1 Ch.2 - Manifolds and Tensors

Definition 1.1. A differentiable manifold is a set $M$ together with a collection of subsets $O_\alpha$ such that
1. $\cup_\alpha O_\alpha = M$, i.e., the subsets $O_\alpha$ cover $M$.
2. For each $\alpha$ there is a one-to-one and onto map $\phi_\alpha : O_\alpha \rightarrow U_\alpha$ where $U_\alpha$ is an open subset of $\mathbb{R}^n$. $\phi_\alpha$ are called charts and the set $\{\phi_\alpha\}$ is called an atlas.
3. If $O_\alpha$ and $O_\beta$ overlap, i.e., $O_\alpha \cap O_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \subset \mathbb{R}^n \rightarrow \phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta \subset \mathbb{R}^n$.

We require that this map be smooth (infinitely differentiable).

Definition 1.2. A function $f : M \rightarrow \mathbb{R}$ is smooth iff for any chart $\phi : O \rightarrow U$, $F := f \circ \phi^{-1} : U \rightarrow \mathbb{R}$ is a smooth function.

Definition 1.3. Let $\lambda : I \rightarrow M$ be a smooth curve with $\lambda(0) = p$. The tangent vector to $\lambda$ at $p$ is the linear map $X_p$ form the space of smooth functions on $M$ to $\mathbb{R}$ defined by $X_p(f) = \frac{d}{dt}
|_{t=0} f(\lambda(t))$.

Proposition 1.4. The set of all tangent vectors at $p$ forms an $n$-dimensional vector space, the tangent space $T_pM$.

Definition 1.5. Let $\{e^\mu, \mu = 1,...,n\}$ be a basis for $T_pM$. We can expand any vector $X \in T_pM$ as $X = X^\mu e^\mu$. We call the numbers $X^\mu$ the components of $X$ with respect to this basis.

Definition 1.6. Let $V$ be a real vector space. The dual space $V^*$ of $V$ is the vector space of linear maps from $V$ to $\mathbb{R}$.

Lemma 1.7. If $V$ is $n$-dimensional then so is $V^*$. If $\{e_\mu, \mu = 1,...,n\}$ is a basis for $V$ then $V^*$ has a basis $\{f^\mu, \mu = 1,...,n\}$, the dual basis defined by $f^\mu e_\nu = \delta_\mu^\nu$.

Theorem 1.8. If $V$ is finite dimensional then $(V^*)^*$ is naturally isomorphic to $V$. The isomorphism is $\Phi : V \rightarrow (V^*)^*$ where $\Phi(X)(\omega) = \omega(X)$ for all $\omega \in V^*$.

Definition 1.9. The dual space of $T_pM$ is denoted $T^*_pM$ and called the cotangent space at $p$. If $\{e_\mu\}$ is a basis for $T_pM$ and $\{f^\mu\}$ is the dual basis then we can expand a covector $\eta$ as $\eta_\mu f^\mu$.

Definition 1.10. Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Define a covector $(df)_p$ by $(df)_p(X) = X(f)$ for any vector $X \in T_pM$. $(df)_p$ is the gradient of $f$ at $p$. 1
Definition 1.11. A tensor of type \((r, s)\) at \(p\) is a multilinear map

\[ T : T_p^r M \times \ldots \times T_p^r M \times T_p M \times \ldots \times T_p M \to \mathbb{R} \]

where there are \(r\) factors of \(T_p^r M\) and \(s\) factors of \(T_p M\). In other words, given \(r\) covectors and \(s\) vectors, a tensor of type \((r, s)\) produces a real number.

Definition 1.12. Let \(T\) be a tensor of type \((r, s)\) at \(p\). If \(\{e_\mu\}\) is a basis for \(T_p M\) with dual basis \(\{f^\mu\}\) then the components of \(T\) in this basis are the numbers

\[ T^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s} = T(f^{\mu_1}, \ldots, f^{\mu_r}, e_{\nu_1}, \ldots, e_{\nu_s}). \]

Definition 1.13. If \(S\) is a tensor of type \((p, q)\) and \(T\) is a tensor of type \((r, s)\) then the outer product of \(S\) and \(T\), denoted \(S \otimes T\) is a tensor of type \((p + r, q + s)\) defined by

\[ (S \otimes T)(\omega_1, \ldots, \omega_p, \eta_1, \ldots, \eta_r, X_1, \ldots, X_q, Y_1, \ldots, Y_s) = S(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q)T(\eta_1, \ldots, \eta_r, Y_1, \ldots, Y_s). \]

Definition 1.14. A vector field is a map \(X\) which maps any point \(p \in M\) to a vector \(X_p\) at \(p\). Given a vector field \(X\) and a function \(f\) we can define a new function \(X(f) : M \to \mathbb{R}\) by \(X(f) : p \mapsto X_p(f)\). The vector field \(X\) is smooth if this map is a smooth function for every smooth function \(f\).

