1.

(a) Explain the following terms: covariant derivative, connection components.

A covariant derivative $\nabla$ on a manifold $\mathcal{M}$ is a map sending every pair of smooth vector fields $X, Y$ to a smooth vector field $\nabla_X Y$, with properties

$$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$$

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$$

$$\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$$

In a basis $e_\mu$ the connection components $\Gamma^\mu_{\nu\rho}$ are defined by

$$\nabla_\rho e_\sigma = \nabla_{e_\rho} e_\sigma = \Gamma^\mu_{\rho\sigma} e_\mu$$

The Christoffel symbols are the coordinate basis components of the Levi-Civita connection, which is defined on any manifold with a metric.

(b) What does it mean for a connection to be torsion-free? Define the Levi-Civita connection and derive the formula

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}).$$

A connection $\nabla$ is torsion-free if $[\nabla_a, \nabla_b]f = 0$ for any function $f$. For a torsion-free connection, if $X$ and $Y$ are vector fields then

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

On a manifold with a metric, the metric singles out a preferred connection. Let $M$ be a manifold with a metric $g$. There exists a unique torsion-free connection $\nabla$, the Levi-Civita connection, which is metric compatible, i.e., $\nabla g = 0$. Hence we have

$$\nabla_i g_{jk} = 0 = \partial_i g_{jk} - \Gamma^l_{ij} g_{kl} - \Gamma^l_{ik} g_{jl}$$

$$\Rightarrow \partial_i g_{jk} = \Gamma^l_{ij} g_{kl} + \Gamma^l_{ik} g_{jl}$$

$$\Rightarrow \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = \Gamma^l_{ij} g_{kl} + \Gamma^l_{ik} g_{jl} + \Gamma^l_{jk} g_{il} - \Gamma^l_{ki} g_{jl} - \Gamma^l_{kj} g_{il} = 2\Gamma^l_{ij} g_{kl},$$

since torsion free implies $\Gamma^k_{[ij]} = 0$.

$$\Rightarrow \Gamma^l_{ij} g_{kl} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

$$\Rightarrow \Gamma^l_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}).$$


(c) A 2d Riemannian manifold has metric
\[ ds^2 = dr^2 + f(r)^2 d\phi^2 \]
where \( \phi \) is periodically identified with period \( 2\pi \) (so curves of constant \( r \) are circles).

(i) Determine the necessary and sufficient conditions on \( f(r) \) for the circle \( r = r_0 \) to have the property that all vectors are invariant under parallel transport around the circle.

We want that for any vector field \( X \),
\[ \nabla_V X = 0, \quad \text{or,} \quad V^a \nabla_a X^b = 0 \Rightarrow V^\mu \partial_\mu X^\nu + V^\mu \Gamma^\nu_{\mu\rho} X^\rho = 0, \]
for some \( V \) indicating “around the circle”. Since \( \frac{\partial}{\partial \phi} \) is tangent to the circle, we take
\[ V = \frac{\partial}{\partial \phi}. \]

Hence our equation reduces to
\[ \partial_\phi X^\nu + \Gamma^\nu_{\phi\rho} X^\rho = 0. \]

To find the \( \Gamma \)'s, we start with the Lagrangian
\[ L = \dot{r}^2 + f(r)^2 \dot{\phi}^2. \]

The equations of motion are
\[ f^2 \ddot{\phi} + 2f f' \dot{r} \dot{\phi} = 0 \quad \Rightarrow \quad \ddot{\phi} + \frac{2f'}{f} \dot{r} \dot{\phi} = 0 \quad \Rightarrow \quad \Gamma^\phi_{\phi \phi} = \Gamma^\phi_{\phi r} = \frac{f'}{f}; \]

\[ \ddot{r} - f f' \dot{\phi}^2 = 0 \quad \Rightarrow \quad \Gamma^r_{\phi \phi} = -f f'. \]

\[ \Rightarrow \nu = \phi: \quad \partial_\phi X^\phi + \Gamma^\phi_{\phi \phi} X^\phi = 0; \]
\[ \Rightarrow \nu = r: \quad \partial_\phi X^r + \Gamma^r_{\phi \phi} X^\phi = 0, \]

which give
\[ \partial_\phi X^\phi + \frac{f'}{f} X^r = 0; \quad \partial_\phi X^r - f f' X^\phi = 0. \]

