• There are 4 questions on the exam. Attempt as many as you see fit.

• Keep in mind it is better to answer one or a few questions thoroughly than to provide many partial solutions. We are looking for quality over quantity.

• This is a 4 hour exam. If you wish to leave early, please be respectful of others still working.

<table>
<thead>
<tr>
<th>Question order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Quantum Field Theory</td>
</tr>
<tr>
<td>2 General Relativity</td>
</tr>
<tr>
<td>3 Cosmology</td>
</tr>
<tr>
<td>4 Symmetries, Fields and Particles</td>
</tr>
</tbody>
</table>
1 Quantum Field Theory

Suppose that one has a real scalar field $\phi$ with Lagrangian given by

$$L = -\frac{1}{2} \partial_a \phi \partial_b \phi \eta^{ab} - \frac{1}{2} m^2 \phi^2$$

where $m$ is the mass of the field.

What is the canonical momentum $\pi_\phi$ conjugate to $\phi$?

What is the Hamiltonian $H$ for this system?

What are the Poisson brackets in classical mechanics between $H$, $\phi$ and $\pi_\phi$.

Explain how one moves from the classical theory to the quantum theory.

By using canonical methods, evaluate

$$K(x, y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle.$$

explaining carefully the physical meaning of your result.

Starting from the time evolution operator for this system, show how to derive the path integral representation of the same object.

Suppose that the scalar field is massless, $m^2 = 0$, find an explicit expression for $K(x, y)$ for the case where $x$ and $y$ are spacelike separated. Comment on your result and Lorentz invariance.

*You may find it useful to use the fact that*

$$\int_0^\infty du \ e^{iuA - e^{iuB}} = -i \left( \frac{A - B}{AB} \right) \text{ for } A, B \text{ both real}$$
2 General Relativity

Let $\mathcal{M}$ be a manifold with metric $g$ and with covariant derivative given by the Levi-Civita connection.

(a) How is parallel transport of a tensor $T$ along a vector field $V$ defined in terms of the covariant derivative?

(b) Let $\phi = \{x^\mu\}$ be a coordinate chart on $\mathcal{M}$. Derive the geodesic equations expressed in coordinates from the property that the tangent vector of a geodesic is parallel propagated along itself.

For the remainder of this questions consider the specific metric

$$ds^2 = -\frac{1}{z} dt^2 + z^2 (dx^2 + dy^2) + z \, dz^2,$$

where $t$, $x$, $y$, $z \in \mathbb{R}$ and $z > 0$.

(c) Calculate the non-vanishing Christoffel symbols from the Euler-Lagrange equations of the geodesic Lagrangian.

(d) Determine four independent constants of motion along the geodesic and use these to show that the geodesic equations can be reduced to a single equation that describes the motion of a particle with energy $E$ in a potential $U(z)$, i.e.

$$z^2 + U(z) = E,$$

where the “dot” denotes a derivative $d/ds$ with respect to the parameter $s$ along the geodesic. What are the energy $E$ and potential $U(z)$ in terms of the constants of motion?

(e) Discuss the qualitative behaviour of a massless (null) particle falling in from large but finite $z_0$ with positive energy and an initially non-vanishing velocity component in the $x$ or $y$ direction, i.e. initial velocity components $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ such that $\dot{x}_0^2 + \dot{y}_0^2 > 0$ and $\dot{z}_0 < 0$.

(f) Calculate the geodesic for a massless particle with $\dot{x}_0 = \dot{y}_0 = 0$ initially (but $\dot{z}_0 < 0$) and contrast the behaviour with the case discussed in (e).
3 Cosmology

Consider the Robertson-Walker line element for a flat universe

\[ ds^2 = a^2(\tau) \left[ d\tau^2 - \delta_{ij} dx^i dx^j \right] , \]

where \( \tau \) is conformal time.

(a) If the universe is filled with a perfect fluid whose equation of state \( w = P/\rho \) is a constant, show that \( a \propto \tau^{2/(1+3w)} \), for \( w \neq -1/3 \), and hence \( a^2 \bar{\rho} \propto \tau^{-2} \), where \( \bar{\rho} \) is the background energy density.

(b) Now consider scalar perturbations in the fluid. If the perturbed line element is written as

\[ ds^2 = a^2(\tau) \left[ (1 + 2\Phi) d\tau^2 - (1 - 2\Phi) \delta_{ij} dx^i dx^j \right] , \]

then the linearised Einstein equations are

\begin{align*}
\nabla^2 \Phi - 3H(\Phi' + H\Phi) &= 4\pi G a^2 \bar{\rho} \delta , \\
\Phi' + H\Phi &= -4\pi G a^2 (\bar{\rho} + \bar{P}) v , \\
\Phi'' + 3H\Phi' + (2H' + H^2)\Phi &= 4\pi G a^2 \delta P ,
\end{align*}

(E1) (E2) (E3)

where primes denote derivatives with respect to \( \tau \), overbars denote homogeneous background quantities, \( \delta \equiv \delta \rho/\rho \) is the fractional density perturbation, \( \delta P \) is the pressure perturbation, \( \partial_i v \) is the peculiar velocity and \( H \equiv a'/a \).

Use the Einstein equations to show that the density contrast in comoving gauge, \( \Delta \equiv \delta - 3H(1 + \bar{P}/\bar{\rho}) v \), evolves as

\[ \Delta'' + \frac{2(1 - 3w)}{1 + 3w} \frac{1}{\tau} \Delta' - \frac{6(1 - w)}{1 + 3w} \frac{1}{\tau^2} \Delta - w \nabla^2 \Delta = 0 \]

(c) Find the growing mode solution for \( \Delta \) in the case of a pressureless fluid \( (w = 0) \). How do \( \Phi \) and \( v \) evolve with the scale factor \( a \)?

(d) For a radiation fluid \( (w = \frac{1}{3}) \), find the solution for \( \Delta \) in Fourier space for a mode well inside the Hubble radius. How does the gravitational potential \( \Phi \) evolve in this case?

(e) The real universe is filled with a mixture of radiation \( (r) \), matter \( (m) \) and dark energy \( (\Lambda) \). Describe qualitatively the evolution of short-wavelength dark matter perturbations from early times (radiation domination) to late times (dark energy domination).
4 Symmetries, Fields and Particles

Let $L(G)$ be the Lie algebra of a matrix Lie group $G$ for which the inner product on $L(G)$,

$$(X, Y) = \text{Tr}(XY),$$

is negative definite. Explain why

$$ds^2 = -\text{Tr}(dgg^{-1}dgg^{-1})$$

is a Riemannian metric on $G$, and establish (as far as you can) its symmetry properties. Write down the Lagrangian for the free motion of a particle on $G$, using this metric. Derive the equation of motion, and show that the general solution is

$$g(t) = g_0 \exp(tX_0)$$

where $g_0 \in G$ and $X_0 \in L(G)$. How do $g_0$ and $X_0$ change under the left and right action of $G$?

For $G=SU(2)$, and any non-zero $X_0$, show that the solution is always periodic.

For $G=SU(3)$, is the solution ever periodic? Is it always periodic? Illustrate your answers by examples.

Describe briefly some further mathematical and physical consequences of the inner product $(X, Y)$ being negative definite.