Dynamic Models for Selling Multiple Items

Rebecca Dizon-Ross† Sheldon M. Ross‡

Abstract

We study the optimal strategy for selling multiple items in a setting where bidders can bid for individual items and for subsets of items. There is a large literature examining search problems with a single good, but few papers that generalize the problem to vector offers for subsets of items. One of the challenges in extending to multiple goods is that, as the state space expands, the problem can become computationally intractable. We show how to solve the dynamic optimization problem so that it becomes computationally feasible when the number of items is not too large. We also consider special cases of the model, including an “additive” case and a “single item” case, and present several intuitive structural results about the optimal policy and value functions for these specialized cases. Finally, we consider extensions to a stopping rule problem.

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†Abdul Latif Jameel Poverty Action Lab, MIT, E53-389, 30 Wadsworth St., Cambridge, MA 02142. rdr@mit.edu.
‡Epstein Department of Industrial and Systems Engineering, University of Southern California, 3715 McClintock Ave., Los Angeles, CA 90089. smross@usc.edu
1 Introduction

We study the problem of selling multiple goods. The setup is a generalization of the classic job search model: a seller with multiple items receives a series of bids from potential buyers. Bidders can bid for individual items and for subsets of items, and the seller can accept a maximum of one bid per bidder. The seller incurs a cost for each period until all items are sold, and tries to maximize expected discounted revenue.

The setting is general, with many real-world applications. We allow for synergies between different goods by placing no restriction on the bids for subsets relative to those for individual items. For example, consider a salesman selling multiple tickets in a concert hall. Some customers may only want to purchase pairs of adjacent seats, and not be interested in purchasing a solo seat. We allow for this in the model: these bidders would make a positive bid for the pair of seats but bid zero for the seats sold separately. We also allow sellers to restrict which subsets are available. For example, the seller can restrict bidders to bid only for single-good subsets. This restricted model could apply to a firm hiring to fill multiple positions: the firm receives a series of applicants whose match qualities vary by position, and must choose which applicants to hire and, if hired, to which job to assign the applicant.

Although there is a large literature in economics and other fields examining search problems with a single good and an infinite horizon, there are few papers that generalize the problem to multiple goods and vector offers. One of the challenges in extending to multiple goods is that, as the state space expands, the problem can become computationally intractable. We present a new, computational approach that can be used to solve for the optimal policy and value function when the number of items is of moderate size. The approach relies on iteration: we provide a method for solving for the value function if there are \( n \) items to sell that depends on the value functions for each of the subsets with \( n - 1 \) items. Since it is straightforward to calculate the value function with just 1 item, this approach enables us to “build up” from the 1-item value functions to calculate the value function for larger subsets.

After providing the computational approach, we examine two special cases of the general model. The first special case we consider is the case where bidders bid for all subsets, and the bids for subsets are restricted to be equal to the sum of the bids for the individual items in the subset (hereafter: the additive model). This model
would be appropriate if there are limited complementarities or substitution patterns between the different items in a subset. This model was previously considered in Bruss and Ferguson (1997), but we prove several intuitive structural results that go beyond what was obtained in that paper. For example, we show that sellers sell more items when bids are high (i.e., that, for a given set of unsold items, the optimal set of items to sell is an increasing function of the offer vector). We also show that, holding the offer vector constant, sellers retain more items when they have more items remaining. (That is, if, for a given offer vector, item \( i \) should not be sold when \( S \) is the set of unsold items, then it should not be sold when the set of unsold items is a superset of \( S \).) In addition, we consider the special case where the bids for all individual items are independent and identically distributed (i.e., the offer vector is a vector of \( n \) independent and identically distributed random variables). In this special case model, we show that the value function is convex in the number of items to be sold.

The next special case we consider is the case where only single items can be sold (hereafter: the single item model): the bidder offers a bid for each individual item, and the seller can accept at most one bid. We prove several results about the convexity or concavity of the value function under restricted assumptions. We then consider the stopping rule problem in which assignments are only made after deciding that no additional offers should be considered. That is, we suppose that buyers are patient, that all offers “remain on the table,” and that only after the stopping decision has been made are the buyers matched with their purchased items. We show how to explicitly determine the optimal stopping policy, and demonstrate that, in this setting, a “one stage lookahead policy,” or a policy that stops if and only if stopping is better than going exactly one more stage and then stopping, is optimal. We also show that, if an offer would be rejected if we were to stop at the present time, then that offer would also be rejected if we stopped at a later time.

Our model is a generalization of the majority of the search problems in the literature that assume that successive potential buyers make independent offers from a common probability distribution. In the economics literature, the problem is known as the job search problem, whereas in the applied probability literature, such problems are generally referred to as asset selling models. These papers can be traced back to the early papers of Stigler (1961) and Stigler (1962) as well as those of MacQueen and Miller Jr (1960) and Derman and Sacks (1960). See Lippman and McCall (1976)
and McMillan and Rothschild (1994) for reviews of the early literature.

Despite the large body of papers on this topic, however, with the exception of Bruss and Ferguson (1997), almost all papers assume that the sequentially arriving offers are single values rather than vector offers. We aim to begin to fill this gap.

There is also a classic literature that looks at the sequential stochastic assignment problem, introduced in Derman, Lieberman, and Ross (1972), where only a single item is sold in each period, but there are vector offers for all items. In that problem, the relative value of the items is fixed across periods. We see the “single item” model presented in this paper as a generalization of the sequential stochastic assignment model to a setting where the relative value of the items is allowed to vary across different bidders.

This paper, and especially the stopping time problem we present, is also related to the literature looking at stopping time problems and how to sell to patient buyers entering over time (e.g., Boshuizen and Gouweleeuw (1993), Zuckerman (1986)). Like the job search literature, these papers almost all address single items. In recent years, papers in the revenue management literature have extended similar models to accommodate multiple items, but, unlike us, they study settings with private information and strategic buyers (see, e.g., Board and Skrzypacz (2010), Gallien (2006), and Gershkov and Moldovanu (2009)).

The remainder of the paper proceeds as follows. Section 2 presents the model. Section 3 describes the computational approach. Section 4 considers the special case additive model. In section 5, we examine the special case single item model. Section 6 presents the stopping rule problem. Section 7 notes extensions to continuous time and to more general cost functions.

## 2 The Model

The seller has a set $N$ of items to sell, $N = \{1, \ldots, n\}$. Potential buyers can place bids for specified nonempty subsets $S_1, \ldots, S_k$ of items; these $k$ subsets include all $n$ one point sets.\footnote{We take the set of specified subsets to be predetermined.} Thus, the bid is a vector $X = (X_{S_1}, \ldots, X_{S_k})$, meaning the buyer is bidding $X_{S_i}$ for subset $S_i$. At most one of the subsets can be sold to each buyer. Thus, someone could bid for the subsets $\{1\}$, $\{2\}$, and $\{1, 2\}$, meaning that they are willing to buy, at their specified prices, either item 1 by itself, item 2 by itself, or...
both 1 and 2. The setup is meant to capture the fact that, in some settings, potential buyers may only be willing to purchase certain subsets of the goods, e.g., if the items were tickets and buyers only wanted to purchase two adjacent tickets. Bids can also be negative, with $X_{S_i} \leq 0$ indicating that that potential buyer would require payment to accept the subset $S_i$. In addition, we suppose that recall of previous offers is not allowed.

