1 Ch.1 - Manifolds and Tensors

Definition 1.1. A differentiable manifold is a set $M$ with a collection of open sets $\mathcal{O}_\alpha$ such that

i. $\bigcup_\alpha \mathcal{O}_\alpha$ cover $M$,

ii. For each $\alpha$ there is an injective map $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \subset \mathbb{R}^n$ such that $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset U_\alpha \subset \mathbb{R}^n \to \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset U_\beta \subset \mathbb{R}^n$

is smooth.

Theorem 1.2 (Whitney). All $n$-dimensional manifolds are surfaces in $\mathbb{R}^N$ for suitably large $N \geq n$, $N \leq 2n+1$.

Definition 1.3. Let $M, \tilde{M}$ be manifolds of dimension $n$ and $\tilde{n}$ respectively. A map $f : M \to \tilde{M}$ is said to be smooth if $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^{\tilde{n}}$ is smooth for all $\alpha, \beta$.

Definition 1.4. Let $\gamma : \mathbb{R} \to M$ be a smooth curve with $\gamma(0) = p \in M$. The tangent vector $V$ to $\gamma$ at $p$ is defined by

$$\left. \frac{d\gamma(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = V|_p \in T_p M.$$

Definition 1.5. The tangent space is the vector space of all tangents to curves in $M$ passing through point $p$. As a vector space of dimension $n$, $T_p M \cong \mathbb{R}^n$.

Definition 1.6. The disjoint union of the tangent spaces at each point on a manifold is known as the tangent bundle,

$$TM = \bigcup_{p \in M} T_p M.$$

Definition 1.7. A vector field assigns a tangent vector at each point $p \in M$. It can be written as

$$V = V^a(x) \frac{\partial}{\partial x^a},$$

where $x^a$ are the coordinate functions on $\gamma$.

Definition 1.8. An integral curve or flow $\gamma = \gamma(\epsilon)$ of a vector field $V$ is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}$$

or equivalently in components by $\dot{\gamma}^a(\epsilon) = V^a(x(\epsilon))$.

Definition 1.9. An invariant of a vector field $V$ is a function $f$ that is constant along the flow, i.e., along the integral curves of $V$.

$$f(x^a(0)) = f(x^a(\epsilon)) \iff V(f) = 0.$$
Definition 1.10. The Lie bracket of two vector fields $V, W$ is a vector field $[V, W]$ defined by its action on functions

$$[V, W]f = V(W(f)) - W(V(f)).$$

It satisfies antisymmetry and the Jacobi identity.

Definition 1.11. A Lie algebra is a vector space $g$ equipped with an anti-symmetric bilinear operation $[\ , \ ] : g \times g \to g$ which satisfies the Jacobi identity.

Definition 1.12 (Ado’s theorem). Every finite dimensional Lie algebra is isomorphic to some matrix Lie algebra.

2 Ch.2 - Lie Groups

Definition 2.1. A Lie group is a group with the structure of a smooth manifold such that group operation $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ and inverse $G \to G, g \mapsto g^{-1}$ are smooth maps between manifolds.

Definition 2.2. Let $G$ be a group and $M$ a manifold. A group action on a manifold is a smooth map $G \times M \to M$, $(g, p) \mapsto g(p)$ such that

i. $e(p) = p$ for all $p \in M$ where $e$ is the identity in $G$.

ii. $g_2(g_1(p)) = (g_2 g_1)(p)$ for all $p \in M, g_1, g_2 \in G$.

Definition 2.3. Let $f : M \to N$ be a smooth map between manifolds. Let $\gamma : I \to M$ be a curve in $M$ and suppose that $V \in T_p M$ is a vector field tangent to $\gamma$. The tangent map or pushforward $f_* : T_p M \to T_{f(p)} N$ is defined by

$$f_*(V) = \frac{d}{d\epsilon} f(\gamma(\epsilon)) \bigg|_{\epsilon = 0}. $$

In components we have

$$(f_* V)^a = V^i \frac{\partial y^a}{\partial x^i}$$

where $x^i$ are local coordinates on $M$ and $y^i$ are local coordinates on $M$.

