Note that we use the metric convention \((-+++)\).

1.

The free Klein-Gordon field obeys the equation

\[-\partial_a \partial^a \phi + m^2 \phi = 0.\]

Using Noether's theorem find the expressions for the conserved energy and conserved three-momentum.

The Klein-Gordon Lagrangian is given by

\[\mathcal{L} = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2.\]

For a coordinate transformation \(x^a \rightarrow x^a - \epsilon^a\), we have the induced transformations

\[\phi(x) \rightarrow \phi(x) + \epsilon^a \partial_a \phi(x); \quad \mathcal{L} \rightarrow \mathcal{L} + \epsilon^a \partial_a \mathcal{L},\]

since \(\phi\) and \(\mathcal{L}\) are scalars. Then Noether's theorem tells us that

\[T^{ab} = -\frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \partial^b \phi + \eta^{ab} \mathcal{L}.\]

So here we have

\[T^{ab} = \partial^a \phi \partial^b \phi - \frac{1}{2} \eta^{ab} (\partial_c \phi \partial^c \phi + m^2 \phi^2).\]

Note that \(\partial_c = (\partial_0, \partial_i)\) and \(\partial^c = (-\partial^0, \partial^i)\). Then the conserved energy

\[E = \int d^3x T^{00} = \frac{1}{2} \int d^3x \left(\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2\right).\]

And the conserved momentum

\[\vec{P} = \int d^3x T^{0i} = \int d^3x \dot{\phi} \nabla \phi.\]

Note that indeed,

\[\partial_a \int d^3x T^{0a} = \frac{1}{2} \int d^3x \partial_a \left(\dot{\phi} \partial^a \phi - \eta^{0a} \left(-\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2\right)\right)\]

\[= \int d^3x \left(\frac{1}{2} \ddot{\phi} \phi - \frac{1}{2} \dot{\phi} \phi + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} \dot{\phi} \nabla^2 \phi - \nabla \phi \cdot \nabla \dot{\phi} + m^2 \phi \phi\right)\]

\[= \int d^3x \left(-\ddot{\phi} + \nabla^2 \phi - m^2 \phi\right) \dot{\phi}\]

\[= 0,\]

using the KG equation.
In the quantised theory the field $\phi(x)$ and the conjugate field $\pi(x)$ can be expressed as

$$
\phi(x) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left( a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right);
$$

$$
\pi(x) = (-i) \int \frac{d^3 \vec{p}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \left( a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right),
$$

where $E_{\vec{p}} = (\vec{p}^2 + m^2)^{1/2}$. Write down the commutation relations satisfied by $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$. Show that the Hamiltonian and $\vec{P}$ can be expressed as

$$
H = \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}};
$$

$$
\vec{P} = \int \frac{d^3 \vec{p}}{(2\pi)^3} \vec{p} a_{\vec{p}}^\dagger a_{\vec{p}}
$$

in the quantised theory.

We have

$$
[a_{\vec{p}}, a_{\vec{q}}] = 0 = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger];
$$

$$
[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).
$$

Now,

$$
\pi := \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}.
$$

Hence

$$
H = \int d^3 x \left( \pi^2 - \mathcal{L} \right)
$$

$$
= \frac{1}{2} \int d^3 x \left( \pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right),
$$

where

$$
\int d^3 x \pi^2 = - \int d^3 x \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \sqrt{\frac{E_{\vec{q}}}{2}} \left( a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \left( a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}} \right)
$$

$$
= - \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \sqrt{\frac{E_{\vec{q}}}{2}} \left( a_{\vec{p}} a_{\vec{q}} \delta^3(\vec{p} + \vec{q}) - a_{\vec{p}} a_{\vec{q}}^\dagger \delta^3(\vec{p} - \vec{q}) - a_{\vec{p}}^\dagger a_{\vec{q}} \delta^3(\vec{p} - \vec{q}) + a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger \delta^3(\vec{p} + \vec{q}) \right)
$$