An independent offer vector arrives each period. We assume that the offer vectors have a known distribution, which does not depend on which items are currently available. We also suppose that the offer distribution is such that $E[X_{S_i}^2] < \infty$ for all $i$. Time is infinite, and the seller incurs a cost $c$ per period until all items are sold. The seller’s objective is to maximize the total expected discounted net return, where $0 < \beta \leq 1$ is the discount factor. (Equivalently, the objective is to maximize the expected present value return when the interest rate is $\frac{1}{\beta} - 1$.)

For $S'$ a subset of $S$, we use the notation $S - S'$ to indicate the set containing all elements of $S$ that are not in $S'$. If $i \in S$, then we use the notation $S - i$ to stand for the set $S - \{i\}$. We also let $S^c = N - S$.

### 3 Computational Approach for the General Model

We begin by stating and providing a solution for the optimality equation. The solution leads directly to our computational approach.

We say that the state of the system is $(S, \mathbf{x})$ if $S \neq \emptyset$ is the set of items that remain to be sold and the offer vector $\mathbf{x} = (x_{S_1}, \ldots, x_{S_k})$ has just been received. If $V(S, \mathbf{x})$ is the maximal expected additional discounted return in this state, then the optimality equation is

$$V(S, \mathbf{x}) = \max \left( \beta V(S), \max_{1 \leq i \leq k : S_i \subset S} [x_{S_i} + \beta V(S - S_i)] \right) - c \quad (1)$$

where $V(\emptyset) = 0$, and for $T$ nonempty

$$V(T) = E[V(T, \mathbf{X})]$$

is the maximal expected additional discounted return when only items in the set $T$ remain to be sold.
Let 
\[ R(S, x) \equiv \max_{1 \leq i \leq k; S_i \subset S} [x S_i + \beta V(S - S_i)] \]
That is, \( R(S, x) - c \) denotes the maximal expected discounted return when in state \((S, x)\) under the condition that one of the subsets \( S_1, \ldots, S_k \) must immediately be sold. Note that \( R(S, x) \) is determined by \( x \) and the set of values \( V(T) \) as \( T \) ranges over all the proper subsets of \( S \).

We now show how to determine \( V(S) \).

**Proposition 1** With \( X \) having the distribution of an offer vector, \( V(S) \) is the unique value \( v \) such that
\[ c + (1 - \beta)v = E[(R(S, X) - \beta v)^+] \] (2)
Furthermore, the policy that, when in state \((S, x)\), does not sell any of the subsets if \( R(S, x) < \beta V(S) \), and, if \( R(S, x) \geq \beta V(S) \), sells one of the subsets \( S_j \subset S \) such that \( R(S, x) = x S_j + \beta V(S - S_j) \), is an optimal policy.

**Proof:** The optimality equation can be rewritten as
\[ V(S, x) = \max \{R(S, x), \beta V(S)\} - c \]
Now, with
\[ I = \begin{cases} 1, & \text{if } R(S, x) \geq \beta V(S) \\ 0, & \text{if } R(S, x) < \beta V(S) \end{cases} \]
we have
\begin{align*}
V(S) & = E[V(S, X)] \\
& = E[V(S, X)|I = 1]P(I = 1) + E[V(S, X)|I = 0](1 - P(I = 1)) \\
& = E[R(S, X) - c|I = 1]P(I = 1) + (\beta V(S) - c)(1 - P(I = 1)) \\
& = E[R(S, X) - \beta V(S)|I = 1]P(I = 1) + \beta V(S) - c
\end{align*}
\[ E[(R(S, X) - \beta V(S))^+] + \beta V(S) - c \]

Hence,
\[ c + (1 - \beta)V(S) = E[(R(S, X) - \beta V(S))^+] \]

(3)

Because \( c + (1 - \beta)v \) is increasing in \( v \), and \( E[(R(S, X) - \beta v)^+] \) is strictly decreasing in \( v \) (at least until it hits 0), it follows that there is a unique value \( V(S) \) that satisfies the preceding equation. That the optimal policy is as stated follows from the optimality equation.

For a one point set \( S = \{i\} \), Proposition 1 yields the known result that \( V(\{i\}) \) is such that
\[ c + (1 - \beta)V(\{i\}) = E[(X_i - \beta V(\{i\}))^+] \]

and the policy that accepts the offer \( x \) for item \( i \) if and only if \( x \geq \beta V(\{i\}) \) is an optimal policy.

Proposition 1 leads directly by our computational approach by yielding a way to solve the problem recursively, at least when \( n \) is of moderate size. Starting with \( R(\{i\}, x) = x_{\{i\}} \), determine \( V(S) \) for all one point sets \( S \). Using these values enables us to determine, for any vector \( x \), \( R(S, x) \) for all two point sets \( S \). For example, for \( S = \{s_1, s_2\} \), \( R(S, x) = \max\{x_{s_1,s_2}, x_{s_1} + \beta V(s_2), \beta V(s_1) + x_{s_2}\} \). We can then use Proposition 1 to determine \( V(S) \) for all two point sets \( S \); and so on, until you obtain \( V(\{1,2,\ldots,n\}) \).

To determine the values of \( V(S) \) we can either use numerical computations or an analytic solution to determine the values \( V(\{i\}), i = 1,\ldots,n \). (If neither is feasible, then simulation can be used.) Then we can use simulation to determine the other values \( V(S) \), first for the 2 point sets, then the 3 point sets, and so on. The simulation should be done by generating \( m \) iid random offer vectors \( X_j, j = 1,\ldots,m \), which will be used in all the determinations. Suppose you have determined \( V(S) \) for all sets of size \( j \) or less. Now, let \( S \) be of size \( j + 1 \). Note first that the function \( R(S, x) \) can be determined for any \( x \). To find that value \( v \) that satisfies (2), note that if \( c + (1 - \beta)v > (<)E[(R(S, X) - \beta v)^+] \), then \( v > (<)V(S) \). Thus, a binary search can be used to find \( V(S) \). To begin, start with known values \( v_l \) and \( v_u \) which are such that \( v_l \leq V(S) \leq v_u \). Let \( v^* = (v_l + v_u)/2 \) and compute \( E[(R(S, X) - \beta v^*)^+] \) and see whether it is greater than \( c + (1 - \beta)v^* \). If so, change the value of \( v_l \) to now equal \( v^* \)
and repeat; if not, change the value of \( v_u \) to \( v^* \) and repeat. Continue until \( v_u - v_l \) is sufficiently small. In the preceding, the value of \( E[(R(S, X) - \beta v^*)^+] \) should be taken equal to \( \sum_{j=1}^{m} \frac{1}{m} (R(S, X^j) - \beta v^*)^+ \).