Definition 1.15. A covector field is a map \(\omega\) which maps any point \(p \in M\) to a covector \(\omega_p\) at \(p\). Given a covector field \(\omega\) and a vector field \(X\) we can define a function \(\omega(X) : M \to \mathbb{R}\) by \(\omega(X) : p \mapsto \omega_p(X_p)\). The covector field \(\omega\) is smooth for any smooth vector field \(X\).

Definition 1.16. An \((r, s)\) tensor field is a map \(T\) which maps any point \(p \in M\) to an \((r, s)\) tensor \(T_p\) at \(p\). Given \(r\) covector fields \(\eta_1, \ldots, \eta_r\) and \(s\) vector fields \(X_1, \ldots, X_s\), we can define a function \(T(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) : M \to \mathbb{R}\) by \(p \mapsto T_p((\eta_1)_p, \ldots, (\eta_r)_p, (X_1)_p, \ldots, (X_s)_p)\). The tensor field \(T\) is smooth if this function is smooth for any smooth covector field \(\eta_1, \ldots, \eta_r\) and vector fields \(X_1, \ldots, X_s\).

Definition 1.17. The commutator of two vector fields \(X\) and \(Y\) is the vector field \([X, Y]\) defined by

\[ [X, Y](f) = X(Y(f)) - Y(X(f)) \]

for any smooth function \(f\).

Definition 1.18. Let \(X\) be a vector field on \(M\) and \(p \in M\). An integral curve of \(X\) through \(p\) whose tangent at every point is \(X\). If we let \(\lambda\) denote an integral curve of \(X\) with \(\lambda(0) = p\), then in a coordinate chart, this definition reduces to the initial value problem

\[ \frac{dx^\mu(t)}{dt} = X^\mu(x(t)), \quad x^\mu(0) = x^\mu_p. \]

2 Ch.3 - The Metric Tensor

Definition 2.1. A metric tensor at \(p \in M\) is a \((0, 2)\) tensor \(g\) with the following properties:

1. It is symmetric: \(g(X, Y) = g(Y, X)\) for all \(X, Y \in T_p M\), i.e., \(g_{ab} = g_{ba}\).
2. It is non-degenerate: \(g(X, Y) = 0\) for all \(Y \in T_p M\) iff \(X = 0\).

Definition 2.2. A Riemannian (Lorentzian) manifold is a pair \((M, g)\) where \(M\) is a differentiable manifold and \(g\) is a Riemannian (Lorentzian) metric tensor field. A Lorentzian manifold is called a space-time.

Definition 2.3. Since \(g_{ab}\) is non-degenerate, it must be invertible. The inverse metric is a symmetric \((2, 0)\) tensor field denoted \(g^{ab}\) and obeys \(g^{ab}g_{bc} = \delta^a_c\).
Definition 2.4. A metric determines a natural isomorphism between vectors and covectors. Given a vector \( X^a \) we can define a covector \( X_a = g_{ab}X^b \). Given a covector \( \eta_a \) we can define a vector \( \eta^a = g^{ab}\eta_b \). These maps are clearly inverses of each other.

Definition 2.5. On a Lorentzian manifold \((M, g)\), a non-zero vector \( X \in T_pM \) is timelike if \( g(X, X) < 0 \), null if \( g(X, X) = 0 \), and spacelike if \( g(X, X) > 0 \).

Definition 2.6. On a Riemannian manifold, the norm of a vector \( X \) is \( |X| = \sqrt{g(X, X)} \) and the angle between two non-zero vectors \( X \) and \( Y \) is \( \theta \) where \( \cos \theta = g(X, Y)/(|X||Y|) \). The same definitions apply to space-like vectors on a Lorentzian manifold.