The first gives \( \partial_\phi X^r = -\frac{f}{f'} \partial^2_\phi X^\phi \). Substituting it into the second, we have
\[ \partial^2_\phi X^\phi + f f' X^\phi = 0. \]

And the second gives \( \partial_\phi X^\phi = \frac{1}{f f'} \partial^2_\phi X^r \). Substituting it into the first, we have
\[ \partial^2_\phi X^r + f f' X^r = 0. \]

Hence
\[ X^\phi(\phi) = X^\phi(0) \cos(f' \phi) + X^{\prime \phi}(0) \sin(f' \phi); \]
\[ X^r(\phi) = X^r(0) \cos(f' \phi) + X^r(0) \sin(f' \phi). \]

So we require \( X^\phi(0) = X^\phi(2\pi) \) and \( X^r(0) = X^r(2\pi) \). This is true iff \( f' = n, \quad n \in \mathbb{Z} \).
(ii) Deduce that there is a 2-parameter family of functions $f(r)$ for which all circles of constant $r$ have this property.

From (i), we have

$$f(r) = nr + c, \quad c = \text{const.}$$

(iii) For this 2-parameter family, show that the metric is locally isometric to the Euclidean metric. Is it globally isometric?

Locally isometric means we can find coordinates such that the metric looks like

$$ds^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2, \quad \tilde{r} = \tilde{r}(r), \quad \tilde{\phi} = \tilde{\phi}(\phi).$$

Indeed we have

$$ds^2 = dr^2 + (nr + c)^2 d\phi^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2$$

where we have set $\tilde{r} = r + c/n$ and $\tilde{\phi} = n\phi$. Hence we have a local isometry. We have a global isometry only when $n = 1$, since now $\tilde{\phi}$ is periodically identified with period $2\pi n$.

2.

Let $T_a$ be tangent to a 1-parameter family of timelike geodesics of the Levi-Civita connection, parameterized by proper time. Let $S^a$ be a deviation vector for this family.

(a) Explain the term deviation vector. Explain why $[S,T] = 0$.

A 1-parameter family of geodesics is a map $\gamma : I \times I' \to \mathcal{M}$ where $I$ and $I'$ both are open intervals in $\mathbb{R}$, such that for fixed $s$, $\gamma(s,t)$ is a geodesic with affine parameter $t$, and $s$ is the parameter that labels the geodesic. Further, the map $(s,t) \to \gamma(s,t)$ is smooth and one-to-one with a smooth inverse.

A deviation vector $S^a$ is the displacement of two objects travelling along two such neighbouring geodesics such that

$$x^a(s + \delta s, t) \simeq x^a(s, t) + \delta s S^a(s, t)$$

where $S^a(s, t) = \frac{\partial x^a(s, t)}{\partial s}$. Note that $T^a(s, t) = \frac{\partial x^a(s, t)}{\partial t}$.

The family of geodesics forms a two-dimensional surface $\Sigma \subset \mathcal{M}$. Treating $s$ and $t$ as coordinates, we can extend them to $(s, t, u, ...)$ defined in a neighbourhood of $\Sigma$. This gives a coordinate chart on $\Sigma$ where $S = \partial/\partial s$ and $T = \partial/\partial t$ are vector fields satisfying

$$[S, T] = 0.$$
(b) State and prove the geodesic deviation equation. You may assume the definition of the Riemann curvature tensor

\[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

The geodesic deviation equation is

\[ \nabla_T \nabla_T S = R(T,S)T \]

if \( \nabla \) has vanishing torsion.

**Proof:** Vanishing torsion implies \( \nabla_X Y - \nabla_Y X = [X,Y] \); we have from above \( [S,T] = 0 \); and we have \( \nabla_T T = 0 \) since \( T \) is tangent to affinely parametrised geodesics. Hence

\[ R(T,S)T = \nabla_T \nabla_S T - \nabla_S \nabla_T T - \nabla_{[T,S]} T = \nabla_T \nabla_T S. \]

(c) Prove that if \( S^a \) and \( T^a \) are orthogonal at one point along a geodesic \( \gamma \) (belonging to the 1-parameter family) then they are orthogonal everywhere along \( \gamma \).

\[ \nabla_T (T \cdot S) = (\nabla_T T) \cdot S + T \cdot (\nabla_T S) = T \nabla_T S = T \nabla_S T = \frac{1}{2} \nabla_S (T \cdot T) = \frac{1}{2} \nabla_S (-1) = 0 \]

since \( T \) is timelike. Hence there is no change in the value of \( T \cdot S \) along the geodesic.