In the preceding approach, because all values \( V(T) \) for \( T \) a proper subset of \( S \) would be known when we need to determine \( V(S) \), it is easy to obtain a lower bound for \( V(S) \).

**Proposition 2**

\[
V(S) \geq \max_{1 \leq i \leq k, S_i \subset S} [E[X_{S_i}] + \beta V(S - S_i)] - c
\]

**Proof:** Because \( E[X_{S_i}] + \beta V(S - S_i) - c \) is the expected return from the policy that, no matter what the initial offer vector, will initially sell the subset \( S_i \) and then proceed optimally from then on, it follows that \( V(S) \geq E[X_{S_i}] + \beta V(S - S_i) - c \). ■

## 4 A Special Case: The Additive Model

We now consider the special case “additive” model in which we restrict the bids for subsets to be equal to the sum of the bids in the subset. This model would be appropriate if there are limited complementarities or substitution patterns between the different items in a subset.

Specifically, we suppose that all subsets are for sale and that each buyer makes an offer vector \( Y_1, \ldots, Y_n \), with the interpretation that they are willing to buy any set \( T \) of items for the price \( \sum_{i \in T} Y_i \). That is, a buyer is simultaneously making an offer for each of the \( n \) individual items, offering \( Y_i \) for item \( i \), and is willing to buy any set of items at a price equal to the sum of the individual offers of items of this subset. If \( S \) is the set of unsold items and \( y \) is the current offer vector then the optimality equation for this special case of the general model is

\[
V(S, y) = \max_{S' \subset S} \left[ \sum_{j \in S'} y_j + \beta V(S - S') \right] - c
\]

which can be rewritten as

\[
V(S, y) = \max (\beta V(S), R(S, y)) - c
\]
where $R(S, y) = \max_{\emptyset \neq S' \subset S} [\sum_{j \in S'} y_j + \beta V(S - S')]$.

This model, with $\beta = 1$, was previously considered in Bruss and Ferguson (1997). However, although the optimality equation (4) was derived in Bruss and Ferguson (1997), its solution as given by Proposition 1 was not obtained in that paper.

Let $\alpha_i(c) \equiv \beta V(\{i\})$. That is, $\alpha_i(c)$ is that value $v$ for which

$$c + (1 - \beta)v = E[ (Y_i - \beta v)^+]$$

and is also the critical value such that, when only item $i$ remains to be sold, it is optimal to sell at the first offer that is at least $\alpha_i(c)$.

One may obtain some insight on our problem by imagining that there are $n$ separate individuals, each with a single item to sell, each trying to maximize his total expected discounted return, and with these individuals being tied together by the fact that they jointly have to pay the amount $c$ for each period one of their items is unsold. This leads to the following proposition.

**Proposition 3** If the current state is $(S, y)$, with $i \in S$, then an optimal policy would never sell item $i$ for the offered value $y_i$ whenever $y_i < \alpha_i(c)$.

**Proof:** Suppose an optimal policy, call the policy $\pi$, calls for selling the set of items $S'$ in state $(S, y)$ where $i \in S'$ and $y_i < \alpha_i(c)$. Now, rather than using $\pi$ consider the policy $\pi^*$ that starts by selling the items $S' - i$. For selling the items in $S - S'$ the policy $\pi^*$ then does exactly what policy $\pi$ would do, and for selling item $i$ the policy $\pi^*$ accepts the first offer that is at least $\alpha_i(c)$. Thus, if under the policy $\pi^*$ we attribute the period costs to the elements in $S - i$ until they are all sold (and only then attribute the cost to $i$, provided it is unsold at that time) it follows that the expected discounted net returns from selling the set of items $S - i$ is the same for policies $\pi$ and $\pi^*$. However, under $\pi^*$ the additional expected net discounted return from selling item $i$ is at least as large as $\alpha_i(c) = \beta V(\{i\})$, which follows because if item $i$ is the only item to be sold then the expected return from the policy that accepts the first offer that is at least $\alpha_i(c)$ is a decreasing function of the costs per period, and thus is higher when the initial period costs are 0 and then become $c$ after some time as opposed to when the period costs are always $c$. Because, in the latter case, the expected return of the $\alpha_i(c)$ critical value policy is $\alpha_i(c)/\beta$, it follows that the expected discounted return under $\pi^*$ is strictly larger than under $\pi$, a contradiction.
that proves the result. ■

Supposing that the periodic cost \( c \) is equally shared by those whose items remain unsold might lead one to conjecture that the optimal policy would sell item \( i \) in state \((S, y)\) when \( i \in S \) and \( y_i > \alpha_i(c/|S|) \), where \(|S|\) is the number of elements in \( S \). The following counterexample shows that this need not be the case when one of the sellers will, with high probability, receive a “free ride” from another seller.

**Example 1:** Suppose there are 2 items to sell, that \( \beta = c = 1 \), and that \( Y_1 \) and \( Y_2 \) are independent with

\[
P(Y_1 = 1) = .99, \quad P(Y_1 = 10) = .01
\]

\[
P(Y_2 = 1) = 1 - 10^{-10}, \quad P(Y_2 = 10^{20}) = 10^{-10}
\]

It is easily checked that \( \alpha_1(1/2) = .59 \). However, it is also easily checked that, until an offer of \( 10^{20} \) appears for item 2, the offer of 1 for item 1 should be rejected. ■

Although it may not be optimal in state \((S, y)\) to sell item \( i \) when \( y_i > \alpha_i(c/|S|) \), if the preceding inequality holds for all \( i \in S \) then all items should be sold.

**Proposition 4**

\[
V(S) \leq \sum_{i \in S} \alpha_i(c/|S|)/\beta
\]

Moreover, it is optimal in state \((S, y)\) to sell all items in \( S \) when \( y_i \geq \alpha_i(c/|S|) \) for all \( i \in S \).

**Proof:** Suppose \( S \) is the set of items still unsold. Consider a new problem with initial set of items \( S \) in which offers appear as before but in which one must pay in each period the amount \( c/|S| \) for each item that has not yet been sold. (That is, if \( r \) items are still unsold then that period’s cost is \( rc/|S| \).) Clearly the maximal expected return in this new problem is at least as large as in the original one in which the amount \( c \) is paid in each period for which there is an unsold item. Because the new problem is equivalent to the \(|S|\) individual problems in which there is a single item to sell and where the cost per period is \( c/|S| \), it follows that \( V(S) \) is at most the sum of the maximal expected returns of these \(|S|\) problems, which proves the inequality.
To prove that the optimal policy sells all items when in state \((S, y)\) whenever 
\[ y_i \geq \alpha_i(c/|S|) \] for all \( i \in S \), note that the maximal expected return if only the set \( S' \) is sold is 
\[ \sum_{i \in S'} y_i + \beta V(S - S'). \] The result now follows since 
\[
\sum_{i \in S'} y_i + \beta V(S - S') \leq \sum_{i \in S} y_i + \beta V(S - S') \leq \sum_{i \in S} y_i + \sum_{i \notin S} \alpha_i(c/|S|) \leq \sum_{i \in S} y_i
\] where the second inequality follows because \( \alpha_i(c) \) is a decreasing function of \( c \).