Definition 2.7. A curve in a Lorentzian manifold is timelike/null/spacelike if its tangent vector is everywhere timelike/null/spacelike.

Definition 2.8. Let \( \lambda(u) \) be a timelike curve with \( \lambda(0) = p \). Let \( X^a \) be the tangent vector to the curve. The proper time \( \tau \) from \( p \) along the curve is defined by

\[
\frac{d\tau}{du} = \sqrt{-\left(g_{ab}X^aX^b\right)_{\lambda(u)}}, \quad \tau(0) = 0.
\]

In a coordinate chart, \( X^\mu = dx^\mu/du \) so this definition can be rewritten as \( d\tau^2 = -g_{\mu\nu}x^\mu x^\nu \), evaluated along the curve.

Definition 2.9. If proper time \( \tau \) is used to parametrise a timelike curve then the tangent to the curve is called the 4-velocity of the curve. In a coordinate basis, it has components \( u^\mu = dx^\mu/d\tau \).

Definition 2.10. The Christoffel symbols are defined by

\[
\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}),
\]

giving the geodesic equation

\[
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad \text{or,} \quad \nabla_X X = 0.
\]

3 Ch.4 - Covariant Derivative

Definition 3.1. A covariant derivative \( \nabla \) on a manifold \( M \) is a map sending every pair of smooth vector fields \( X, Y \) to a smooth vector field \( \nabla_X Y \), with the properties

\[
\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z;
\]
\[
\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z;
\]
\[
\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y = f\nabla_X Y + X(f)Y.
\]

Definition 3.2. Let \( Y \) be a vector field. The covariant derivative of \( Y \) is the \((1,1)\) tensor field \( \nabla Y \). In abstract index notation we write \((\nabla Y)^a_b\) as \( \nabla_b Y^a \) or \( Y^a_{;b} \).

Definition 3.3. In a basis \( \{e_\mu\} \) the connection components \( \Gamma^\mu_{\nu\rho} \) are defined by

\[
\nabla_{e_\rho} e_\nu := \nabla_{e_\rho} e_\nu = \Gamma^\mu_{\nu\rho} e_\mu
\]

Definition 3.4. A connection \( \nabla \) is torsion-free if \( \nabla_a \nabla_b f = \nabla_b \nabla_a f \) for any function \( f \). This is equivalent to \( \Gamma^\rho_{[\mu\nu]} = 0 \).
Lemma 3.5. For a torsion-free connection, if $X$ and $Y$ are vector fields, then
\[ \nabla_X Y - \nabla_Y X = [X, Y]. \]

Proof: In coordinate basis,
\[
\nabla_X Y^\mu - \nabla_Y X^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu \\
= X^\nu \partial_\nu Y^\mu + X^\nu \Gamma^\mu_{\rho\nu} Y^\rho - Y^\nu \partial_\nu X^\mu - Y^\nu \Gamma^\mu_{\rho\nu} X^\rho \\
= X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu + X^\nu Y^\rho \Gamma^\mu_{[\rho\nu]} \\
= [X, Y]^\mu.
\]

Definition 3.6. Let $M$ be a manifold with a metric $g$. There exists a unique torsion-free connection $\nabla$ such that the metric is covariantly constant, i.e., $\nabla g = 0$ or $g_{abc} = 0$. This is the Levi-Civita connection.

Definition 3.7. Let $M$ be a manifold with a connection $\nabla$. An affinity parametrised geodesic is an integral curve of a vector field $X$ satisfying $\nabla_X X = 0$.

Theorem 3.8. Let $M$ be a manifold with a connection $\nabla$. Let $p \in M$ and $X_p \in T_p M$. Then there exists a uniquely parametrised geodesic through $p$ with tangent vector $X_p$ at $p$.

Definition 3.9. Let $M$ be a manifold with a connection $\nabla$. Let $p \in M$. The exponential map from $T_p M$ to $M$ is defined as the map which sends $X_p$ to the point unit affine parameter distance along the geodesic through $p$ with tangent $X_p$ at $p$.

Definition 3.10. Let $\{ e_\mu \}$ be a basis for $T_p M$. Normal coordinates at $p$ are defined in a neighbourhood of $p$ as follows. Pick $q$ near $p$. Then the coordinates of $q$ are $X^\mu$ where $X^\mu$ is the element of $T_p M$ that maps to $q$ under the exponential map.