(d) Suppose that spacetime is four-dimensional with Riemann tensor

\[ R_{abcd} = \frac{1}{12} R(g_{ac}g_{bd} - g_{ad}g_{bc}) \]

Show that \( R \) must be constant.

We first contract the given Riemann tensor to find

\[ R_{bd} = \frac{1}{12} R(4g_{bd} - g_{bd}) = \frac{1}{4} R_{g_{bd}}. \]

Using this, we have the Einstein tensor

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = -\frac{1}{4} R g_{ab}. \]

But \( \nabla_a G_{ab} = 0 \). Hence

\[ \nabla_a R = 0, \]

i.e., \( R \) is constant.

(e) Assume that \( S^a \) is orthogonal to \( T^a \). Let \( f = S_a S^a \). Show that, in the above spacetime,

\[ K = (\nabla_T S)_a (\nabla_T S)^a - \frac{1}{12} R f \]

is constant along a geodesic \( \gamma \) in the family.
\[ \nabla_T K = \nabla_T (\nabla_T S \cdot \nabla_T S) - \frac{1}{12} (\nabla_T R) S \cdot S - \frac{1}{12} R \nabla_T (S \cdot S) \]
\[ = 2 \nabla_T \nabla_T S \cdot \nabla_T S - \frac{1}{6} RS \cdot \nabla_T S \quad \text{(since \( R \) here is constant)} \]
\[ = 2R(T, S) T \cdot \nabla_T S - \frac{1}{6} RS \cdot \nabla_T S \]
\[ = 2R_{abcd} T^b T^c S^d T^e \nabla_e S^a - \frac{1}{6} RS^a T^e \nabla_e S^a \]
\[ = \frac{1}{6} R (g_{ac} g_{bd} - g_{ad} g_{bc}) T^b T^c S^d T^e \nabla_e S^a - \frac{1}{6} RS^a T^e \nabla_e S^a \]
\[ = \frac{1}{6} RT_d T_a S^d T^e \nabla_e S^a - \frac{1}{6} RT_c T_a T^e \nabla_e S^a - \frac{1}{6} R (T_c T^e) S_a T^e \nabla_e S^a - \frac{1}{6} RS_a T^e \nabla_e S^a \]
\[ = \frac{1}{6} RS_a T^e \nabla_e S^a - \frac{1}{6} RS_a T^e \nabla_e S^a \quad \text{(since \( T_d S^d = 0 \) by orthogonality and \( T \) is timelike)} \]
\[ = 0. \]

(f) Obtain a second order differential equation for the evolution of \( f \) along \( \gamma \). Hence deduce that if \( R > 0 \) then geodesics which are close initially will diverge exponentially. What happens if \( R < 0 \)?

\[
\nabla_T \nabla_T f = \nabla_T \nabla_T (S \cdot S)
\]
\[ = 2 \nabla_T (S \cdot \nabla_T S)
\[ = 2 (\nabla_T S) \cdot (\nabla_T S) + 2S \cdot \nabla_T \nabla_T S
\]
\[ = 2K + \frac{1}{12} Rf + 2S \cdot R(T, S) T
\]
\[ = 2K + \frac{1}{6} Rf + 2S^a \frac{1}{12} R (g_{ac} g_{bd} - g_{ad} g_{bc}) T^b T^c S^d
\]
\[ = 2K + \frac{1}{6} Rf + \frac{1}{6} RS_a T^a T^d T^e S^d - \frac{1}{6} RS_a T^e T^c S^d
\]
\[ = 2K + \frac{1}{3} Rf \quad \text{(since \( S^d T_d = 0 \)).} \]

So the differential equation for the evolution of \( f \) is
\[
\frac{d^2 f}{dt^2} - \frac{1}{3} Rf - 2K = 0,
\]
from which we can see that if \( R > 0 \), the late-time solution is dominated by an exponential growth \( \sim \exp(\sqrt{R/3} t) \), but if \( R < 0 \) there are oscillations.

3.

In the study of linearised perturbations of Minkowski spacetime, it is assumed that there exist global coordinates \( x^\mu \) with respect to which the metric has components \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \) where the components of \( h_{\mu \nu} \) have absolute values much smaller than 1.