$$
= - \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{E_{\vec{p}}}{2} \left( a_{\vec{p}} a_{\vec{p}} - a_{\vec{p}} a_{\vec{p}}^\dagger - a_{\vec{p}}^\dagger a_{\vec{p}} + a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger \right),
$$

since $E_{\vec{p}} = E_{|\vec{p}|}$. Similarly,

$$
\int d^3 x (\nabla \phi)^2 = - \int d^3 x \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}} \sqrt{2E_{\vec{q}}}} \left( \vec{p} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - \vec{p} a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \cdot \left( \vec{q} a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - \vec{q} a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}} \right)
$$

$$
= - \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}} \sqrt{2E_{\vec{q}}}} \vec{p} \cdot \vec{q} \left( a_{\vec{p}} a_{\vec{q}} \delta^3(\vec{p} + \vec{q}) - a_{\vec{p}} a_{\vec{q}}^\dagger \delta^3(\vec{p} - \vec{q}) - a_{\vec{p}}^\dagger a_{\vec{q}} \delta^3(\vec{p} - \vec{q}) + a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger \delta^3(\vec{p} + \vec{q}) \right)
$$

$$
= \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{E_{\vec{p}}}{2} \left( a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} + a_{\vec{p}}^\dagger a_{\vec{p}}^\dagger \right),
$$

2
and
\[\int d^3x \, m^2 \phi^2 = m^2 \int d^3x \, \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}} \sqrt{2E_{\vec{q}}}} \left( a_{\vec{p}} e^{i\vec{q} \cdot \vec{x}} + a_{\vec{p}}^+ e^{-i\vec{q} \cdot \vec{x}} \right) \left( a_{\vec{q}} e^{i\vec{p} \cdot \vec{x}} + a_{\vec{q}}^+ e^{-i\vec{p} \cdot \vec{x}} \right)\]
\[= m^2 \int d^3\vec{p} \, \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}} \sqrt{2E_{\vec{q}}}} \left( a_{\vec{p}} a_{\vec{q}} \delta^3(\vec{p} + \vec{q}) + a_{\vec{p}} a_{\vec{q}}^+ \delta^3(\vec{p} - \vec{q}) + a_{\vec{p}}^+ a_{\vec{q}} \delta^3(\vec{p} - \vec{q}) + a_{\vec{p}}^+ a_{\vec{q}}^+ \delta^3(\vec{p} + \vec{q}) \right)\]
\[= m^2 \int d^3\vec{p} \frac{1}{2E_{\vec{p}}} \left( a_{\vec{p}} a_{-\vec{p}}^+ + a_{\vec{p}} a_{\vec{p}} + a_{\vec{p}}^+ a_{-\vec{p}} + a_{\vec{p}}^+ a_{\vec{p}}^+ \right).\]

Hence
\[H = \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left( \left( a_{\vec{p}} a_{-\vec{p}}^+ + a_{\vec{p}} a_{-\vec{p}}^+ \right) \left( -E_{\vec{p}}^2 + p^2 + m^2 \right) + \left( a_{\vec{p}} a_{\vec{p}}^+ + a_{\vec{p}} a_{\vec{p}}^+ \right) \left( E_{\vec{p}}^2 + p^2 + m^2 \right) \right)\]
\[= \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left( a_{\vec{p}} a_{\vec{p}}^+ + a_{\vec{p}} a_{\vec{p}}^+ \right) 2E_{\vec{p}}^2\]
\[= \frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \left( a_{\vec{p}} a_{\vec{p}}^+ + a_{\vec{p}}^+ a_{\vec{p}} \right)\]
\[= \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \left( a_{\vec{p}} a_{\vec{p}}^+ + \frac{1}{2} (2\pi)^3 \delta^3(0) \right)\]
\[= \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} a_{\vec{p}} a_{\vec{p}}^+ ,\]

ignoring the infinite term. Note that we could have also proceeded with
\[H = \int d^3x \, T^{00}.\]