Lemma 3.11. $\Gamma^\mu_{(\nu\rho)} (p) = 0$ in normal coordinates at $p$. Hence for a torsion-free connection, $\Gamma^\mu_{\nu\rho}(p) = 0$ in normal coordinates at $p$.

Proof: Affinely parametrised geodesics through $p$ are given in normal coordinates by $X^\mu(t) = t X^\mu_p$, hence the geodesic equation reduces to $\Gamma^\mu_{\nu\rho}(X(t)) X^\nu_p X^\rho_p = 0$. At $t = 0$, we have $\Gamma^\mu_{\nu\rho}(p) X^\nu_p X^\rho_p = 0$. The antisymmetric part of $\Gamma$ vanishes when contracted with the $X_p$’s and we are left with $\Gamma^\mu_{(\nu\rho)} (p) = 0$.

Lemma 3.12. On a manifold with a metric, if the Levi-Civita connection is used to define normal coordinates at $p$ then $\partial_\rho g_{\mu\nu} = 0$ at $p$.

Proof: At $p$, $\Gamma^\rho_{\nu\rho} = 0$. Hence
\[ 0 = 2g_{\mu} \Gamma^\rho_{\nu\rho} = \partial_\nu g_{\mu\rho} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}. \]
Now if we symmetrise $\mu, \nu$, then the last two terms cancel and we have $\partial_\rho g_{\mu\nu} = 0$.

Lemma 3.13. On a manifold with metric one can choose normal coordinates at $p$ so that $\partial_\rho g_{\mu\nu}(p) = 0$, and $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (Lorentzian case); $g_{\mu\nu}(p) = \delta_{\mu\nu}$ (Riemannian case).

Definition 3.14. In a Lorentzian manifold a local inertial frame at $p$ is a set of normal coordinates at $p$ with the above properties.

4 Ch.6 - Curvature

Definition 4.1. Let $X^a$ be the tangent to a curve. A tensor field $T$ is parallelly transported along the curve if $\nabla_X T = 0$.
Definition 4.2. The Riemann curvature tensor $R^a_{bcd}$ of a connection $\nabla$ is defined by $R^a_{bcd}Z^bX^cY^d = (R(X,Y)Z)^a$, where $X, Y, Z$ are vector fields and $R(X,Y)Z$ is the vector field $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$.

In coordinate basis, $R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\sigma} - \Gamma^\tau_{\nu\sigma} \Gamma^\mu_{\tau\rho}$.

Definition 4.3. The Ricci curvature tensor is the $(0, 2)$ tensor defined by $R_{ab} = R^c_{acb}$.

Proposition 4.4. If $\nabla$ is torsion-free then $R^a_{[bcd]} = 0$.

Proof: Let $p \in M$ and choose normal coordinates at $p$. Vanishing torsion implies $\Gamma^\mu_{\nu\rho}(p) = 0$. So $R^\mu_{\nu\rho}(p) = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$. Antisymmetrising on $\rho\sigma$ gives $R^\mu_{[\nu\rho]} = 0$. But if components of a tensor vanish in one basis then they vanish in any basis, so our result is not limited to normal coordinates. On top, $p$ is an arbitrary point, so our result is also not limited to the point $p$. Hence $R^a_{[bcd]} = 0$.

Proposition 4.5 (Bianchi identity). If $\nabla$ is torsion-free then

$$R^a_{b[cd,e]} = 0.$$ 

Proof: Let $p \in M$ and choose normal coordinates at $p$. Vanishing torsion implies $\Gamma^\mu_{\nu\rho}(p) = 0$. Now,

$$R^\mu_{\nu\rho,\tau} = \partial_\tau \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \partial_\tau \Gamma^\mu_{\nu\rho}.$$ 

Antisymmetrising on $\rho\sigma$ gives $R^\mu_{[\nu\rho,\tau]} = 0$, and by same reasons as above, we have $R^a_{b[cd,e]} = 0$.

Definition 4.6. Let $M$ be a manifold with a connection $\nabla$. A 1-parameter family of geodesics is a map $\gamma : I \times I' \to M$ where $I$ and $I'$ are both open intervals in $\mathbb{R}$, such that

1. for fixed $s$, $\gamma(s,t)$ is a geodesic with affine parameter $t$ (so $s$ is the parameter that labels the geodesic);
2. the map $(s,t) \mapsto \gamma(s,t)$ is smooth and one-to-one with a smooth inverse.