Definition 2.4. Let $V, W$ be vector fields on $M$ such that $V = \dot{\gamma}$, i.e., $\gamma$ is the integral curve of $V$. The Lie derivative is defined by

$$\mathcal{L}_V W = \lim_{\epsilon \to 0} \frac{W(p) - (\gamma(\epsilon), W(p))}{\epsilon}.$$

Definition 2.5. Let $dx^i, dx^j$ be covector fields. The wedge product is the antisymmetrised product defined by

$$dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

Definition 2.6. A one-form is a covector field, $\Omega = \Omega_i dx^i$. A k-form is an antisymmetric tensor field of the form

$$\Omega = \frac{1}{r!} \Omega_{i_1 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r}.$$

Definition 2.7. The differential or exterior derivative $d$ of a one-form is defined by

$$d\Omega = \frac{\partial \Omega}{\partial x^j} dx^j \wedge dx^i.$$

Theorem 2.8 (Cartan’s formula).

$$\mathcal{L}_V \Omega = d(V \lhd \Omega) + V \lhd d\Omega.$$
Definition 2.9. The Lie algebra \( g \) of a Lie group \( G \) is the tangent space \( T_e G \) to \( G \) at the identity element \( e \in G \). The bracket in \( g \) is the bracket of vector fields in \( G \).

Definition 2.10. The function known as left translation is defined by

\[
L_g : G \rightarrow G \quad \text{with} \quad L_g(h) = gh
\]

for any \( g, h \in G \). This function also induces a pushforward map

\[
(L_g)_*: g = T_e G \rightarrow T_p G, \quad \forall \in T_e G \mapsto (L_g)_* \in T_p G.
\]

Right translation is defined analogously with \( R_g(h) = gh^{-1} \).

Definition 2.11. Let \( X \) be a vector field on \( G \). \( X \) is a left-invariant vector field if \((L_g)_* X_{g'} = X_{g'g}\) for all \( g, g' \in G \).

Lemma 2.12. Let \( \Omega \) be a 1-form and let \( V, W \) be vector fields. Then

\[
d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]).
\]

Proof:

\[
d\Omega(V, W) = (d\Omega)_{\alpha b} V^a W^b
\]

\[
= (\nabla_\alpha \Omega_b)V^a W^b - (\nabla_b \Omega_\alpha)V^a W^b
\]

\[
= \nabla_V(\Omega_b W^b) - \Omega_b \nabla_V W^b - \nabla_W(\Omega_b V^a) + \Omega_a \nabla_W V^\alpha
\]

\[
= V(\Omega(W)) - W(\Omega(V)) - \Omega_a(V^b \nabla_b W^a - W^b \nabla_b V^a)
\]

\[
= V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]).
\]

Proposition 2.13 (Maurer-Cartan relations). Let \( \{ L_\alpha, \alpha = 1, ..., \text{dim}(g) \} \) be a basis of left-invariant vector fields such that \([L_\alpha, L_\beta] = f_{\alpha \beta}^\gamma L_\gamma\). Let \( \{ \sigma^\alpha \} \) be the dual basis of left-invariant one-forms, i.e., \( L_\alpha \cdot \sigma^\beta = \delta_\alpha^\beta \). Then

\[
d\sigma^\alpha + \frac{1}{2} f_{\beta \gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma = 0.
\]

Proof: Using the previous lemma,

\[
(d\sigma^\alpha)(L_\beta, L_\gamma) = L_\beta(\sigma^\alpha(L_\gamma)) - L_\gamma(\sigma^\alpha(L_\beta)) - \sigma^\alpha([L_\beta, L_\gamma])
\]

\[
= L_\beta \delta_\gamma^\alpha - L_\gamma \delta_\beta^\alpha - \sigma^\alpha([L_\beta, L_\gamma])
\]

\[
= -\sigma^\alpha f_{\beta \gamma}^\rho L_{\rho}
\]

\[
= -f_{\beta \gamma}^\alpha.
\]

Antisymmetrising, we have

\[
d\sigma^\alpha = -\frac{1}{2} f_{\beta \gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma.
\]

Definition 2.14. Let \( g \in G \). The left-invariant Maurer-Cartan one-form on \( G \) is the Lie algebra valued 1-form defined on vectors in \( T_g G \) by

\[
\rho : T_g G \rightarrow T_e G = g;
\]

\[
\rho(v) = (L_g^{-1})_* v, \quad v \in T_g G;
\]

\[
\rho : = g^{-1} dg = \sigma^\alpha \otimes T_\alpha.
\]