Whereas Proposition 4 yields an upper bound on \( V(S) \), a lower bound can be obtained by choosing an item \( i \in S \), selling the items in \( S - i \) according to the optimal policy when these are the only items, and selling item \( i \) at the first offer that is at least \( \alpha_i(c) \). By the argument of Proposition 3 the expected return of this policy is at least \( \alpha_i(c)/\beta + V(S - i) \). Hence, in conjunction with Proposition 4, we have the following lower and upper bound on \( V(S) \).

**Proposition 5** For \( |S| \geq 2 \),

\[
\max_{i \in S} \{ \alpha_i(c)/\beta + V(S - i) \} \leq V(S) \leq \sum_{i \in S} \alpha_i(c/|S|)/\beta
\]

Assuming that the values \( \alpha_i(x) \) can be quickly computed, the preceding bounds can be used as the initial values of \( v_l \) and \( v_u \) in the binary search procedure for \( V(S) \) that was presented in the previous section. (Recall that at the moment when \( V(S) \) is to be computed the algorithm supposes we already know all the values \( V(S') \) for \( S' \) a proper subset of \( S \).)

It is not true, for a given set of unsold items \( S \), that if it is optimal to sell item \( 1 \in S \) when the offer vector is \( x_1, \ldots, x_n \) that it would necessarily be optimal to sell item \( 1 \) if the offer vector were \( y_1, \ldots, y_n \) whenever \( y_1 > x_1 \). For a counterexample, consider Example 1 with the modification that

\[
P(Y_1 = 1) = .98, \quad P(Y_1 = 2) = .01, \quad P(Y_1 = 10) = .01
\]
Then it is easy to check that whereas it would be optimal to sell item 1 (and item 2) if the offer vector were $(1, 0, 20)$ it would be optimal to sell neither if the offer vector were $(2, 1)$. However, if it is optimal to sell item 1 when the offer vector is $x_1, \ldots, x_n$ then it would be optimal to sell item 1 if the offer vector were $y_1, \ldots, y_n$ provided that $y_i \geq x_i$ for all $i \in S$. To prove this, we need the following lemma from Bruss and Ferguson (1997), for which we provide a simpler proof than the one of Bruss and Ferguson (1997).

**Lemma 1** For $S$ and $T$ subsets of $\{1, \ldots, n\}$

$$V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$$

**Proof:** Suppose there are $n$ types of items and let $S$ and $T$ denote subsets of types. Consider a seller who has $|S| + |T|$ items to sell, consisting of one each of types in $S \cap T^c$, one each of types in $S^c \cap T$, and two each of types in $S \cap T$. The seller arranges these items into 2 collections, the first consisting of one item of each of the types of $S$ and the second of one item of each of the types of $T$. Also, consider a second seller with exactly the same $|S| + |T|$ items to sell, but who arranges them into two collections with the first collection consisting of one item of each of the types of $S \cup T$, and the second collection of one item of each of the types of $S \cap T$. Suppose that offer vectors $y_1, \ldots, y_n$ appear each period with the interpretation that a buyer is willing to buy from any seller any number of units of item $i$ at a price $y_i$ per unit, for each $i = 1, \ldots, n$. Suppose further that each of the sellers must pay in each period either the amount $c$ if they have unsold items in only one of their collections or $2c$ if they have unsold items in both of their collections. Suppose that the first seller uses an optimal policy to decide which items to sell when offers appear, resulting in the expected total discounted return of $V(S) + V(T)$. Suppose further that the second seller follows the first seller in always selling exactly the same group of items, but choosing whenever possible an item from its second collection. (That is, for instance, if the first seller sells only one of the two type $i$ items in its two collections, then the second seller will sell the type $i$ item that is in its second collection.) It is easy to see that if the remaining unsold items in the collections of the first seller are $S'$ and $T'$, then the remaining unsold items in the collections of the second seller are $S' \cup T'$ and $S' \cap T'$. Consequently, the second seller will have sold his second collection at a time no later than when the first collector had sold either of her collections. Hence,
the return of the second seller is at least as large as that from the first seller, showing that there is a policy for seller 2 with an expected discounted return that is at least $V(S) + V(T)$. Consequently, $V(S \cup T) + V(S \cap T)$, the maximal expected discounted return for seller 2, is at least $V(S) + V(T)$.

We now show, for a given set of unsold items, that the optimal set of items to sell is an increasing function of the offer vector. To be precise, say that the action of selling the items in $S \subset R$ is an optimal action for state $x, R$ if

$$\sum_{i \in S} x_i + \beta V(R - S) = \max_{S' \subset R} \left\{ \sum_{i \in S'} x_i + \beta V(R - S') \right\}$$

**Proposition 6** Let $x \leq y$. If $S$ is an optimal action for state $x, R$ and if $T$ is an optimal action for state $y, R$, then $S \cup T$ is also an optimal action for state $y, R$.

**Proof:** Using the notation $AB = A \cap B$ for sets $A$ and $B$, we need to show that

$$\sum_{i \in S \cup T} y_i + \beta V(RS^cT^c) \geq \sum_{i \in T} y_i + \beta V(RT^c)$$

or, equivalently, that

$$\sum_{i \in ST^c} y_i \geq \beta V(RT^c) - \beta V(RT^cS^c)$$

Because $S$ is an optimal action for state $x, R$ we have

$$\sum_{i \in S} x_i + \beta V(R - S) \geq \sum_{i \in ST} x_i + \beta V(R - ST)$$

Hence,

$$\sum_{i \in ST^c} x_i \geq \beta V(RS^c \cup RT^c) - \beta V(RS^c) \geq \beta V(RT^c) - \beta V(RT^cS^c)$$

where the second inequality followed from applying Lemma 1 to the sets $RS^c$ and $RT^c$. Because $\sum_{i \in ST^c} y_i \geq \sum_{i \in ST^c} x_i$ the preceding establishes (6) and the result is proven. ■
It follows from Proposition 6, upon letting \( \mathbf{x} = \mathbf{y} \), that the union of sets of optimal actions for a given state is also an optimal action. Thus, we can define the **maximal optimal policy** as the optimal policy that at every state sells all items that are contained in any optimal action for that state.

The following corollary of Proposition 6 shows that the optimal set of items that are left unsold in state \((S, y)\) is an increasing function of \( S \).

