Propositional logic

1 Vocabulary of propositional logic

• Vocabulary

(1) a. Propositional letters: \( p, q, r, s, t, p_1, q_1, ..., p_2, q_2, ... \)

b. Propositional connectives: \( \neg, \land, \lor, \rightarrow, \leftrightarrow \)

c. Parentheses: ( )

<table>
<thead>
<tr>
<th>Connectives</th>
<th>Compose proposition with connectives</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>negation</td>
<td>( \neg p ) (the negation of ( p ))</td>
<td>it is not the case that ( p )</td>
</tr>
<tr>
<td>conjunction</td>
<td>( p \land q ) (conjunction of ( p ) and ( q ))</td>
<td>( p ) and ( q )</td>
</tr>
<tr>
<td>disjunction</td>
<td>( p \lor q ) (disjunction of ( p ) and ( q ))</td>
<td>( p ) or ( q )</td>
</tr>
<tr>
<td>implication</td>
<td>( p \rightarrow q ) (implication of ( p ) and ( q ))</td>
<td>if ( p ), then ( q )</td>
</tr>
<tr>
<td>equivalence</td>
<td>( p \leftrightarrow q ) (equivalence of ( p ) and ( q ))</td>
<td>( p ) if and only if ( q )</td>
</tr>
</tbody>
</table>

Exercise: Translate the following sentences into propositional logic.

(2) ex. John didn’t hand in the exam.

\( h = \) John handed in the exam. \( \neg h. \)

a. It is not the case that Guy comes if Peter or Harry comes.

b. John is not only stupid but also nasty.

c. Nobody laughed or applauded.

d. Charles and Elsa are brother and sister or nephew and niece.

2 Syntax of propositional logic

• Well-formed formulas (wffs): ‘grammatically correct expressions’

(3) A recursive definition for wff in a language \( L \)

a. Every propositional letter in the vocabulary of \( L \) is a wff in \( L \).

b. If \( p \) is a wff in \( L \), then \( \neg p \) is too.

c. If \( p \) and \( q \) are wffs in \( L \), then so are \( (p \land q), (p \lor q), (p \rightarrow q), \) and \( (p \leftrightarrow q) \).

d. Nothing else is a wff in \( L \).

Exercise: Following the recursive rules above, identify whether each of the following strings is a wff.

(4)

a. \( (p) \)

b. \( \neg(\neg p) \)

c. \( \neg \neg p \)

d. \( (p \land q) \)
• **Abbreviations** (to make formulas easier to read while do not carry any danger of ambiguity):

(5) Abbreviation rules
   a. The outermost parentheses need not be explicitly mentioned.
   \[ p \land q \] is short for \( (p \land q) \)
   b. The conjunction and disjunction symbols apply to as little as possible.
   \[ p \land q \rightarrow \neg r \lor s \] is short for \( ((p \land q) \rightarrow (\neg r \lor s)) \)

**Exercise:** Add parentheses to the following wffs.

(6) a. \( p \rightarrow (q \rightarrow r \lor s) \)
   b. \( (p \rightarrow q) \rightarrow r \lor s \)

3 **Semantics of propositional logic**

3.1 **Truth values and truth functional**

- Every declarative sentence has exactly one *truth value*. In a two-valued logic, every sentence is assigned one of the following truth values: 1 (TRUE) or 0 (FALSE).

- The truth value of a complex sentence is computed based on (i) the truth-values of the connected simple sentences and (ii) the truth-functional properties of the connective(s).

(7) a. There is a blizzard *and* I feel good.
   b. There is a blizzard *but* I feel good.

3.2 **Truth tables**

- Truth tables of propositional connectives

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( p \land q )</th>
<th>( p \lor q )</th>
<th>( p \rightarrow q )</th>
<th>( p \leftrightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- More on disjunctions:

The disjunction \( p \lor q \) is an *inclusive disjunction*. In natural languages, however, \( p \) or \( q \) is more likely to be interpreted exclusively: an *exclusive disjunction* is true iff exactly one of the disjuncts is true.

(8) Your money or your life!

**Exercise:** Make a truth table for the exclusive disjunction.
More on implications:

In propositional logic, an implication \( p \rightarrow q \) is defined truth-functionally. The antecedent \( p \) and the consequent \( q \) do not necessarily have a causal relation.

(9) a. If Harvard is in Cambridge, then Cambridge has no university.
   b. If Gennaro is a professor at Harvard, then Boston is in MA.

When the antecedent is false, the implication is vacuously true regardless of whether the consequent is true or false.