Similarly, the right-invariant Maurer-Cartan one-form

\[
\tilde{\rho} : = dgg^{-1} = \tilde{\sigma}^\alpha \otimes T_\alpha.
\]
Theorem 2.15 (Maurer-Cartan equation).

\[ d\rho + \rho \wedge \rho = 0. \]

Proof: From \( \rho := g^{-1}dg \),

\[ d\rho = -g^{-1}(dg)g^{-1} \wedge dg + g^{-1}d\rho - \rho \wedge \rho. \]

Definition 2.16. The left-invariant metric on a Lie group is

\[ h = g_{\alpha\beta} \sigma^\alpha \otimes \sigma^\beta \]

where \( g_{\alpha\beta} \) is a symmetric non-degenerate constant matrix, and \( \otimes \) denotes a symmetrised tensor product.

3 Ch.3 - Hamiltonian Mechanics and Symplectic Geometry

Definition 3.1. Let \( M \) be a manifold of dimension \( 2n \) and let \( f, g \) be smooth functions on \( M \). A Poisson structure is a map \( \{ , \} : C^\infty(M) \times C^\infty(M) \to \mathbb{R} \) defined in coordinates using \( \omega^{ij} = \omega^{[ij]} \) according to

\[ \{ f, g \} = \sum_{i,j=1}^{n} \omega^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \]

such that the Jacobi identity holds. For functions \( f, g \), the Poisson structure

\[ \{ f, g \} := X_g f = \omega(X_g, X_f) = -\omega(X_f, X_g) = -X_f g = -\{ g, f \}. \]

Definition 3.2. A symplectic manifold is a smooth manifold \( M \) of even dimension \( \dim(M) = 2n \) with a two-form \( \omega \) that is closed \( (d\omega = 0) \) and non-degenerate \( (\omega \wedge \cdots \wedge \omega \neq 0, \text{with } n \text{ number of } \omega \text{'s}). \)

Definition 3.3 (Hamiltonian vector field).

- Because \( \omega \) is non-degenerate, there is an isomorphism between \( TM \) and \( T^*M \) given by

\[ v \in TM \implies v \cdot \omega \in T^*M. \]

Hence 1-forms on a symplectic manifold \( M \) may be identified with vector fields, and every differentiable function \( H : M \to \mathbb{R} \) determines a unique vector field \( X_H \), the Hamiltonian vector field, by requiring that

\[ X_H \cdot \omega = -dH. \]

We can now give another definition of the Poisson structure,

\[ \{ f, g \} := X_g f = \omega(X_g, X_f) = -\omega(X_f, X_g) = -X_f g = -\{ g, f \}. \]

- There is a Lie algebra homomorphism between the Lie algebra of functions on \( M \) with the Poisson bracket, and Hamiltonian vector fields with the Lie bracket. In particular,

\[ [X_f, X_g] = -X_{\{ f, g \}}. \]
Theorem 3.4 (Darboux’s theorem). Let $(M, \omega)$ be a 2n-dimensional symplectic manifold. There exists local coordinates $x^1 = q^1, \ldots, x^n = q^n, x^{n+1} = p_1, \ldots, x^{2n} = p_n$ around any point on $M$ such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

and the Poisson bracket takes the standard form.

Definition 3.5. Let $M, N$ be manifolds and let $f : M \to N$ be a smooth function between the manifolds. Then the pullback $f^* : T^*_f(p)N \to T^*_p M$. In components we have

$$f^*(dy^i) = \frac{\partial y^i}{\partial x^a} dx^a,$$

where $x^a$ and $y^i$ are coordinates on $M$ and $N$ respectively. Note that $(f^*(dy))(V) = dy(f_*(V))$.

Definition 3.6. Let $(M, \omega)$ be a symplectic manifold of dimension 2n. The one-parameter groups of $\omega$-preserving transformations generated by Hamiltonian vector fields are known as canonical transformations.

Definition 3.7. Let $K = K^{i_1 \ldots i_r} p_{i_1} \ldots p_{i_r}$, where $K^{i_1 \ldots i_r}(x)$ is a symmetric $(r,0)$ tensor on $M$. If $K^{i_1 \ldots i_r}$ satisfies the Killing tensor equation

$$\{K, H\} = 0 \iff \nabla_{(i_1} K_{i_2 \ldots i_r+1)} = 0,$$

then we say that $K^{i_1 \ldots i_r}$ is a Killing tensor.