(a) Let \( h = h^\rho_\rho \) and \( \tilde{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \). By imposing the gauge condition \( \partial^\mu \tilde{h}_{\mu \nu} = 0 \), derive the linearised Einstein equation in the form
\[
\partial^\rho \partial_\rho h_{\mu \nu} = -16\pi T_{\mu \nu}.
\]
You may use
\[ R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\sigma} - \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\sigma}, \]
valid in a coordinate basis.

To first order, the Christoffel symbols are
\[ \Gamma^\mu_{\nu\rho} = \frac{1}{2} \eta^{\mu\sigma} \left( \partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho} \right) \]

And to first order, the Riemann tensors are
\[ R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} \]
\[ = \frac{1}{2} \eta^{\mu\rho} \left( \partial_\nu h_{\sigma\rho} + \partial_\sigma h_{\nu\rho} - \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho} - \partial_\nu h_{\sigma\rho} + \partial_\sigma h_{\nu\rho} \right) \]
\[ \Rightarrow R^\rho_{\nu\rho\sigma} = \frac{1}{2} \left( \partial_\nu h_{\rho\sigma} + \partial_\sigma h_{\rho\nu} - \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\rho\nu} - \partial_\nu h_{\rho\sigma} + \partial_\sigma h_{\rho\nu} \right) \]
\[ = \partial^\rho \partial_\nu h_{\sigma\rho} - \frac{1}{2} \partial^\rho \partial_\sigma h \]
\[ = R_{\nu\sigma} \]

Now,
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \partial^\rho \partial_\mu (h_{\nu\rho}) - \frac{1}{2} \partial^\rho \partial_\rho h_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \eta_{\mu\nu} (\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial^\rho \partial_\rho h) \]
\[ = \frac{1}{2} \partial^\rho \partial_\mu (h_{\nu\rho} - \frac{1}{2} h_{\nu\rho}) + \frac{1}{2} \partial^\rho \partial_\nu (h_{\mu\rho} - \frac{1}{2} h_{\mu\rho}) - \frac{1}{2} \partial^\rho \partial_\rho (h_{\mu\nu} - \frac{1}{2} h_{\mu\nu}) \]
\[ + \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \eta_{\mu\nu} (\partial^\rho \partial^\sigma (h_{\rho\sigma} - \frac{1}{2} h_{\rho\sigma}) + \partial^\rho \partial_\rho h) \]
\[ = \partial^\rho \partial_\mu (h_{\nu\rho}) - \frac{1}{2} \partial^\rho \partial_\rho \eta_{\mu\nu} - \frac{1}{2} \partial^\rho \partial_\nu h_{\mu\rho} + \frac{1}{2} \partial^\rho \partial_\rho \eta_{\mu\nu} \]
\[ + \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial_\rho h - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial_\rho h \]
where \( h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \) gives \( h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu} \) and \( \bar{h} = -h \). Hence from
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \]
we get the linearised Einstein equation
\[ \partial^\rho \partial_\mu (\bar{h}_{\nu\rho}) - \frac{1}{2} \partial^\rho \partial_\rho \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}. \]

Imposing the gauge condition \( \partial^\mu \bar{h}_{\mu\nu} = 0 \), the first and the third terms disappear and we arrive at
\[ \partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \]

(b) Consider a vacuum plane gravitational wave solution with
\[ \bar{h}_{\mu\nu} = \text{Re}(H_{\mu\nu} e^{ik_{\nu} x}) \]
where \( H_{\mu\nu} \) is a constant complex matrix and \( k_{\nu} \) a constant covector. What restrictions must \( H_{\mu\nu} \) and \( k_{\nu} \) obey in the above gauge?
In the gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$, $k^\mu H_{\mu\nu} = 0$

i.e., the waves are transverse.

Explain why there is a residual gauge freedom $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)}$ provided $\xi_\mu$ satisfies a certain condition. Show that this condition is satisfied by

$$\xi_\mu = \text{Re}(X_\mu e^{ik_\rho x^\rho})$$

where $X_\mu$ is constant.