This implies that the family of geodesics forms a 2-dimensional surface $\Sigma \subset M$.

Definition 4.7 (geodesic deviation equation). If $\nabla$ has vanishing torsion then $\nabla_T \nabla_T S = R(T,S)T$.

Or in abstract index notation,

$$T^a \nabla_a (T^b \nabla_b S^a) = R^a_{bcd}T^bT^cS^d.$$ 

Proof: Vanishing torsion gives $\nabla_T S - \nabla_S T = [T, S] = 0$. Hence

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \nabla_T T + \nabla(T, S)T = R(T, S)T$$

since $\nabla_T T = 0$ as $T$ is tangent to affinely parametrised geodesics.

Proposition 4.8. The Riemann tensor satisfies

$$R_{abcd} = R_{cdab}, \quad R_{(ab)cd} = 0.$$ 

Proof: The second follows from the first and the antisymmetry of the Riemann tensor. For the first, introduce normal coordinates at $p$, so $\partial_\mu g_{\nu\rho}(p) = 0$, and $\Gamma^\mu_{\nu\rho}(p) = 0$. Then at $p$,

$$0 = \partial_\mu \delta^\nu_{\rho} = \partial_\mu (g^{\nu\sigma} g_{\sigma\rho}) = g_{\sigma\rho} \partial_\mu g^{\nu\sigma} \quad \Rightarrow \quad \partial_\mu g^{\nu\sigma} = 0.$$ 

$$\Rightarrow \partial_\mu \Gamma^\tau_{\nu\rho} = \frac{1}{2} g^{\nu\sigma} (g_{\mu\rho,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho}).$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma}) = R_{\rho\sigma\mu\nu}.$$ 

This is a tensor equation so is valid not only in normal coordinates, and $p$ is arbitrary. Hence $R_{abcd} = R_{cdab}$. 

Proposition 4.9. The Ricci tensor is symmetric, i.e.,
\[ R_{ab} = R_{ba}. \]

Proof:
\[ R_{ab} = g^{cd} R_{cadb} = g^{cd} R_{dbca} = R_{abc} = R_{ba}. \]

Definition 4.10. The Ricci scalar is
\[ R = g^{ab}R_{ab}. \]

Definition 4.11. The Einstein tensor is the symmetric \((0, 2)\) tensor
\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \]

Proposition 4.12. The Einstein tensor satisfies the contracted Bianchi identity
\[ \nabla^a G_{ab} = \nabla^a R_{ab} - \frac{1}{2} \nabla^b R = 0. \]

Proof: From the Bianchi identity
\[ R_{abcd,e} + R_{abec;dl} + R_{ab;ecd} = 0. \]
Contract \(a\) with \(c\) gives
\[ R_{bd,e} - R_{be;d} + g^{ac} R_{ab;ce} = 0. \]
Contract \(b\) with \(d\) gives
\[ R_{;e} - g^{bd} R_{be;d} + g^{bd} g^{ac} R_{ab;ce} = 0. \]
But \(g^{bd} g^{ac} R_{ab;ce} = -g^{ac} R_{ac;e} = -g^{bd} R_{be;d}.\) Hence,
\[ \nabla^e R - 2\nabla^b R_{be} = 0 \Rightarrow \nabla^a R_{ab} - \frac{1}{2} \nabla^b R = 0. \]

Theorem 4.13 (Lovelock). Let \(H_{ab}\) be a symmetric tensor such that
1. in any coordinate chart, at any point, \(H_{\mu\nu}\) is a function of \(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma};\)
2. \(\nabla^a H_{ab} = 0;\)
3. either space-time is four-dimensional or \(H_{\mu\nu}\) depends linearly on \(g_{\mu\nu,\rho\sigma}\). Then there exist constants \(\alpha, \beta\) such that
\[ H_{ab} = \alpha G_{ab} + \beta g_{ab}. \]
Hence we have the Einstein equation
\[ G_{ab} + \Lambda g_{ab} = 8\pi GT_{ab}. \]

5 Ch. 7 - Diffeomorphisms and Lie derivative

Definition 5.1. Let \(M, N\) be differentiable manifolds of dimension \(m, n\). A function \(\phi : M \to N\) is smooth iff \(\psi_A \circ \phi \circ \psi_A^{-1}\) is smooth for all charts \(\psi_A\) of \(M\) and all charts \(\psi_A\) of \(N\). Note that this is a map from a subset of \(\mathbb{R}^m\) to a subset of \(\mathbb{R}^n\).