We also have from earlier,
\[\vec{P} = \int d^3x \, \vec{\phi} \nabla \phi = \int d^3x \, \vec{\pi} \nabla \phi\]
\[= \int d^3x \, \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2 \sqrt{2E_{\vec{q}}}}} \left( a_{\vec{p}} e^{i\vec{q} \cdot \vec{x}} - a_{\vec{p}}^+ e^{-i\vec{q} \cdot \vec{x}} \right) \left( \bar{q} a_{\vec{q}} e^{i\vec{p} \cdot \vec{x}} - \bar{q} a_{\vec{q}}^+ e^{-i\vec{p} \cdot \vec{x}} \right)\]
\[= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2 \sqrt{2E_{\vec{q}}}}} \left( a_{\vec{p}} a_{\vec{q}} \delta^3(\vec{p} + \vec{q}) - a_{\vec{p}} a_{\vec{q}}^+ \delta^3(\vec{p} - \vec{q}) - a_{\vec{p}}^+ a_{\vec{q}} \delta^3(\vec{p} - \vec{q}) + a_{\vec{p}}^+ a_{\vec{q}}^+ \delta^3(\vec{p} + \vec{q}) \right)\]
\[= -\int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} \left( a_{\vec{p}} a_{-\vec{p}}^+ + a_{\vec{p}} a_{\vec{p}} + a_{\vec{p}}^+ a_{-\vec{p}} + a_{\vec{p}}^+ a_{\vec{p}}^+ \right)\]
\[= -\int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} \left( a_{\vec{p}} a_{\vec{p}}^+ + \frac{1}{2} (2\pi)^3 \delta^3(0) \right) ,\]

where the two terms in the second last line drop out since we’re integrating over all space and \(\vec{p} a_{\vec{p}} a_{-\vec{p}}\) and \(\vec{p} a_{\vec{p}}^+ a_{-\vec{p}}^+\) make the integral odd.

Calculate \([H, a_{\vec{p}}^+]\) and \([\vec{P}, a_{\vec{p}}^+]\). Discuss the particle content of the theory. Show that the Klein-Gordon field obeys Bose-Einstein statistics.

We have
\[\left[H, a_{\vec{p}}^+ \right] = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} a_{\vec{p}}^+ a_{\vec{p}}^+ = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} a_{\vec{p}}^+ a_{\vec{p}} a_{\vec{p}}^+ = E_{\vec{p}} a_{\vec{p}}^+.\]
Similarly,
\[\left[\vec{P}, a_{\vec{p}}^+ \right] = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} a_{\vec{p}}^+ a_{\vec{p}}^+ = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} a_{\vec{p}}^+ a_{\vec{p}} a_{\vec{p}}^+ = \vec{p} a_{\vec{p}}^+.\]
The theory is about bosons. For example, for a two-particle state \( a_\vec{p}^\dagger a_\vec{q}^\dagger \langle 0 \rangle \),
\[
[a_\vec{p}^\dagger, a_\vec{q}^\dagger] = 0 \Rightarrow a_\vec{p}^\dagger a_\vec{q}^\dagger \langle 0 \rangle = a_\vec{q}^\dagger a_\vec{p}^\dagger \langle 0 \rangle.
\]

The state is symmetric under the exchange of \( p \) and \( q \), so the wavefunction of two identical bosons is symmetric under the interchange of particles, implying Bose-Einstein statistics. And since
\[
\phi(\vec{x}) \mid 0 \rangle = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\vec{p}}} (a_\vec{p}^\dagger \psi(\vec{x}) + a_\vec{p}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \mid 0 \rangle,
\]
basically creates a particle at \( \vec{x} \) using \( a_\vec{p}^\dagger \), the KG field \( \phi(\vec{x}) \) obeys Bose-Einstein statistics. Note that we also have \([\phi(\vec{x}), \phi(\vec{y})] = 0\), so the above argument using commutations relations of \( a_\vec{p}^\dagger \) also applies to \( \phi(\vec{x}) \).