A manifold $\mathcal{M}$ with metric $g$ and energy-momentum tensor $T$ is physically equivalent to $\mathcal{M}$ with $\phi^*(g)$ and energy momentum tensor $\phi^*(T)$ if $\phi$ is a diffeomorphism. Hence

$$(\phi^{-1})_*(g) = g + \mathcal{L}_\xi g + \cdots = \eta + \mathcal{L}_\xi \eta + \cdots$$

is equivalent to

$$g = \eta + h,$$

i.e., $h + \mathcal{L}_\xi \eta$ describe physically equivalent metric perturbations. Since $(\mathcal{L}_\xi \eta)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2 \partial_{(\mu} \xi_{\nu)}$

(think about the Killing equation), there is a residual gauge freedom

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)}.$$

The gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ needs to be satisfied. Note first that

$$h \rightarrow h + 2 \partial^\rho \xi_\rho.$$

$$\Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h_{\eta\mu\nu}$$

$$\rightarrow h_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)} - \frac{1}{2} h_{\eta\mu\nu} - \frac{1}{2} h_{\eta\mu\nu} (2 \partial^\rho \xi_\rho)$$

$$= \bar{h}_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)} - \eta_{\mu\nu} \partial^\rho \xi_\rho.$$

Hence we have

$$\partial^\rho \bar{h}_{\mu\nu} \rightarrow \partial^\rho \bar{h}_{\mu\nu} - \partial^\rho \partial_\mu \xi_\nu - \partial^\rho \partial_\nu \xi_\mu + \partial^\rho \partial^\rho \xi_\rho \eta_{\mu\nu}$$

$$= \partial^\rho \bar{h}_{\mu\nu} - \partial^\rho \partial_\mu \xi_\nu$$

$$= - \partial^\rho \partial_\mu \xi_\nu \quad \text{(by gauge condition)}$$

$$\Rightarrow 0.$$

This condition is satisfied by

$$\xi_\mu = \text{Re}(X_\mu e^{ik_\rho x^\rho})$$

since vacuum $\Rightarrow \partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0 \Rightarrow k_\rho k_\rho = 0$ with the given solution of $\bar{h}_{\mu\nu}$.

Now show that $X_\mu$ can be chosen to bring $H_{\mu\nu}$ to the form

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_{\times} & 0 \\ 0 & H_{\times} & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
First consider the null vector $k^\mu = \omega(1, 0, 0, 1)$. Then from the transverse gauge condition $\partial^\rho \bar{h}_{\rho\nu} = 0$, we have here
\[ i k^\rho H_{\rho\nu} = i k^0 H_{\rho0} + i k^3 H_{\rho3} = 0 \quad \Rightarrow \quad H_{\rho0} + H_{\rho3} = 0. \]

Now, recall that we had
\[ \bar{h}_{\rho\nu} \rightarrow \bar{h}_{\rho\nu} + 2 \partial_\rho \xi_\nu - \eta_{\rho\nu} \partial^\rho \xi_\mu, \]
\[ \Rightarrow H_{\rho\nu} \rightarrow H_{\rho\nu} + i(k^\rho X_\nu + k_\nu X_\rho - \eta_{\rho\nu} k^\rho X_\rho). \]

To achieve the longitudinal gauge $H_{0\nu} = 0$, set
\[ k^0 X_\nu + k_\nu X_0 - \eta_{0\nu} k^\rho X_\rho := i H_{0\nu}, \]
i.e.,
\[ k_0 X_0 + k_1 X_1 + k_2 X_2 + k_3 X_3 = i H_{00}; \]
\[ k_0 X_1 + k_1 X_0 = i H_{01}; \]
\[ k_0 X_2 + k_2 X_0 = i H_{02}; \]
\[ k_0 X_3 + k_3 X_0 = i H_{03}. \]

Or,
\[ \begin{pmatrix} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & 0 & 0 \\ k_2 & 0 & k_0 & 0 \\ k_3 & 0 & 0 & k_0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = i \begin{pmatrix} H_{00} \\ H_{10} \\ H_{20} \\ H_{30} \end{pmatrix}. \]

After row reduction, the last equation is 0 on both sides. Hence there exists multiple solutions for $X_\mu$. We can impose the traceless condition $H^\mu_\mu = 0$ to fix the gauge. Combining the results of the transverse and the longitudinal gauge conditions gives $H_{3\nu} = 0$. Hence,
\[ H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_x & 0 \\ 0 & H_x & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

Gravitational wave travels at the speed of light since $\partial^\rho \partial_\rho \bar{h}_{\rho\nu} = 0$ gives $k^\rho k_\rho = 0$. It is transverse since, as mentioned earlier, $\partial^\nu \bar{h}_{\rho\nu} = 0$ gives $k^\nu H_{\rho\nu} = 0$. And it has two independent polarisations as represented by $H_x$ and $H_+$.

(c) 
(i) Explain briefly why it is not possible to define an energy-momentum tensor for the gravitational field in General Relativity.