Definition 5.2. Let \(\phi : M \to N\) and \(f : N \to \mathbb{R}\) be smooth functions. The pull-back of \(f\) by \(\phi\) is the function \(\phi^*(f) : M \to \mathbb{R}\) defined by \(\phi^*(f) = f \circ \phi\), i.e., \(\phi^*(f)(p) = f(\phi(p))\).

Definition 5.3. Let \(\phi : M \to N\) be smooth. Let \(p \in M\) and \(X \in T_pM\). The push-forward of \(X\) with respect to \(\phi\) is the vector \(\phi_*(X) \in T_{\phi(p)}N\) defined as follows. Let \(\lambda\) be a smooth curve in \(M\) passing through \(p\) with tangent \(X\) at \(p\). Then \(\phi_*(X)\) is the tangent vector to the curve \(\phi \circ \lambda\) in \(N\) at the point \(\phi(p)\).
Lemma 5.4. Let \( f : N \to \mathbb{R} \). Then
\[
(\phi_*(X))(f) = X(\phi^*(f)).
\]

Proof: Let \( \lambda(0) = p \) wlog. Then
\[
(\phi_*(X))(f) = \frac{d}{dt} [(f \circ (\phi \circ \lambda)(t))]|_{t=0} = \frac{d}{dt} [(f \circ \phi \circ \lambda)(t)]|_{t=0} = X(\phi^*(f)).
\]

Definition 5.5. Let \( \phi : M \to N \) be smooth. Let \( p \in M \) and \( \eta \in T^*_p M \) defined by \((\phi^*(\eta))(X) = \eta(\phi_*(X))\) for any \( X \in T_p M \).

Lemma 5.6. Let \( f : N \to \mathbb{R} \). Then
\[
\phi^*(df) = d(\phi^*(f)).
\]

Proof: Let \( X \in T_p M \). Then
\[
(\phi^*(df))(X) = df(\phi_*(X)) = (\phi_*(X))(f) = X(\phi^*(f)) = d(\phi^*(f))(X).
\]

Definition 5.7. A map \( \phi : M \to N \) is a diffeomorphism iff it is one-to-one, onto, smooth, and has a smooth inverse.

Definition 5.8. Let \( \phi : M \to N \) be a diffeomorphism and \( T \) a tensor of type \((r,s)\) on \( M \). Then the push-forward of \( T \) is a tensor \( \phi_*(T) \) of type \((r,s)\) on \( N \) defined by
\[
\phi_*(T)(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) = T(\phi^*(\eta_1), \ldots, \phi^*(\eta_r), \phi^*^{-1}(X_1), \ldots, \phi^*^{-1}(X_s)).
\]

Note that pull-back can be defined in a similar way, with the result \( \phi^* = (\phi^{-1})_* \).

Definition 5.9. Let \( \phi : M \to N \) be a diffeomorphism. Let \( \nabla \) be a covariant derivative on \( M \). The push-forward of \( \nabla \) is a covariant derivative \( \nabla_\phi \) on \( N \) defined by
\[
\nabla_\phi X \phi = \phi_*(\nabla_{\phi^*}(X))(\phi^*(T))
\]
where \( X \) is a vector field and \( T \) a tensor field on \( N \).

Definition 5.10. A diffeomorphism \( \phi : M \to M \) is a symmetry transformation of a tensor field \( T \) iff \( \phi_*(T) = T \) everywhere. A symmetry transformation of the metric tensor is called an isometry.

Definition 5.11. Let \( X \) be a vector field on a manifold \( M \). Let \( \phi_t \) be the map which sends a point \( p \in M \) to the point parameter distance \( t \) along the integral curve of \( X \) through \( p \). It can be shown that \( \phi_t \) is a diffeomorphism.

