• There are 4 questions on the exam. Attempt as many as you see fit.

• Keep in mind it is better to answer one or a few questions thoroughly than to provide many partial solutions. We are looking for quality over quantity.

• This is a 3 hour exam. If you wish to leave early, please be respectful of others still working.

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1 Quantum Field Theory

The massless Dirac equation is
\[ i \gamma^\mu \partial_\mu \psi(x) = 0 \]
where \( \psi(x) \) is the spinor wave function and
\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}
\end{align*}
\]
where \( 1 \) is the \( 2 \times 2 \) unit matrix and \( \sigma^i \) are the Pauli matrices. Compute \( \gamma^5 \) where
\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \]
Show that it anticommutes with \( \gamma^0 \) and \( \gamma^i \).

Consider the eigenstates of \( \gamma^5 \) and show that the Dirac equation becomes two, two-component equations.

Show that the left-handed spinor has plane wave solutions of the form
\[ \psi(x) = \lambda(p) e^{-ip \cdot x}. \]
By considering the condition on the four momentum, \( p^\mu \), for such a solution to exist, compute \( \lambda(p) \).

Consider spatial rotations about the three-momentum axis \( \mathbf{p} \) on your solution. What can you infer about the spin states of the particles of the quantised theory?
2 General Relativity

(a) Let $\omega_a$ be a 1-form and $V^a$ a vector field. Prove that

$$(\omega_b V^b)_a = (\mathcal{L}_V \omega)_a - V^b (d\omega)_{ab}.$$ 

You may assume that $(\mathcal{L}_V \omega)_\mu = V^\nu \omega_{\mu, \nu} + \omega_b V^b \omega_{\nu, \mu}$ in a coordinate basis.

(b) Show that a Killing vector field $K^a$ satisfies the equation

$$\nabla_a \nabla_b K^c = R^c_{\text{bad}} K^d.$$ 

You may assume the Ricci identity: $\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a_{\text{bcd}} Z^b$.

(c) In a four-dimensional spacetime, let $K^a$ and $V^a$ be commuting Killing vector fields and let $\omega_a$ be the twist of $K^a$, defined by

$$\omega_a = \epsilon_{abcd} K^b \nabla^c K^d$$

where the RHS involves the volume form $\epsilon_{abcd}$. Use the above results to prove that, in a vacuum spacetime, (i) $*\omega = 0$ and (ii) $\omega_a V^a$ is constant.

[You may assume that $(\star d \star X)_{ab} = \nabla^c X_{abc}$ for a 3-form $X_{abc}$.]
3 Cosmology

(a) Show that a shrinking comoving Hubble sphere, \( d(aH)^{-1}/dt < 0 \), is equivalent to

\[ \varepsilon \equiv -\frac{\dot{H}}{H^2} < 1 , \]

where overdots denote derivatives with respect to physical time \( t \), the function \( a(t) \) is the FRW scale factor and \( H \equiv \dot{a}/a \).

(b) A homogeneous scalar field \( \tilde{\phi}(t) \) in a flat Robertson-Walker spacetime satisfies the following equation of motion

\[ \ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} = -\frac{dV}{d\tilde{\phi}} , \quad \text{where} \quad H^2 = \frac{1}{3M_{pl}^2} \left[ \frac{1}{2} \dot{\tilde{\phi}}^2 + V(\tilde{\phi}) \right] . \]

\((*)\)

Show that

\[ \varepsilon = \frac{\frac{1}{2} \dot{\tilde{\phi}}^2}{M_{pl}^2 H^2} . \]

(c) Recall that the (dimensionless) power spectrum of comoving curvature perturbations is

\[ \Delta_s^2(k) = \left( \frac{H}{\tilde{\phi}} \right)^2 \times \left( \frac{H}{2\pi} \right)^2 , \]

\((**\))

where the right-hand side is evaluated at \( k = aH \). In spatially flat gauge, explain the qualitative origin of the two factors in eq. \((**\))

Use the slow-roll approximation of eq. \((*)\), to show that

\[ \Delta_s^2(k) = \frac{1}{12\pi^2} \frac{V^3}{(V')^2} , \]

where \( V' \equiv dV/d\tilde{\phi} \) and \( M_{pl} \equiv 1 \).

Using the horizon exit condition \( (k = aH, \text{ with slowly varying } H) \) to relate \( k \) to \( a \), and the slow-roll approximation to relate \( a \) to \( \dot{\tilde{\phi}} \), show that

\[ k \frac{d}{dk} \approx a \frac{d}{da} \approx -\frac{V'}{V} \frac{d}{d\tilde{\phi}} . \]

Use this to derive the following expression for the scalar spectral index

\[ n_s - 1 \equiv \frac{d\ln \Delta_s^2}{d\ln k} = 2 \frac{V''}{V} - 3 \left( \frac{V'}{V} \right) ^2 , \]

where \( V'' \equiv d^2V/d\tilde{\phi}^2 \).
4 Symmetries, Fields and Particles

Let $G$ be a matrix Lie group which is non-abelian and simple. Define the Lie algebra $L(G)$, and the (Lie) bracket on $L(G)$. What does it mean to say that $L(G)$ is of compact type?

A field theory, defined in Minkowski space, has a field $U$ with values in $G$ (i.e. $U(x) \in G$ for all $x$). The action is

$$S = -\int \text{Tr}(U^{-1} \partial_\mu U U^{-1} \partial^\mu U) \, d^4 x.$$ 

Determine, as far as you can, the symmetries of this action. Why is it important that $L(G)$ is of compact type?

The action $S$ can be modified so as to be gauge invariant under the transformations $U(x) \to g(x)U(x)g(x)^{-1}$, with $g(x) \in G$. One needs to introduce a gauge potential $A_\mu(x)$ and a covariant derivative

$$D_\mu U = \partial_\mu U + \alpha A_\mu U + \beta U A_\mu$$

where $\alpha$ and $\beta$ are real constants. Find the gauge transformation formula for $A_\mu(x)$ and the values of $\alpha$ and $\beta$ which ensure that $U^{-1}D_\mu U$ transforms to $g(U^{-1}D_\mu U)g^{-1}$. Deduce the modified form of $S$ that is gauge invariant. Describe the additional gauge invariant term that should be added to $S$ so that both $U$ and $A_\mu$ are dynamical.