The energy in the gravitational field should be quadratic in the first derivatives of the metric, which in turn can always be set to zero at any point by a change of coordinates.

(ii) Explain how to define a symmetric tensor $t_{\mu\nu}$ that is quadratic in the linearized gravitational field $h_{\mu\nu}$ and conserved, i.e., $\partial^\rho t_{\mu\nu} = 0$.

The Einstein tensor to second order is
\[ G_{\mu\nu}[g] = G^{(1)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h]. \]

Assume that no matter is present, i.e., $G_{\mu\nu}[g] = 0$ and at first order, the linearised Einstein equation is $G^{(1)}[h] = 0$. At second order,
\[ G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi t_{\mu\nu}[h]. \]
where
\[ t_{\mu\nu}[h] := -\frac{1}{8\pi} G^{(2)}_{\mu\nu}[h] \]

Consider the contracted Bianchi identity \( \nabla^\mu G_{\mu\nu} = 0 \). At first order,
\[ \partial^\mu G^{(1)}_{\mu\nu}[h] = 0 \]
for arbitrary first order perturbation \( h \), i.e., if we replace \( h \) with \( h^{(2)} \) then \( \partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0 \). At second order,
\[ \partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] + \partial^\mu G^{(2)}_{\mu\nu}[h] + h G^{(1)}_{\mu\nu}[h] = 0 \]
where the third term vanished since we assumed that the equation of motion \( G^{(1)}[h] = 0 \) is obeyed.
\[ \Rightarrow \partial^\mu t_{\mu\nu} = 0 \]

Hence \( t_{\mu\nu} \) is a symmetric tensor that is (i) quadratic in the linear perturbation \( h \), (ii) conserved if \( h \) satisfies its equation of motion, and (iii) appears on the RHS of the second order Einstein equation.

(iii) Why is \( t_{\mu\nu} \) unsatisfactory as a definition of an energy-momentum tensor for \( h_{\mu\nu} \)?

\( t_{\mu\nu} \) is unsatisfactory as it is not invariant under the gauge transformation \( h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial(\xi) \). This is how the impossibility of localising gravitational energy arises in linearised theory.

4.

(a) (i) What is a Killing vector field?

A vector field \( X \) is a Killing vector field if
\[ \mathcal{L}_X g = 0, \]
i.e.,
\[ \nabla_a X_b + \nabla_b X_a = 0 \]

(ii) Show that if a spacetime admits a Killing vector field then along any geodesic there is a conserved quantity.

Let \( X^a \) be a Killing vector field and let \( V^a \) be tangent to an affinely parametrised geodesic. Then \( X_a V^a \) is constant along the geodesic.

**Proof:**
\[ \nabla_b (X_a V^a) = V^b \nabla_b (X_a V^a) = X_a V^b \nabla_b V^a + V^b V^a \nabla_b X_a = 0 \]
where \( V^b \nabla_b V^a = 0 \) as \( V^b \) is tangent to an affinely parametrised geodesic (it is the geodesic equation!), and the second term vanishes as \( V^b V^a \) is symmetric but \( \nabla_b X_a \) is antisymmetric by the Killing equation.

(iii) Write down 3 linearly independent Killing vector fields of the metric
\[ ds^2 = A(z)^2(-dt^2 + dx^2 + dy^2) + dz^2 \]
where \( A(z) \) is a positive function.

Let \( T := \frac{\partial}{\partial t} \). We claim that \( T \) is a Killing vector.

**Proof:**
\[ (\mathcal{L}_X g)_{\mu\nu} = X^a \partial_\mu g_{a\nu} + g_{ab} \partial_\mu X^b + g_{ab} \partial_\nu X^b = 0 \]
where the later two terms vanish since \( X^t = 1 \) while other \( X^\rho = 0 \), and the first term vanishes since \( \partial_{\rho \neq z} g_{\mu \nu} = 0 \) as the entries of \( g_{\mu \nu} \) depend only on \( z \), and for \( \rho = z \), this term vanishes still since \( X^z = 0 \). Similarly, \( X := \frac{\partial}{\partial z} \) and \( Y := \frac{\partial}{\partial y} \) are Killing vectors.

Note that if a Killing vector field \( X \) generates isometries, then \( \mathcal{L}_X g = 0 \). And since

\[
\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = 0,
\]

we can find a third Killing vector field by taking the commutator of the first two, given that \( X \) and \( Y \) are independent Killing vector fields, i.e., \( [X,Y] \neq 0 \).