Definition 3.8. An integrable system is a symplectic manifold $(M, \omega)$ of dimension 2n together with $n$ functions $f_1 : M \to \mathbb{R}$ such that

i. $\{f_j, f_k\} = 0$, i.e., the functions are in involution;

ii. $df_1 \wedge \ldots \wedge df_n \neq 0$ at all points on $M$, i.e., the $n$ functions are functionally independent.

Theorem 3.9 (Arnold-Liouville). Let $(M, \omega, f_1)$ be an integrable system with Hamiltonian $f_1$. Then

i. provided $M_f = \{ x \in M | f_1 = c + 1, \ldots, f_n = c_n \}$ is connected, it is diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \leq k \leq n$, where $T^k = S^1 \times \ldots \times S^1$ with $k$ copies of $S^1$ is a torus;

ii. in a neighbourhood of $M_f$ there exists a canonical transformation to action angle variables $\phi_1, \ldots, \phi_k, I_1, \ldots, I_{n-k}$ such that $\phi_i$ are coordinates on $M_f$ and $I_i$ are first integrals;

iii. Hamiltonian’s equations of motion are solvable by quadrature, i.e., by solving a finite number of integrals. Further,

$$\dot{I}_i = \frac{\partial H}{\partial \phi_i} = 0;$$

$$\dot{\phi}_i = \frac{\partial H}{\partial I_i} = \omega(I_1, \ldots, I_n)$$

so that $I_i(t) = I_i(0)$ and $\phi_i(t) = \omega_i t + \phi_i(0)$.
4 Ch.4 - Topological Charges in Field Theory

**Definition 4.1.** Solitons are non-singular, static, finite-energy solutions of the Euler-Lagrange equations.

**Definition 4.2.** Let \( \phi_- = \lim_{x \to -\infty} \phi \) and \( \phi_+ = \lim_{x \to +\infty} \phi \). Then we define the topological charge \( N \) to be the difference of these two values
\[
N = \phi_+ - \phi_- = \int_\mathbb{R} dx \phi_x.
\]

**Definition 4.3.** Let \( M, M' \) be oriented manifolds of the same dimension with volume forms \( \omega, \omega' \) and without boundaries. Let \( f : M \to M' \) be a smooth function. The degree of \( f \), denoted \( \deg(f) \), is given by
\[
\int_M f^*(\omega') = \deg(f) \int_{M'} \omega'.
\]

**Theorem 4.4.** \( \deg(f) \) is an integer given by
\[
\deg(f) = \sum_{x \in f^{-1}(y)} \text{sgn}(J(x)),
\]
where \( y \in M' \), \( x \in U \subset M \) has coordinates \( x^i, \ i = 1, \ldots, \dim(M) \) and if \( y \in f(U) \subset M' \) has coordinates \( y^j = y^j(x^1, \ldots, x^{\dim(M)}) \), \( j = 1, \ldots, \dim(M') \), then \( J = \det \left( \frac{\partial y^j}{\partial x^i} \right) \).

5 Ch.5 - Gauge Theory

**Definition 5.1.** With \( \alpha, \beta \in \Lambda^p(\mathbb{R}^n) \), we define the inner product on \( p \)-forms according to
\[
(\alpha, \beta) = (\beta, \alpha) = \frac{1}{p!} \alpha^{a_1 \ldots a_p} \beta_{a_1 \ldots a_p}.
\]

**Definition 5.2.** Consider a manifold with a metric and a volume form \( \epsilon \). Let \( \lambda \in \Lambda^p \) and \( M \in \Lambda^{n-p} \). The Hodge operator \( \ast : \Lambda^p \to \Lambda^{n-p} \) is defined by
\[
\lambda \wedge M = (\ast \lambda, M) \epsilon.
\]
If the metric has a signature with \((n-t)\) temporal components \((+)\) and \(t\) spatial components \((-)\), then in components
\[
(\ast \lambda)_{b_1 \ldots b_q} = (-1)^t \frac{1}{p!} \epsilon^{a_1 \ldots a_p} b_1 \ldots b_q \lambda_{a_1 \ldots a_p}, \quad p + q = n.
\]

