Note that we use the metric convention (− + ++).

1 Feynman Rules

1. Draw a Feynman diagram of the process and put momenta on each line consistent with momentum conservation.

2. Associate with each internal propagator
   \[ \frac{-i}{p^2 + m^2 - i\epsilon} \quad \text{(scalar propagator);} \]
   \[ \frac{-i(-p + m)}{p^2 + m^2 - i\epsilon} \quad \text{(fermion propagator);} \]
   \[ \frac{-i\eta_{ab}}{p^2 - i\epsilon} \quad \text{(photon propagator).} \]

3. Associate vertices with coupling constants obtained from the interaction term in the Lagrangian, and impose momentum conservation
   \[ (2\pi)^4 \delta^{(4)} \left( \sum p_{\text{in}} - \sum p_{\text{out}} \right). \]

4. Integrate over momenta associated with loops, with the measure
   \[ \frac{d^4k}{(2\pi)^4}, \]
   and multiplied by -1 for fermions and 1 for bosons.

5. Add in wavefunction terms for external particles of momentum \( p \) and spin \( s \):
   \[ u(p, s) \quad \text{incoming fermions;} \quad \bar{u}(p, s) \quad \text{outgoing fermions;} \]
   \[ \bar{v}(p, s) \quad \text{incoming antifermions;} \quad v(p, s) \quad \text{outgoing antifermions;} \]
   \[ \epsilon_a \quad \text{incoming photons;} \quad \epsilon^*_a \quad \text{outgoing photons.} \]

6. Take into account symmetry factors, which originate from various combinatorial factors counting the different ways in which the \( \frac{\delta}{\delta J} \)'s can hit the \( J \)'s in the path-integral formulation.

Note that we look at the Lagrangian for information. For example, if it contains the term \( g\phi \bar{\psi}\gamma^5\psi \) then
- \( g\gamma^5 \Rightarrow \text{add } ig\gamma^5 \) to vertices;
- \( \phi \Rightarrow \text{the propagator is scalar;} \)
- \( \bar{\psi}\psi \Rightarrow \text{particles are fermionic so we need to subtract one diagram from the other when calculating } A. \)
2 Examples

- Fermion propagating through \( n \) vertices labelled \( a_1 \) to \( a_n \), emitting a photon at each vertex
  \[
  (-ie)^n \bar{u}(p', s')\gamma^{a_n} - i(-p_{n} + m) \frac{1}{p_{n}^2 + m^2 - i\epsilon} \gamma^{a_{n-1}} - i(-p_{n-1} + m) \frac{1}{p_{n-1}^2 + m^2 - i\epsilon} \gamma^{a_{n-2}} \ldots \gamma^{a_1} u(p, s).
  \]

- Anti-fermion propagating through \( n \) vertices labelled \( a_n \) to \( a_1 \), emitting a photon at each vertex
  \[
  (-ie)^n \bar{v}(p, s)\gamma^{a_n} - i(-p_{n} + m) \frac{1}{p_{n}^2 + m^2 - i\epsilon} \gamma^{a_{n-1}} - i(-p_{n-1} + m) \frac{1}{p_{n-1}^2 + m^2 - i\epsilon} \gamma^{a_{n-2}} \ldots \gamma^{a_1} v(p', s').
  \]

- Fermion loop with \( n \) vertices emitting photons
  \[
  (-ie)^n\gamma^{a_n} - i(-p_{n} + m) \frac{1}{p_{n}^2 + m^2 - i\epsilon} \gamma^{a_{n-1}} - i(-p_{n-1} + m) \frac{1}{p_{n-1}^2 + m^2 - i\epsilon} \gamma^{a_{n-2}} \ldots \gamma^{a_1} - i(-p_{1} + m) \frac{1}{p_{1}^2 + m^2 - i\epsilon} \gamma^{a_{1}}.
  \]

- \( e^- \mu^- \rightarrow e^- \mu^- \) (photon internal line)
  \[
  (-ie)^2 \bar{u}_e(p', s')\gamma^a u_e(p, s) \frac{-i\eta_{ab}}{(p - p')^2 - i\epsilon} \bar{u}_\mu(q', r')\gamma^b u_\mu(q, r).
  \]

- \( e^- e^- \rightarrow e^- e^- \) (photon internal line)
  \[
  A_1 = (-ie)^2 \bar{u}_e(p', s')\gamma^a u_e(p, s) \frac{-i\eta_{ab}}{(p - p')^2 - i\epsilon} \bar{u}_e(q', r')\gamma^b u_e(q, r);
  
  A_2 = (-ie)^2 \bar{u}_e(p', s')\gamma^a u_e(q, r) \frac{-i\eta_{ab}}{(p - p')^2 - i\epsilon} \bar{u}_e(p', s')\gamma^b u_e(q, r).
  \]

So due to Fermi-Dirac statistics,

\[
A = A_1 - A_2.
\]

And scattering cross-section

\[
|\mathcal{M}|^2 = A_1 A_1^* + A_2 A_2^* - (A_1 A_2^* + A_2 A_1^*).
\]

- \( e^- e^+ \rightarrow \mu^- \mu^+ \) (photon internal line)
  \[
  (-ie)^2 \bar{u}(p')\gamma^a v(q') \frac{-i\eta_{ab}}{(p + q)^2 - i\epsilon} \bar{v}(q)\gamma^b u(p)
  \]

- Compton scattering \( e^- \gamma \rightarrow e^- \gamma \) (fermion internal line)
  \[
  A_1 = (-ie)^2 \bar{u}(p', s')\gamma^b \epsilon_b^* \frac{-i(-p + k) + m}{(p + k)^2 + m^2 - i\epsilon} \gamma^a u(p, s);
  
  A_2 = (-ie)^2 \bar{u}(p', s')\gamma^b \epsilon_b^* \frac{-i(-p - k') + m}{(p - k')^2 + m^2 - i\epsilon} \gamma^a u(p, s).
  \]

\[
A = A_1 + A_2,
\]

\[
|\mathcal{M}|^2 = A_1 A_1^* + A_2 A_2^* + (A_1 A_2^* + A_2 A_1^*).
\]
- Two scalars to two scalars (scalar internal line)

\[ A_1 = (2\pi)^4 \delta(p + q - p' - q')(-i\lambda)^2 \frac{-i}{(p + q)^2 + m^2 - i\epsilon}; \]

\[ A_2 = (2\pi)^4 \delta(p + q - p' - q')(-i\lambda)^2 \frac{-i}{(p - p')^2 + m^2 - i\epsilon}; \]

\[ A_3 = (2\pi)^4 \delta(p + q - p' - q')(-i\lambda)^2 \frac{-i}{(p - q')^2 + m^2 - i\epsilon}; \]

We use Mandelstam variables

\[ s = (p + q)^2, \quad t = -(p - p')^2 \quad u = -(p - q')^2. \]

Our definitions of \( A \) and \( M \) have been ambiguous as to whether \( i(2\pi)^4 \delta(\cdots) \) is included. I guess \( A \) does but \( M \) doesn’t. Also we defined them to include the factor of \( i \). Some people don’t. Anyway, here

\[ A = A_1 + A_2 + A_3 := (2\pi)^4 \delta(p + q - p' - q')(-i\lambda)^2 \left[ \frac{-i}{s + m^2 - i\epsilon} + \frac{-i}{t + m^2 - i\epsilon} + \frac{-i}{u + m^2 - i\epsilon} \right]. \]