Definition 5.12. The Lie derivative of a tensor field \( T \) with respect to a vector field \( X \) at \( p \) is
\[
(\mathcal{L}_X T)_p \equiv \lim_{t \to 0} \frac{((\phi_0)_* T)_p - T_p}{t}.
\]

Definition 5.13. If \( \phi_t \) is a symmetry transformation of \( T \) for all \( t \) then \( \mathcal{L}_X T = 0 \). If \( \phi_t \) are a 1-parameter group of isometries then \( \mathcal{L}_X g = 0 \), i.e.,
\[
\nabla_a X_b + \nabla_b X_a = 0.
\]
This is Killing’s equation and solutions are called Killing vector fields.

Proof \( \mathcal{L}_X g = 0 \iff \nabla_a X_b + \nabla_b X_a = 0 \):
\[
(\mathcal{L}_X g)(Y, Z) = \mathcal{L}_X (g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z).
\]

\[
\Rightarrow (\mathcal{L}_X g)_{\mu \nu} Y^\mu Z^\nu = X^\rho \partial_\rho (g_{\mu \nu} Y^\mu Z^\nu) - g_{\mu \nu} (X^\rho \partial_\rho Y^\mu - Y^\rho \partial_\rho X^\mu) Z^\nu - g_{\mu \nu} Y^\mu (X^\rho \partial_\rho Z^\nu - Z^\rho \partial_\rho X^\nu)
\]
\[
= X^\rho \partial_\rho (g_{\mu \nu} Y^\mu Z^\nu) - g_{\mu \nu} (Y^\rho \partial_\rho (X^\mu Z^\nu) + g_{\mu \nu} (Z^\rho Y^\nu \partial_\rho X^\mu + Y^\rho Z^\nu \partial_\rho X^\nu)
\]
\[
= (X^\rho \partial_\rho g_{\mu \nu} + g_{\rho \mu} \partial_\rho X^\rho + g_{\rho \nu} \partial_\rho X^\rho) Y^\mu Z^\nu.
\]
⇒ \((\mathcal{L}Xg)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\nu + \partial_\nu X^\mu\).

Now, in normal coordinates, \(\partial_\rho g_{\mu\nu} = 0\) and \(\partial \iff \nabla\) since \(\Gamma = 0\).

⇒ \((\mathcal{L}Xg)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu\).

This is a tensor equation so is valid not only in normal coordinates, and \(p\) is arbitrary. Hence

\[(\mathcal{L}Xg)_{ab} = \nabla_a X_b + \nabla_b X_a.\]

**Lemma 5.14.** Let \(X^a\) be a Killing vector field and let \(V^a\) be tangent to an affinely parametrised geodesic. Then \(X^a V^a\) is constant along the geodesic.

*Proof:*

\[\nabla_V(X^a V^a) = V^a V^b \nabla_b X_a + X^a V^b \nabla_b V^a = 0\]

since the first term vanishes by the antisymmetry of \(\nabla_b X^a\) and the second vanishes since \(V\) is parallelly transported along itself.

### 6 Ch.9 - Differential forms

**Definition 6.1.** Let \(M\) be a differentiable manifold. A \(p\)-form on \(M\) is an antisymmetric \((0,p)\) tensor field on \(M\).

**Definition 6.2.** The wedge product of a \(p\)-form \(X\) and a \(q\)-form \(Y\) is the \((p+q)\)-form \(X \wedge Y\) defined by

\[ (X \wedge Y)_{a_1...a_p b_1...b_q} = \frac{(p+q)!}{p!q!} X_{a_1...a_p} Y_{b_1...b_q}; \]

**Definition 6.3.** The exterior derivative of a \(p\)-form \(X\) is the \((p+1)\)-form \(dX\) defined in a coordinate basis by

\[ (dX)_{\mu_1...\mu_{p+1}} = (p+1)\partial_{[\mu_1} X_{\mu_2...\mu_{p+1}]]. \]

**Definition 6.4.** \(X\) is closed if \(dX = 0\) everywhere. \(X\) is exact if there exists a \((p-1)\)-form \(Y\) such that \(X = dY\) everywhere. Exact implies closed, but the converse is true only locally.

**Lemma 6.5** (Poincaré). If \(X\) is a closed \(p\)-form, \(p \geq 1\), then for any \(r \in M\), there exists a neighbourhood \(O\) of \(r\) and a \((p-1)\)-form \(Y\) such that \(X = dY\) in \(O\).

### 7 Others

**Definition 7.1.** The energy-momentum tensor is defined in general relativity as

\[ T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{ab}}. \]