**Lemma 5.3.** For a \( p \)-form \( \lambda \), \( \ast \ast \lambda = (-1)^t (-1)^{(n-p)} \lambda \).

**Proof:**
\[
(\ast \ast \lambda)_{c_1 \ldots c_p} = (-1)^t (-1)^{t} \frac{1}{p!} \epsilon^{a_1 \ldots a_p b_1 \ldots b_q} c_1 \ldots c_p \lambda_{a_1 \ldots a_p}
= \frac{1}{p!} \epsilon^{a_1 \ldots a_p b_1 \ldots b_q} c_1 \ldots c_p \lambda_{a_1 \ldots a_p}
= (-1)^{pq} \frac{1}{p!} \epsilon^{a_1 \ldots a_p b_1 \ldots b_q} c_1 \ldots c_p \lambda_{a_1 \ldots a_p}
= (-1)^{pq} \frac{1}{p!} \epsilon^{a_1 \ldots a_p b_1 \ldots b_q} c_1 \ldots c_p \lambda_{a_1 \ldots a_p}
= (-1)^t (-1)^{(n-p)} \lambda_{c_1 \ldots c_p}.
\]
Definition 5.4. Instantons are non-singular finite energy solutions of classical equations of motion (e.g., Yang-Mills equations) on a Euclidean manifold whose action, e.g.,

\[ S = -\int_{\mathbb{R}^4} \text{Tr}(F \wedge *F) \]

is finite.

Definition 5.5. Let \( G \) be a Lie group with Lie algebra \( g \). Let \( A \) be a \( g \)-valued 1-form, the gauge potential. Then we define the Maxwell 2-form

\[ F := dA + A \wedge A. \]

Note that \( \omega_p \wedge \omega_q = (-1)^{pq} \omega_q \wedge \omega_p \).

Definition 5.6. A polynomial \( P(A) \) is said to be invariant if it is invariant under the adjoint action of some Lie group, i.e.,

\[ P(\text{Ad}_g A) = P(g^{-1}Ag) = P(A). \]

Definition 5.7. A fibre bundle is a quintuple

\((E, B, \pi, F, G) = (\text{total space, surjective map, base space, fibre} = \text{vector space or Lie group, Lie group})\)

such that

- \( E \) and \( B \) are manifolds with smooth surjective map \( \pi : E \to B \).
- For each \( x \in B \), there exists an open set \( U_a \subset B \) such that \( x \in U_a \) and there exists a diffeomorphism \( \phi_a : U_a \times F \to \pi^{-1}(U_a) \) such that \( \pi(\phi_a(x, f)) = x \), where \( (x, f) \in U_a \times F \). \( \phi_a \) is called a trivialisation.
- There are transition functions \( \phi_{ab} : F \to F \)
  i. \( \phi_{ab} : F \to F; \)
  ii. \( \phi_{ab} \in G; \)
  iii. \( \phi_{ab} = \phi_{ab}(x), x \in U_a \cap U_b; \)
  iv. \( \phi_{aa} = I; \)
  v. \( \phi_{ab} \circ \phi_{bc} = \phi_{ac} \) for all \( x \in U_a \cap U_b \cap U_c \).

A principle bundle is a fibre bundle with \( E = P \) and \( F = G \), and \( G \) acts on \( F \) by left translations.

Definition 5.8. A section of a bundle \( \pi : E \to B \) is a map \( S : B \to E \) such that \( \pi \circ S \) is the identity on \( B \).

Definition 5.9. The connection on a principle bundle \((E, B, \pi, G)\) is a Lie-algebra valued one-form \( \omega \) on \( E \) such that its vertical component (the component of the fibres) is the Maurer-Cartan 1-form \( \gamma^{-1}d\gamma \).

Since any bundle is locally trivial (i.e., looks like a product manifold), we may write in local coordinates

\[ \omega = \gamma^{-1}A\gamma + \gamma^{-1}d\gamma, \]

where \( A \) is a Lie algebra valued one-form on \( B \) and \( \gamma : B \to G \) takes values in the group.

Definition 5.10. The curvature is a Lie algebra-valued two-form on the total space \( E \) given by

\[ \Omega = d\omega + \omega \wedge \omega = \gamma^{-1}F\gamma, \]

where \( F \) is a Lie algebra-valued two-form on the base space \( B \).