
\[Z(\lambda, J) = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{2}q^4 + Jq}\]
\[= \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 + Jq} \left[ 1 - \frac{\lambda}{4!}q^4 + \frac{1}{2} \left( \frac{\lambda}{4!} \right)^2 q^8 + \cdots \right] \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 + Jq}\]
\[= e^{-\frac{\lambda}{4!(d/dJ)^4}} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2(q - \frac{J}{m})^2 + \frac{J^2}{2m^2}}\]
\[= Z(0,0) e^{-\frac{\lambda}{4!(d/dJ)^4}} e^{\frac{J^2}{2m^2}}\]
\[= Z(0,0) \left[ 1 - \frac{\lambda}{4!} (d/dJ)^4 + \frac{1}{2} \left( \frac{\lambda}{4!} \right)^2 (d/dJ)^8 + \cdots \right] \left[ 1 + \frac{J^2}{2m^2} + \frac{J^2}{2m^2}^2 + \cdots \right],\]
where \(Z(0,0) = \sqrt{2\pi/m}\).

\(Z(\lambda, J)\) corresponds to Feynman diagrams.

1. diagrams are made of lines and vertices at which four lines meet;
2. for each vertex assign a factor of \(-i\lambda\);
3. for each line assign \(1/m^2\);
4. for each external end assign \(J\).

So the term in \(Z(J)\) corresponds to 7 lines, 2 vertices, and 6 ends, is
\[\left[ \left( \frac{1}{m^2} \right)^7 (-\lambda)^2 \right] J^6,\]
which can be derived from the “writing terms as derivatives” trick above by taking the \(J^{14}\) term from \(e^{J^2/2m^2}\), which is \(\frac{1}{7!} \left[ \frac{J^2}{2m^2} \right]^{7}\), and the \(\lambda^2\) term from \(e^{-\frac{\lambda}{4!}(d/dJ)^4}\), which is \(\frac{1}{7} \left[ -\frac{\lambda}{4!} \left( \frac{d}{dJ} \right)^4 \right]^2\). Acting the latter on the former, we get
\[\frac{1}{7} \left[ (\frac{d}{dJ})^4 \right]^2 \frac{J^6}{2 \cdot 4!^2 \cdot 7!^6 \cdot 2^7 m^{14}}.\]

2.

The massless Dirac equation is
\[\gamma^a \partial_a \psi(x) = 0,\]
where
\[\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\]
Compute \(\gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3\).
\[ \gamma^5 = -i \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{array} \right) \]
\[ = i \left( \begin{array}{cc} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{array} \right) \left( \begin{array}{cc} \sigma^2 \sigma^3 & 0 \\ 0 & \sigma^2 \sigma^3 \end{array} \right) \]
\[ = \left( \begin{array}{cc} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{array} \right) \left( \begin{array}{cc} \sigma^1 & 0 \\ 0 & \sigma^1 \end{array} \right) \]
\[ = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]

Show that it anticommutes with \( \gamma^0 \) and \( \gamma^i \).

\[ \{ \gamma^5, \gamma^0 \} = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 = 0. \]
\[ \{ \gamma^5, \gamma^1 \} = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 = 0. \]

Similarly for other \( \gamma^i \)’s.

We guess that the solutions to the Dirac equation are

\[ \psi = e^{ip \cdot x} u(p), \quad \psi = e^{-ip \cdot x} v(p), \]

where \( u(p) \) is a positive frequency spinor wavefunction, and \( v(p) \) is a negative frequency spinor wavefunction.

Compute \( u(p) \).

Let \( u(p) := \left( \begin{array}{c} A \\ B \end{array} \right) \) and plug it into the Dirac equation. Then,

\[ \gamma^a p_a u(p) = 0 \quad \Rightarrow \quad \left( \begin{array}{cc} 0 & -p_0 + \sigma^i p_i \\ -p_0 - \sigma^i p_i & 0 \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right) = 0. \]

If we work in the frame \((\omega, 0, 0, \omega)\), then

\[ (-\omega + \sigma^3 \omega) B = \left( \begin{array}{cc} -\omega + \omega & 0 \\ 0 & -\omega - \omega \end{array} \right) \left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) = 0. \]

\[ \Rightarrow B_2 = 0, \quad B_1 = \text{anything} \quad \Rightarrow \quad B := \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \]

Similarly,

\[ A := \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \]

4.

Give an account of the canonical quantisation of the electromagnetic field. You should give the Hamiltonian and the canonical commutation relations. You should discuss gauge invariance and the need to gauge fix.

We start by discussing gauge invariance in the Maxwell theory. From the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} + J_a A^a, \]
we obtain the Maxwell equation
\[ \partial_a F^{ab} + J^b = 0, \]
which gives
\[ \partial_b \partial_a F^{ab} + \partial_b J^b = 0 \quad \Rightarrow \quad \partial_b J^b = 0. \]
due to symmetry of partial derivatives and antisymmetry of \( F_{ab} \). Now, by Noether’s theorem, conserved currents are associated with symmetries. Here we have a gauge symmetry,
\[ A^a \rightarrow A^a + \partial^a \Lambda(x), \]
which leaves the action unchanged, since
\[ S \rightarrow S + \int d^4x \left( -\frac{1}{4} (\partial_a \partial_b A^a - \partial_b \partial_a A^b)(\partial^c \partial^d \Lambda - \partial^d \partial^c \Lambda - (\partial^a J_a) \Lambda) \right) = S \]
since \( J \) is conserved. Now, \( A^a \) determines \( F_{ab} \) uniquely but not vice versa. We need to impose some condition on \( A^a \) such that knowing \( F_{ab} \) determines \( A^a \) uniquely.