**Corollary 1** Let \( N(S, y) \) denote the set of items that are not sold when the state is \( S, y \) and the maximal optimal policy is employed. If \( S \subset T \) then \( N(S, y) \subset N(T, y) \).

**Proof:** Let \( M \) be sufficiently large so that an offer of size \( M \) or larger for any item will always be accepted by an optimal policy. Suppose \( S \subset T \). For a given offer vector \( y \), let \( y^* = (y^*_1, \ldots, y^*_n) \) where

\[
y^*_i = \begin{cases} 
\max(y_i, M), & \text{if } i \in S^cT \\
y_i, & \text{otherwise} 
\end{cases}
\]

Noting that \( N(S, y) = N(T, y^*) \) the corollary reduces to proving that \( N(T, y^*) \subset N(T, y) \), which follows from Proposition 6 because \( y \leq y^* \). \( \blacksquare \)

**Remark:** A set function \( W \) that satisfies \( W(S \cup T) + W(ST) \geq W(S) + W(T) \) is said to be **supermodular**. Thus, Lemma 1 states that \( V \) is supermodular. Now,

\[
R(S, y) = \max_{\emptyset \neq S' \subset S} \left[ \sum_{j \in S'} y_j + \beta V(S - S') \right] = \max_{S^* \subset S, S^* \neq S} \left[ \sum_{j \in S^*} y_j - \sum_{j \in S^c} y_j + \beta V(S^*) \right] = \max_{S^* \subset S, S^* \neq S} H(S^*)
\]

where, for a fixed set \( S \),

\[
H(S^*) = \sum_{j \in S} y_j - \sum_{j \in S^*} y_j + \beta V(S^*)
\]

Assuming that all offers are nonnegative, is easy to check that \( H \) will also be a
supermodular function. Thus, after all of the values \( V(S^*), S^* \subset S, S^* \neq S \), have been determined, in order to find \( R(S, y) \) it is necessary to find the maximum of a supermodular function. An efficient polynomial time algorithm for doing this is given in Iwata (2002).

\[ \text{4.1 Case of Offer Vectors being Vectors of Independent and Identically Distributed Random Variables} \]

Suppose that the offer vector \( Y_1, \ldots, Y_n \) is a vector of \( n \) iid random variables having distribution function \( F \). In this case, we can take the state to be \( r, y_1, \ldots, y_r \) when there are currently \( r \) unsold items and the buyer offers the amounts \( y_1, y_2, \ldots, y_r \) for these \( r \) items. With \( y = (y_1, \ldots, y_r) \), let \( V(r, y) \) be the maximal expected additional discounted return when the state is \( r, y \). Also, let \( V(r) = E[V(r, Y_1, \ldots, Y_r)] \) where \( Y_1, \ldots, Y_r \) are iid with distribution \( F \), so that \( V(r) \) is the maximal expected additional discounted return when there are \( r \) unsold items. The optimality equation is

\[
V(r, y) = \max\{R(r, y), \beta V(r)\} - c
\]

where

\[
R(r, y) = \max_{1 \leq j \leq r} \left\{ \sum_{i=1}^{j} y(i) + \beta V(r-j) \right\}
\]

where \( y(i) \) is the \( i^{\text{th}} \) largest of the values \( y_1, \ldots, y_r \), and \( V(0) = 0 \).

It is an immediate consequence of Lemma 1 that \( V(r) \) is a convex function:

**Corollary 2** \( V(r) - V(r - 1) \) is nondecreasing in \( r \geq 1 \).

**Proof:** Let \( S \) and \( T \) both be sets of \( r - 1 \) items that have \( r - 2 \) items in common and apply Lemma 1. \( \blacksquare \)

**Remarks:** (a) Because

\[
V(r) = V(r - 1) + V(r) - V(r - 1) \geq V(r - 1) + V(r - 1) - V(r - 2)
\]

we can use \( 2V(r - 1) - V(r - 2) \) as a lower bound for \( V(r) \) when using the compu-
tational procedure for $V(r)$.

(b) If we supposed that the $n$ items are separately owned, with those owners whose items have not yet sold equally sharing the cost for the coming period then we might suspect that the socially optimal policy we’ve been considering is a Nash equilibrium policy in this game theoretic context. But this is not the case. To see why not, let $\beta = 1, n = 2$ and note that $V(2) > 2V(1)$ (indeed, $V(2) \geq 2V(1) + c$). Now, suppose the offer $x_1, -\infty$ is received, where $V(2)/2 < x_1 < V(2) - V(1)$. Because $x_1 + V(1) < V(2)$ the socially optimal policy would call for the owner of item 1 to reject that offer. However, because under the socially optimal policy the total expected return of both items is $V(2)$, it follows by symmetry that if both owners employed that policy then each would have an expected return of $V(2)/2$. Hence, because $x_1 > V(2)/2$, the owner of item 1 would increase his expected return by accepting the offer $x_1$.

**Example 2.** Table 1 gives the values of $V(n)$ obtained by the computational procedure when $X_1, \ldots, X_n$ are independent uniform $(0, 1)$ random variables and $c = .1$. The values for $n = 1, 2, 3$ agree with those obtained in Bruss and Ferguson (1997). (No numerical results are given in Bruss and Ferguson (1997) when $n > 3$.)

Table 1: $V(n)$ when $X_1, \ldots, X_n$ are independent uniform $(0, 1)$ and $c = .1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.553</td>
</tr>
<tr>
<td>2</td>
<td>1.273</td>
</tr>
<tr>
<td>3</td>
<td>2.035</td>
</tr>
<tr>
<td>4</td>
<td>2.826</td>
</tr>
<tr>
<td>5</td>
<td>3.637</td>
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<td>6</td>
<td>4.463</td>
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<tr>
<td>7</td>
<td>5.302</td>
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<tr>
<td>8</td>
<td>6.150</td>
</tr>
<tr>
<td>9</td>
<td>7.007</td>
</tr>
<tr>
<td>10</td>
<td>7.871</td>
</tr>
</tbody>
</table>
5 A Special Case: The Single Item Model

In this section, we consider the special case model where a maximum of one item can be sold each period. That is, we suppose that $k = n$ and that an offer is a vector $x_1, \ldots, x_n$, meaning that the buyer is willing to pay $x_i$ for item $i$ for each $i = 1, \ldots, n$, but only wants to purchase at most one item. The optimality equation is

$$V(S, x) = \max (\beta V(S), R(S, x)) - c$$

where

$$R(S, x) = \max_{i \in S} (x_i + \beta V(S - i))$$

One might imagine that if it is optimal to accept one of the offers from the offer vector $x$ when the set of unsold items is $S$ then it would be optimal to accept one of the offers of $x$ when the set of unsold items is a superset of $S$. That is, one might suppose that “if it is optimal to accept an offer when the state is $(S, x)$, then it is optimal to accept an offer when the state is $(S + j, x)$, $j \notin S$”. However this need not be true, as shown by the following example.