(10) a. If \( 1+1 = 0 \), then Boston is in MA. \( \text{TRUE} \)
    b. If \( 1+1 = 0 \), then Boston is not in MA. \( \text{TRUE} \)

3.3 Logical equivalence

- \( p \) and \( q \) are said to be (logically) equivalent iff for every valuation \( V \), we have: \( V(p) = V(q) \)

For example, \( \neg p \) and \( p \rightarrow \neg p \) are logically equivalent.

<table>
<thead>
<tr>
<th>( V )</th>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \rightarrow \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( V_2 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Exercise: Are the following pairs of wffs/sentences (logically) equivalent?

(11) a. \( p \rightarrow q \)
    b. \( \neg p \lor q \)

(12) a. ‘If Gennaro is a professor of Harvard Linguistics, then his office is in Boylston.’
    b. ‘If Jim is a professor of Harvard Linguistics, then his office is in Boylston.’

(13) a. ‘If Gennaro is a professor of Harvard Linguistics, then his office is in Boylston.’
    b. ‘It is not the case that [Gennaro is a professor of Harvard linguistics, and his office isn’t in Boylston].’
3.4 Tautology, contradiction, contingency

- **Terminology:**
  - **Tautology:** Always true \( p \lor \lnot p \)
  - **Contradiction:** Always false \( p \land \lnot p \)
  - **Contingency:** Sometimes true and sometimes false \( p \)

- **A selected list of tautologies (redundant brackets are omitted)**

  (14) Associative and commutative laws for \( \land, \lor, \leftrightarrow \)

  (15) Distributive laws
    a. \((p \land (q \lor r)) \leftrightarrow ((p \land q) \lor (p \land r))\)
    b. \((p \lor (q \land r)) \leftrightarrow ((p \lor q) \land (p \lor r))\)

  (16) Negation
    a. \(\lnot \lnot p \leftrightarrow p\)
    b. \(\lnot (p \rightarrow q) \leftrightarrow (p \land \lnot q)\)
    c. \(\lnot (p \leftrightarrow q) \leftrightarrow ((p \land \lnot q) \lor (\lnot p \land q))\)

  (17) De Morgan’s laws
    a. \(\lnot (p \land q) \leftrightarrow (\lnot p \lor \lnot q)\)
    b. \(\lnot (p \lor q) \leftrightarrow (\lnot p \land \lnot q)\)

  (18) Others
    a. \(p \lor \lnot p\) Excluded middle
    b. \(\lnot (p \land \lnot p)\) Contradiction
    c. \((p \rightarrow q) \leftrightarrow (\lnot q \rightarrow \lnot p)\) Contraposition
    d. \(((p \land q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))\) Exportation

- **Exercise:** Let \(p\) be a tautology, \(q\) a contradiction, and \(r\) a contingency. Which of the following sentences are (i) tautological, (ii) contradictory, (iii) contingent, (iv) logically equivalent to \(r\).

  (19) a. \(p \land r\)
    b. \(p \lor r\)
    c. \(q \land r\)
    d. \(p \lor q\)
    e. \(r \rightarrow q\)
• Two ways of determining whether a wff. is a tautology.

  – *Truth-table method*
  
  We investigate every possible combination of truth-values for the simple sentences and then check the resulting truth-value of the complex expression.

  **Example:** \( \neg p \rightarrow p \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( \neg \neg p \rightarrow p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

  **Example:** \( p \rightarrow (q ightarrow p) \)

  | \( p \) | \( q \) | \( q ightarrow p \) | \( p \rightarrow (q ightarrow p) \) |
  |-------|-------|----------------|----------------|
  | 1     | 1     | 1              | 1              |
  | 1     | 0     | 1              | 1              |
  | 0     | 1     | 0              | 1              |
  | 0     | 0     | 1              | 1              |

  – *Indirect reasoning*
  
  Assume that the considered wff is false. If this assumption leads to a contradiction (i.e., impossible to be false), this wff is a tautology; otherwise it isn’t.

  **Example:** \([ p \rightarrow (q \land r)] \rightarrow (p \rightarrow r)\)

<table>
<thead>
<tr>
<th>( p \rightarrow (q \land r) )</th>
<th>( \rightarrow )</th>
<th>( p \rightarrow r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
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</tr>
<tr>
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<td>1</td>
</tr>
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<td>4</td>
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<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

• **Exercise:** Determine whether the following sentences are tautologous:

  (20)  a. \( (p \rightarrow q) ightarrow (q \rightarrow p) \)
  
  b. \( p \lor (p \rightarrow q) \)