2.

The Lagrangian density for a scalar particle interacting with a fermion is
\[
\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2} m^2 \phi^2 - i \bar{\psi} (\gamma^\alpha \partial_\alpha + m) \psi + g \bar{\psi} \phi \psi.
\]
Write down the field equations satisfied by \( \phi, \psi \), and \( \bar{\psi} \).

\[
-\partial_\alpha \partial^\alpha \phi + m^2 \phi = 0 \Rightarrow (-\partial_\alpha \partial^\alpha + m^2)\phi = 0;
-\gamma^\alpha \partial_\alpha \psi + g \phi \psi = 0 \Rightarrow (\gamma^\alpha \partial_\alpha - im + g \phi)\psi = 0;
\gamma^\alpha \partial_\alpha \bar{\psi} - im \bar{\psi} + g \phi \bar{\psi} = 0 \Rightarrow (\gamma^\alpha \partial_\alpha - im + g \phi)\bar{\psi} = 0.
\]

State the Feynman rules. Draw the tree-level Feynman diagrams for the processes \( \psi\psi \rightarrow \psi\psi, \psi\bar{\psi} \rightarrow \psi\bar{\psi} \) and \( \psi\bar{\psi} \rightarrow \psi\phi \). Write down the respective scattering amplitudes.

The following is a full list. Those not applicable to this problem are marked with (*).

1. Draw a Feynman diagram of the process and put momenta on each line consistent with momentum conservation.

2. Associate with each internal propagator

\[
\frac{-i}{p^2 + m^2 - i\epsilon} \quad \text{(scalar propagator)};
\]

\[
\frac{-i(-p + m)}{p^2 + m^2 - i\epsilon} \quad \text{(fermion propagator)};
\]

\[
\frac{-i\eta_{ab}}{p^2 - i\epsilon} \quad \text{(photon propagator)}.
\]

3. Associate vertices with coupling constants obtained from the interaction term in the Lagrangian, and impose momentum conservation at each vertex and overall, with
\[
(2\pi)^4 \delta^4 \left( \sum p_{\text{in}} - \sum p_{\text{out}} \right).
\]

We omit writing this every time.
4. Integrate over momenta associated with loops, with the measure

\[ \frac{d^4 k}{(2\pi)^4}. \]

5. Add in wavefunction terms for external particles of momentum \( p \) and spin \( s \):

- \( u(p, s) \) (incoming fermions);
- \( \bar{u}(p, s) \) (outgoing fermions);
- \( \bar{v}(p, s) \) (incoming antifermions);
- \( v(p, s) \) (outgoing antifermions);
- \( \epsilon_a \) (incoming photons);
- \( \epsilon_a^* \) (outgoing photons).

6. Find the overall sign of the diagram from the commutation of fermions.

7. Multiply by the number of ways the diagram can be connected and divide by \( n! \), where \( n \) is the number of vertices in the diagram.

Only t- and u-channel contribute to the scattering amplitudes for \( \psi\psi \rightarrow \psi\psi \). Hence,

\[
iA = (-ig)^2 \bar{u}_3 u_1 \frac{-i}{(p_1 - p_3)^2 + m^2 - i\epsilon} \bar{u}_4 u_2 - (-ig)^2 \bar{u}_4 u_1 \frac{-i}{(p_1 - p_4)^2 + m^2 - i\epsilon} \bar{u}_3 u_2.\]