**Counterexample 1:** Suppose $n = 2$ and that $X_1$ and $X_2$ are independent and such that $P(X_i = 100) = .01 = 1 - P(X_i = 1), i = 1, 2$. Suppose $c = \beta = 1$. Because $E[(X_i - 1)^+] = .99 < 1$, we see that $\alpha_1(1) < 1$, and so it is optimal to accept the offer $(1, 1)$ when the set of unsold items is $S = \{1\}$. However, when the set of unsold items is $S = \{1, 2\}$, the offer vector $(1, 1)$ should be rejected, because a better strategy is to reject that offer and wait for an offer vector that has at least one of its values equal to 100, selling an item at that time for 100 and then selling the other item for the next offer to appear. To show this, note first that the expected return if the offer $(1, 1)$ is accepted is $-1 + 1 + \alpha_1(1) < 1$. On the other hand, because

$$P(\max(X_1, X_2) = 100) = .02 - .0001 = .0199$$

it follows that the expected return under the policy that waits for an offer vector having at least one of its component values equal to 100 is $-1 - \frac{1}{.0199} + 100 + \alpha(1) \approx 49 + \alpha_1(1)$. Hence, whereas it is optimal to accept the offer vector $(1, 1)$ when item 1 is the only unsold item, it is optimal to reject it when both items 1 and 2 are unsold.
Another seemingly reasonable hypothesis is that if is optimal to accept the offer for item $i$ when the state is $(S, x)$ then the optimal decision when the state is $(S+j, x)$ is either to reject the offer entirely or to sell either $i$ or $j$. That this too need not be true is shown in our next counterexample.

**Counterexample 2:** Suppose that $S = \{1, 2, 3\}$, and that $c = \beta = 1$. Let $U$ and $V$ be independent random variables, each being uniform on $(0, 10)$. Suppose that $(X_1, X_2, X_3)$ has the same joint distribution as does $(U, V, 10 - U)$. Note that the marginal distributions of the $X_i$ are identical. Now, if the offer vector $(9, 8.9, 1)$ is received when the set of unsold items is $\{1, 2\}$, then it is optimal to sell item 1. However, if that same offer were to be received when the set of unsold items were $\{1, 2, 3\}$ then, because $\max(U, 10 - U)$ is stochastically larger than $\max(U, V)$, it would now be optimal to sell item 2.

### 5.1 The Case of Exchangeable Offer Vectors

One case where our computational approach can be used, even for large $n$, to find the optimal policy is when the the offer vector $X_1, \ldots, X_n$ is exchangeable. In this case, the state reduces to the pair $(r, x)$ where $r$ is the number of items still unsold and $x$ is the maximum of the current offered values for these $r$ items. With $V(r, x)$ defined as the maximal expected additional net return given that the current state is $(r, x)$, the optimality equation becomes

$$V(r, x) = \max\{x + \beta V(r - 1), \beta V(r)\} - c$$

where

$$V(r) = E[V(r, M_r)]$$

and where $M_j$ has the distribution of $\max_{1 \leq i \leq j} X_i$. It follows that the optimal policy is to accept the offer of value $x$ when in state $(r, x)$ if and only if

$$x \geq \beta[V(r) - V(r - 1)]$$
We now show that when $\beta = 1$ the critical values $V(r) - V(r - 1)$ increase in $r$.

**Proposition 7** If $\beta = 1$, then $V(r)$ is a convex function.

**Proof:** Letting $V_j(1)$ equal the maximal expected non-discounted net return in the single item selling problem in which a cost $c$ is paid when offers are received and the offers have the distribution of $M_j = \max_{1 \leq i \leq j} X_i$, it follows that

$$V(r) = V_r(1) + V_{r-1}(1) + \ldots + V_1(1)$$

Hence,

$$V(r) - V(r - 1) = V_r(1)$$

Because $M_r$ is stochastically increasing in $r$, it follows that the single item selling problem in which the offers have the distribution of $M_r$ has a maximal expected return that increases in $r$. Thus, $V_r(1)$ increases in $r$ and the result is proven. \(\blacksquare\)

**Remark:** $V(r)$ need not be convex when $\beta < 1$. A simple counterexample is obtained when $P(X_i = c + 1) = 1$. For then, $V(r) = 1 + \beta + \ldots + \beta^{r-1}$, yielding that $V(r) - V(r - 1) = \beta^{r-1}$, which decreases in $r$.

### 5.2 Applications

The “single item” model considered in this section, which assumes that only a single item can be sold in each period, has important applications in many areas. For instance, suppose there are $n$ patients who are waiting for an organ transplant. Each arriving organ will have a value for each individual patient, with the value depending on such things as the distance between the donor organ and the patient’s transplant center, the blood type of the donor, tissue typing of the donor organ, as well as on the urgency of the patient’s need. If we let $X_1, \ldots, X_n$ be the value vector of a random organ to these $n$ patients, the problem becomes one of choosing which, if any, of the patients is to receive a newly arrived organ, with the objective being to maximize the total expected discounted net return, where the net return is the sum of the values obtained by the patients minus the total costs incurred until all have received an organ.
By interpreting items as workers and buyers as jobs, our model is a generalization of the *sequential stochastic assignment problem* introduced by Derman, Lieberman, and Ross (1972) (hereafter: DLR). DLR’s model supposed that there are \( n \) workers with worker \( i \) having a specified value \( p_i \). Jobs appear sequentially, with each job having a value \( X \) that is chosen, independently from job to job, according to a known distribution function. If a worker with value \( p_i \) is assigned to a job with value \( x \) then a return \( p_i x \) is earned and the problem was to assign workers to jobs so as to maximize the expected sum of values. (DLR assumed that each job had to be assigned to some worker so the problem ended when \( n \) jobs had appeared.) Thus the model of DLR is a special case of our model in which the random offer vectors are of the special form \((p_1 X, p_2 X, \ldots, p_n X)\) with \( X \) having a specified distribution. Because of this special structure, the paper DLR was able to show that for each \( k = 1, \ldots, n \) there were critical numbers \( x_{1,k} \leq x_{2,k} \leq \ldots \leq x_{k,k} \) such that if a job having value \( x \) appeared when there were \( k \) unassigned workers, then the optimal strategy was to assign the job to the unassigned worker having the largest value \( p \) if \( x \) was greater than \( x_{k,k} \), to assign it to the worker having the second largest \( p \) value if \( x \) was between \( x_{k-1,k} \) and \( x_{k,k} \), and so on. Using that these values \( x_{i,k} \) did not depend on the values \( p_1, \ldots, p_n \), the paper was able to explicitly determine these critical values. DLR also obtained some structural results about the optimal policy when the return earned when a job of value \( x \) was assigned to a worker of value \( p \) was a general return function \( r(p, x) \) such that \( \frac{\partial^2}{\partial p \partial x} r(p, x) \geq 0 \). This special case of our more general sequential stochastic assignment model has previously been applied to a variety of asset selling and transplant decision models (see, for instance, Righter (1987) and Su and Zenios (2005)).