The Lorentz gauge \( \partial_a A^a = 0 \) maintains Lorentz invariance, but the \( A^a \) that satisfies the gauge is not unique, since we only require
\[ \partial_a A^a + \partial_a \partial^a \Lambda := f(x) + \Box \Lambda = 0, \]
i.e., we just need
\[ \Box \Lambda = -f(x), \]
which can be solved using Green’s function. The Green’s function is not unique! If we take
\[ \Lambda' = \Lambda + \phi, \]
then
\[ -f = \Box \Lambda' = \Box \Lambda + \Box \phi = -f + \Box \phi \]
tells us that it works as long as \( \Box \phi = 0 \), i.e., \( \phi \) is a solution of the scalar wave-function. Hence further conditions must be imposed on top of the Lorentz gauge if there is to be a unique \( A^a \) given a fixed \( F_{ab} \).

The Coulomb gauge \( \nabla \cdot \vec{A} = 0 \) does not completely fix the gauge either. To see this, (ignoring the current for the moment) we start by writing the EOM
\[ \partial_a F^{ab} = \partial_a \partial^a A^b - \partial_b \partial^b A^a = 0 \]
in Coulomb gauge.
\[ \Rightarrow b = 0 : \quad \Box A^0 - \partial^0 \partial_a A^a = \Box A^0 - \partial^0 \partial_b A^0 = \nabla^2 A^0 = 0. \]
This only determines \( A^0 \) up to some (possibly time-dependent) constants, so \( \nabla \cdot \vec{A} = 0 \) is insufficient to tackle \( A^a \rightarrow A^a + \partial^a \Lambda \). We can fix the gauge completely by further imposing \( A^0 = 0 \). If \( A^0 = f(t) \), then we can simply take
\[ \partial^0 \Lambda = -f(t); \quad \nabla^2 \Lambda = 0. \]
Hence even if we start from \( A^0 \neq 0 \), we can always make a gauge transformation to \( A^0 = 0 \).

In summary, our gauge conditions are
\[ \nabla \cdot \vec{A} = 0, \quad A^0 = 0. \]

Now, to quantise the electromagnetic field canonically, we start by looking at the Hamiltonian. We use the source free
\[ \mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} = \frac{1}{2} (\partial_0 A_i - \partial_i A_0)(\partial^0 A^i - \partial^i A^0) - \frac{1}{4} (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) = \frac{1}{2}(E^2 - B^2). \]
\[ \Rightarrow \pi^0 = \frac{\partial \mathcal{L}}{\partial A_0} = 0, \quad \pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{0i} = E^i. \]

\[ \Rightarrow H = \int d^3x \left( \pi^i \dot{A}_i - \mathcal{L} \right) = \int d^3x \left( \frac{1}{2} (E^2 + B^2) - A^0 (\nabla \cdot \vec{E}) \right). \]

We can think of \( A^0 \) as a Lagrange multiplier for imposing the constraint \( \nabla \cdot \vec{E} = 0 \), so we only treat \( A^i \) as a quantum operator, with mode expansion

\[ A^i = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_r \epsilon_r(\vec{p}) \left( a_{r,\vec{p}} e^{ip \cdot x} + a^\dagger_{r,\vec{p}} e^{-ip \cdot x} \right), \]

and the momentum operator is

\[ \pi^i = (-i) \int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \sum_r \epsilon_r(\vec{p}) \left( a_{r,\vec{p}} e^{ip \cdot x} - a^\dagger_{r,\vec{p}} e^{-ip \cdot x} \right), \]

where the sum is over polarisations \( r \), and \( E_{\vec{p}} = p^0 = |\vec{p}| \) since \( \Box A^a = 0 \Rightarrow p^2 = 0 \), i.e., the photon is massless! Now, using the mode expansion, the gauge condition \( \nabla \cdot \vec{A} = 0 \) implies \( \epsilon \cdot \vec{E} = 0 \), i.e., it gives polarisations perpendicular to the spatial momentum - two of them.

Photons are bosons, so the commutation relations for the creation and annihilation operators are simply

\[ [a_{r,\vec{p}}, a^\dagger_{s,\vec{q}}] = (2\pi)^3 \delta_{rs} \delta^3(\vec{p} - \vec{q}), \]

and the rest 0. And we have

\[ [A^i(\vec{x}), \pi^j(\vec{y})] = i \left( \delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2} \right) \delta^3(\vec{x} - \vec{y}). \]

Substituting the mode expansions into the classical Hamiltonian gives the quantum one,

\[ H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} E_{\vec{p}} : \sum_r \left( a_{\vec{p},r} a^\dagger_{\vec{p},r} + a^\dagger_{\vec{p},r} a_{\vec{p},r} \right) : = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_r a^\dagger_{\vec{p},r} a_{\vec{p},r} \]

by normal-ordering.