Only s- and t-channel contribute to the scattering amplitudes for \( \psi\bar{\psi} \rightarrow \psi\bar{\psi} \). Hence,

\[
iA = (-ig)^2 \bar{u}_3 v_1 \frac{-i}{(p_1 + p_2)^2 + m^2 - i\epsilon} \bar{v}_2 u_1 - (-ig)^2 \bar{u}_3 u_1 \frac{-i}{(p_1 - p_3)^2 + m^2 - i\epsilon} \bar{v}_3 v_1.\]
For $\psi \phi \rightarrow \psi \phi$,

$$iA = (-ig)^2\bar{u}_3 u_1 \frac{-i(-\not{\bar{p}}_1 + \not{p}_2) + m}{(p_1 + p_2)^2 + m^2 - i\epsilon} + (-ig)^2\bar{u}_3 u_1 \frac{-i(-\not{\bar{p}}_1 - \not{p}_4) + m}{(p_1 - p_4)^2 + m^2 - i\epsilon}.$$ 

Note that we only include a minus sign between diagrams that differ only in the exchange of two identical fermions.

3.

The Lagrangian for a scalar field of charge $e$ interacting with the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} + (D_a \phi)^* D^a \phi - m^2 \phi^* \phi$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and $D_a = \partial_a - iqA_a$. Show that this Lagrangian has a gauge symmetry.

Under a gauge transformation of the field,

$$\phi(x) \rightarrow e^{iq\lambda(x)} \phi(x), \quad \phi^*(x) \rightarrow e^{-iq\lambda(x)} \phi^*(x),$$

we postulate that

$$A_a(x) \rightarrow A_a(x) + \partial_a \lambda(x)$$

guarantees the covariance of the transformations of $D_a \phi$ and $(D_a \phi)^*$. Indeed,

$$D^a \phi \rightarrow (\partial^a - iqA^a - iq\partial^a \lambda) e^{iq\lambda} \phi$$

$$= iq\partial^a A e^{iq\lambda} \phi + e^{iq\lambda} \partial^a \phi - iqA^a e^{iq\lambda} \phi - iq\partial^a \Lambda e^{iq\lambda} \phi$$

$$= e^{iq\lambda} D^a \phi$$

Similarly,

$$(D_a \phi)^* \rightarrow e^{-iq\lambda} (D^a \phi)^*$$

Hence $(D_a \phi)^* D^a \phi \rightarrow (D_a \phi)^* D^a \phi$. Clearly, $\phi^* \phi \rightarrow \phi^* \phi$. And we also have

$$F_{ab} \rightarrow F_{ab} + \partial_a \partial_b \Lambda - \partial_b \partial_a \Lambda = F_{ab}.$$

Hence $\mathcal{L}$ has a gauge symmetry.

Draw the two interaction vertices in this theory, identifying the corresponding interaction terms in the Lagrangian.

We have

$$\mathcal{L} = -\frac{1}{4} (\partial_a A_b - \partial_b A_a) (\partial^a A^b - \partial^b A^a) + (\partial_a + iqA_a) \phi^* (\partial^a - iqA^a) \phi - m^2 \phi^* \phi.$$ 

The two interaction terms are

$$iqA_a [\phi^* \partial^a \phi - (\partial^a \phi^*) \phi] \quad \text{and} \quad q^2 A_a \phi^* \phi.$$

They respectively correspond to
When one quantises the theory in Coulomb gauge, $\nabla \cdot \vec{A} = 0$, the naive photon propagator is

$$\Delta_{ab}(p) = \begin{cases} \frac{i}{|p|^2} & a = b = 0 \quad \text{(time-time)} \\ \frac{i}{p^2 - i\epsilon} \left( \delta_{ij} - \frac{p_i p_j}{|p|^2} \right) & a = i, b = j \quad \text{(space-space)} \\ 0 & \text{otherwise (mixed time/space)} \end{cases}$$

Draw the leading order diagrams for $\phi \bar{\phi} \rightarrow \phi \bar{\phi}$ scattering. Show that your answer can be expressed in terms of the Lorentz invariant propagator

$$\Delta_{ab}(p) = \frac{-i\eta_{ij}}{p^2 - i\epsilon}$$

suitably contracted with external momenta.