6 The Single Item per Buyer Stopping Rule Problem

We now solve the stopping rule problem for the “single item” model. Suppose, as in the previous section, that buyers bid only for the one point subsets. Suppose that no decision as to the item to be sold to a buyer needs be made before the decision is made to stop observing future offers. Let \( R(x_1, \ldots, x_k) \) be the maximal possible return if one stops after having received \( k \) offers with values \( x_1, \ldots, x_k \). (This maximal possible return can be obtained by solving the classical assignment problem.) Also, let \( X \) have the distribution of a random offer vector. The dynamic programming state of the
system is the vector \((x_1, \ldots, x_k)\), meaning that \(k\) buyers have presented offers, with resulting offer vectors \(x_1, \ldots, x_k\). If \(V(x_1, \ldots, x_k)\) represents the maximal expected additional discounted (from time \(k\)) net return if the current state is \((x_1, \ldots, x_k)\), then the optimality equation is

\[
V(x_1, \ldots, x_k) = \max\{R(x_1, \ldots, x_k), \beta E[V(x_1, \ldots, x_k, X)]\} - c
\]

The one stage lookahead policy is defined as that policy that stops when in a state if and only if stopping is better than going exactly one more stage and then stopping. Thus, in our problem, the one stage lookahead policy stops when in state \(x_1, \ldots, x_k\) if and only if

\[
R(x_1, \ldots, x_k) \geq \beta(E[R(x_1, \ldots, x_k, X)] - c)
\]  

(7)

We will prove that the preceding one stage lookahead policy is optimal. To do so, we will use the following result from Shapley (2006), stating that if you must stop after one more offer vector, then the marginal benefit (compared with stopping before that final offer vector) decreases as the set of offers previously received increases.

**Lemma 2**

\[
R(x_1, \ldots, x_k, x) - R(x_1, \ldots, x_k) \geq R(x_1, \ldots, x_k, x_{k+1}, x) - R(x_1, \ldots, x_k, x_{k+1})
\]

**Proposition 8** The one stage lookahead policy is an optimal policy.

**Proof:** To show this we have to show that the set of stopping states of the one-stage lookahead policy is closed in the sense that if this policy would call for stopping in the state \(x_1, \ldots, x_k\) then it would also call for stopping in state \(x_1, \ldots, x_k, x_{k+1}\) for any \(x_{k+1}\) (see Ross (1983)). Thus, we need to show that (7) implies that

\[
R(x_1, \ldots, x_k, x_{k+1}) \geq \beta(E[R(x_1, \ldots, x_k, x_{k+1}, X)] - c)
\]

Now, note that the preceding would follow if we can show that

\[
E[R(x_1, \ldots, x_k, x_{k+1}, X)] - E[R(x_1, \ldots, x_k, X)] \leq \frac{R(x_1, \ldots, x_k, x_{k+1}) - R(x_1, \ldots, x_k)}{\beta}
\]
That is, it suffices to show that

\[ E[R(x_1, \ldots, x_k, x_{k+1}, X) - R(x_1, \ldots, x_k, X)] \leq \frac{R(x_1, \ldots, x_k, x_{k+1}) - R(x_1, \ldots, x_k)}{\beta} \]

which will be true if for all \( x \)

\[ R(x_1, \ldots, x_k, x_{k+1}, x) - R(x_1, \ldots, x_k, x) \leq \frac{R(x_1, \ldots, x_k, x_{k+1}) - R(x_1, \ldots, x_k)}{\beta} \]

Using that \( R(x_1, \ldots, x_k, x_{k+1}) \geq R(x_1, \ldots, x_k) \) and that \( 0 < \beta \leq 1 \), the preceding will hold if

\[ R(x_1, \ldots, x_k, x_{k+1}, x) - R(x_1, \ldots, x_k, x_{k+1}) \leq R(x_1, \ldots, x_k, x) - R(x_1, \ldots, x_k) \]

Thus, the result follows from Lemma 2. \( \square \)

**Remark:** The analogue of Lemma 2 does not hold when \( V \) replaces \( R \). That is, it need not be true that

\[ V(x_1, \ldots, x_k, x) - V(x_1, \ldots, x_k) \geq V(x_1, \ldots, x_k, x_{k+1}, x) - V(x_1, \ldots, x_k, x_{k+1}) \]

For a counterexample, suppose that \( k = n = 2 \), that \( P(X_1 = i) = 1/2, i = 1, 2, \) and \( P(X_2 = 1) = .99 = 1 - P(X_2 = 1000) \). Take \( \beta = c = 1 \), and suppose that \( x_1 = (1, 1), x_2 = (1, 1000) \), and \( x = (2, 1) \). Then

\[ V(x_1, x_2, x) - V(x_1, x_2) = 1 \]

because in both cases it is optimal to stop. On the other hand,

\[ V(x_1, x) - V(x_1) \approx 0 \]

because the optimal policy will not stop before a candidate having a second component score value of 1000 is observed, and almost certainly there would have been a first component score value of 2 by that time. \( \square \)
Before showing how to compute the one stage lookahead policy we first show that any buyer who would not be assigned (that is, not sold an item) if we were currently to stop and make the best possible assignments using only those offers already seen would also not be assigned if we stopped at a future time.

**Proposition 9** If buyer $o \in S$ is unassigned by the optimal policy for selling the $n$ items when only using the set of buyers in $S$, then $o$ will also be unassigned by the optimal policy for selling the $n$ items when only using the set of buyers in $S \cup r$, where $r \notin S$.

**Proof:** The proof is by induction on $n$. It is immediate when $n = 1$ because if buyer $o$ does not have the highest offer of those in $S$ for the single item available, then it would not have the highest offer of those in $S \cup r$. So, assume that the result is true whenever there are $n - 1$ items to be sold, and now suppose that there are $n$ items to be sold. Consider the optimal policy for selling these $n$ items using only buyers in $S$, and let $p_S(i), i = 1, \ldots, n$ be the buyer assigned by this policy to item $i$. Suppose that $p_S(i) \neq o, i = 1, \ldots, n$, so that buyer $o$ is not assigned by the optimal policy, and note that this means that $o$ would not be assigned by the optimal policy for selling items $1, \ldots, n - 1$ when $S - p_S(n)$ is the set of buyers available.

Now consider the optimal assignment when $S \cup r$ is the set of buyers available for the $n$ items. There are two cases:

**Case 1:** Buyer $r$ is unassigned. In this case, the optimal assignment would just be the optimal assignment for items $1, \ldots, n$ using only buyers in $S$. Hence, buyer $o$ would be unassigned in this case.

**Case 2:** Buyer $r$ is assigned, say to item $n$. In this case the rest of the optimal assignment is the optimal assignment of items $1, \ldots, n - 1$ when only using buyers in $S$. Because, as previously noted, $o$ would not be assigned when selling items $1, \ldots, n - 1$ when only using buyers in $S - p_S(n)$, it follows by the induction hypothesis that $o$ would not be assigned when using buyers in $S$. This completes the induction proof.

We now give the algorithm for determining the one stage lookahead policy. In solving the successive deterministic assignment problems the algorithm given in Toroslu and Üçoluk (2007) which efficiently uses the solution of an assignment problem involving
$n$ buyers and $n$ items to solve the problem that results when one additional buyer is added should be employed.

1. Observe the first $n$ buyers. Let $x_1, \ldots, x_n$ be their offer vectors. Use the Hungarian algorithm (see Toroslu and Üçoluk (2007)) to solve for $R(x_1, \ldots, x_n)$ and to find the optimal assignments when only these buyers are available.

2. To determine whether to stop we need to evaluate $E[R(x_1, \ldots, x_n, X)]$, which we will do via simulation. So, simulate $m$ offer vectors, say their values are $y_1, y_2, \ldots, y_m$. Using the algorithm of Toroslu and Üçoluk (2007) compute the values $R(x_1, \ldots, x_n, y_i)$ for $i = 1, \ldots, m$. Now use $\frac{1}{m} \sum_{i=1}^{m} R(x_1, \ldots, x_n, y_i)$ as the value of $E[R(x_1, \ldots, x_k, X)]$. Consequently, if

$$R(x_1, \ldots, x_n) \geq \beta \left( \frac{1}{m} \sum_{i=1}^{m} R(x_1, \ldots, x_n, y_i) - c \right)$$

then stop.

3. Observe the next offer vector, say it is $x_{n+1}$. Use the algorithm of Toroslu and Üçoluk (2007) to find $r = R(x_1, \ldots, x_n, x_{n+1})$, and well as the optimal assignment if you were to stop at that point. Using that the optimal assignment will not assign one of the $n + 1$ buyers, renumber the buyers so that $x_1, \ldots, x_n$ represent the offer vectors of the $n$ buyers that are assigned by the optimal assignment if you were to stop at that point. Note that with this renumbering $R(x_1, \ldots, x_n) = r$. Now, go to Step 2.

**Remark:** Because the simulation is being performed only to determine if

$$R(x_1, \ldots, x_n) \geq \beta (E[R(x_1, \ldots, x_n, X)] - c)$$

and as it probably makes little difference what is done if the two sides of the preceding are roughly equal, the value of $m$, the number of simulation runs done to evaluate $E[R(x_1, \ldots, x_n, X)]$, need not be very large.
7 Extensions

In this section we note that our results extend to continuous time models as well as allowing more general per period cost functions.

7.1 To a Continuous Time Model

Suppose that buyers arrive according to a Poisson process with rate \( \lambda \), and that until all items are sold the seller incurs costs at rate \( a \) per unit time as well as a fixed cost \( b \) each time a buyer appears. Suppose that our objective is to maximize the total expected present value net return under continuous compounding with interest rate \( \alpha \). (That is, \( \alpha \) is the discount factor in the sense that a return \( R \) earned at time \( t \) is given value \( Re^{-\alpha t} \).)

As before, say that the state of the system is \((S, x)\) if \( S \neq \emptyset \) is the set of items that remain to be sold and the offer vector \( x = (x_{S_1}, \ldots, x_{S_k}) \) has just been received. Let \( V_\alpha(S, x) \) be the maximal expected additional discounted return in this state. Also, with \( V_\alpha(S) = E[V_\alpha(S, X)] \), let

\[
V^*_\alpha(S) = \int_0^\infty \lambda e^{-\lambda y}(e^{-\alpha y}V_\alpha(S)) - \int_0^y ae^{-\alpha s}ds \ dy = \frac{\lambda}{\lambda + \alpha}V_\alpha(S) - \frac{a}{\lambda + \alpha}
\]  

be the maximal expected additional discounted return when \( S \) is the set of items that remain to be sold and we are waiting for the next offer.

The optimality equation is

\[
V_\alpha(S, x) = \max (V^*_\alpha(S), \ R_\alpha(S, x)) - b
\]

where \( R_\alpha(S, x) \equiv \max_{1 \le i \le k: S_i \subset S}[x_{S_i} + V^*_\alpha(S - S_i)] \).

Analogous to Proposition 1, we can prove

**Proposition 10** \( V^*_\alpha(S) \) is the unique value \( v \) such that

\[
\frac{\alpha}{\lambda} v + \frac{a}{\lambda} + b = E[(R_\alpha(S, X) - v)^+]
\]

Furthermore, the policy that, when in state \((S, x)\), does not sell any of the subsets if \( R_\alpha(S, x) < V^*_\alpha(S) \); and, if \( R_\alpha(S, x) \ge V^*_\alpha(S) \), sells one of the subsets \( S_j \subset S \) for
which $R_\alpha(S, x) = x_{S_j} + V_\alpha^*(S - S_j)$, is an optimal policy.

All the results for the discrete time problem carry over to the continuous time.

### 7.2 To a More General Per Period Cost Function

If the cost incurred when the set of unsold items is $S$ is a general function $c(S)$, then the computational approach of Section 3 can still be employed. Propositions 1 and 2 of that section remain true when $c$ is replaced by $c(S)$.

Lemma 1, as well as the following Proposition 6 and Corollary 1, of Section 4 remain true if we suppose that $c(S)$ satisfies the submodularity property

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T)$$

where $S$ and $T$ are subsets of $\{1, \ldots, n\}$. For instance, it would suffice if we assumed that, for nonnegative constants $c, c_1, \ldots, c_n$,

$$c(S) = \begin{cases} 
  c + \sum_{i \in S} c_i, & \text{if } S \neq \emptyset \\
  0, & \text{if } S = \emptyset 
\end{cases} \quad (11)$$

Proposition 3 also remains valid when $\alpha_i(c) = \alpha(\{i\})$. (By the submodularity assumption on $C$, the per period cost difference between what is incurred under $\pi^*$ versus under $\pi$ is at most $c(\{i\})$ until item $i$ is sold. Hence, the expected return under $\pi^*$ is at least the expected return under $\pi$ plus $\beta V(\{i\}) - y_i$. As this latter difference is positive, the expected return under $\pi^*$ is greater than it is under $\pi$.) If, in the model of Section 4.1, we supposed that the per period cost is $c(r)$ whenever there remains $r > 0$ unsold items, then Corollary 2 remains valid provided $c(r)$ is assumed concave.

### 8 Conclusion

In this paper, we generalized the classic job search model to a setting where multiple items can be sold in each period. This model has many potential real-world applications, for example, to firms hiring for multiple positions, salesman selling blocks of tickets, organ donation, etc. Despite the general nature of the problem and the ready availability of applications, however, there are few papers analyzing search models
with multiple items. Our main motivation with this paper is to help researchers better use multiple item search models in the future by providing methods for computing the value function, and by providing results about the shape of the value function and the structure of the optimal policy.
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