See Tong’s notes p142 for the proof of the equivalence.

4.

Write an essay on symmetries in field theory. Your essay should include a statement and proof of Noether’s theorem; give examples of important symmetries in different field theories; describe the difference between a global symmetry and a gauge symmetry.

Noether’s theorem connects symmetries with conservation laws. Suppose that there is a continuous symmetry of a Lagrangian system (given a solution of the equations of motion $\phi, \phi + \epsilon \delta \phi$ is also a solution for some small parameter $\epsilon$), then

- There is a conserved current $J^a(x)$ such that $\partial_a J^a = 0$.
- The conserved current gives rise to a conserved charge, $Q = \int_{R^3} d^3 x \cdot J^0$.

**Proof:** A solution of the equations of motion requires that $\delta I = 0$ if $\phi \rightarrow \phi + \delta \phi$. $\delta I = 0$ is the condition for symmetry, and to achieve this, we need $\delta \mathcal{L} = \partial_a F^a$ for some $F^a$, since

$$\int d^3 x \partial_a F^a = \int dt \int_{\Sigma} F \cdot dS = 0,$$
since $S$ vanishes at infinity.

Varying the Lagrangian,

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \delta (\partial_a \phi)$$

$$= \partial_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \right) \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \delta (\partial_a \phi)$$

$$= \partial_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \delta \phi \right),$$

using the Euler-Lagrange equations and the fact that $\phi$ obeys the EOM. Let the Noether current

$$J^a = F^a - \frac{\partial \mathcal{L}}{\partial (\partial_a \phi)} \delta \phi.$$

If we have a symmetry then this must be conserved, i.e., $\partial_a J^a = 0$. Also,

$$0 = \int_D d^4x \partial_a J^a = \int_{t=t_i}^{t_f} d^3x J^0 = Q_f - Q_i$$

where $Q_f, Q_i$ are Noether charges.

A global symmetry is a “real” symmetry where the parameter is independent of space-time coordinates whereas a gauge symmetry can be viewed as merely a redundancy of the system. For example, in Maxwell’s theory, the gauge symmetry $A_a \rightarrow A_a + \partial_a \Lambda$ is not a global symmetry.

**Example: Complex scalar field**

$$\mathcal{L} = -\partial_a \phi^* \partial^a \phi - m^2 \phi \phi^*$$

$$\Rightarrow \partial^2 \phi - m^2 \phi = 0$$

Let $\phi \rightarrow e^{i\lambda} \phi$ for some $\lambda = \text{constant}$. This transformation leaves the Lagrangian invariant, hence there must be a Noether current associated with this symmetry. It is easily found to be

$$J_a = i(\phi^* \partial_a \phi - \partial_a \phi^* \phi)$$

**Example: Energy-momentum tensor**

Carry out a constant coordinate transformation $x^a \rightarrow x^a - \epsilon^a$, so that the induced transformation for the real scalar field is

$$\phi(x) \rightarrow \phi(x) + e^b \partial_b \phi(x).$$

The Lagrangian $\mathcal{L} = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2$ is also a scalar so it transforms as

$$\mathcal{L} \rightarrow \mathcal{L} + e^b \partial_b \mathcal{L}.$$

The Noether current is given by

$$J^a = e^b \frac{\partial \mathcal{L}}{\partial (\partial_b \phi)} \partial_b \phi - e^b \delta^a_b \mathcal{L}.$$
\[ P^i = \text{Total momentum} = \int d^3x T^{0i}. \]

Note that in general relativity, the energy-momentum tensor is usually found by varying the action with respect to the metric.

\[ T_{ab} = -\frac{2}{\sqrt{-\det g}} \frac{\delta I}{\delta g_{ab}}. \]

This is a different definition of the energy-momentum tensor, but coincides with the Noether definition for scalars. For electromagnetism however, the two definitions are not the same.