Let \( \mathcal{M} \) be a manifold with metric \( g \) and with covariant derivative given by the Levi-Civita connection.

(a) How is parallel transport of a tensor \( T \) along a vector field \( V \) defined in terms of the covariant derivative?

\[
\nabla_V T = 0.
\]

Let \( \phi = \{x^\mu\} \) be a coordinate chart on \( \mathcal{M} \). Derive the geodesic equations expressed in coordinates from the property that the tangent vector of a geodesic is parallel propagated along itself.

Let \( X \) be the tangent vector. Then with our coordinates, \( X = X^\mu \frac{\partial}{\partial x^\mu} = \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu} = \frac{d}{ds} \). Next, \( X \) parallel propagating along itself

\[
\Leftrightarrow \nabla_X X = X^\mu \nabla_\mu X = 0.
\]

But

\[
X^\mu \nabla_\mu X^\nu = X^\mu \left( \frac{\partial X^\nu}{\partial x^\mu} + \Gamma^\nu_{\rho\mu} X^\rho \right)
= \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu} \frac{dx^\nu}{ds} + \Gamma^\nu_{\rho\mu} \frac{dx^\rho}{ds} \frac{dx^\mu}{ds}
= \frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\rho\mu} \frac{dx^\rho}{ds} \frac{dx^\mu}{ds}.
\]

Hence we have the geodesic equation in coordinates

\[
\frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\rho\mu} \frac{dx^\rho}{ds} \frac{dx^\mu}{ds} = 0.
\]

For the remainder of this question consider the metric

\[
ds^2 = -\frac{1}{z} dt^2 + z^2 (dx^2 + dy^2) + zdz^2
\]

where \( t, x, y, z \in \mathbb{R} \) and \( z > 0 \).

(c) Calculate the non-vanishing Christoffel symbols from the Euler-Lagrange equations of the geodesic Lagrangian.

From the above metric, the Lagrangian is

\[
\mathcal{L} = \frac{1}{z} l^2 - z^2 (x^2 + y^2) - za^2,
\]

where \( \cdot \) is wrt the parameter \( s \) along the geodesic. So we have four equations of motion using

\[
\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0.
\]

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\[
\frac{d}{ds} \left( \frac{1}{z} \right) = z^{-1} \dot{z} - z^{-2} \ddot{z} = 0 \quad \Rightarrow \quad \dot{z} - z^{-1} \ddot{z} = 0 \quad \Rightarrow \quad \Gamma^t_{zt} = -z^{-1};
\]
\[
\frac{d}{ds} (z^2 \dot{x}) = z^2 \ddot{x} + 2z \dot{x} \dot{z} = 0 \quad \Rightarrow \quad \ddot{x} + 2z^{-1} \dot{x} \dot{z} = 0 \quad \Rightarrow \quad \Gamma^x_{zx} = 2z^{-1};
\]
\[
\frac{d}{ds} (z^2 \dot{y}) = z^2 \ddot{y} + 2z \dot{y} \dot{z} = 0 \quad \Rightarrow \quad \ddot{y} + 2z^{-1} \dot{y} \dot{z} = 0 \quad \Rightarrow \quad \Gamma^y_{zy} = 2z^{-1};
\]
\[
-2z(\dot{x}^2 + \dot{y}^2) - \dot{z}^2 + \frac{d}{ds} (2z \dot{z}) = 2z \ddot{z} + \dot{z}^2 - 2z(\dot{x}^2 + \dot{y}^2) = 0 \quad \Rightarrow \quad \ddot{z} + \frac{1}{2} z^{-1} \dot{z}^2 - \dot{x}^2 - \dot{y}^2 = 0
\]
\[
\Rightarrow \quad \Gamma^z_{zz} = \frac{1}{2} z^{-1}, \quad \Gamma^z_{xx} = -1 = \Gamma^z_{yy}.
\]

(d) Determine four independent constants of motion along the geodesic and use these to show that the geodesic equations can be reduced to a single equation that describes the motion of a particle with energy \( E \) in a potential \( U(z) \), i.e.,
\[
\dot{z}^2 + U(z) = E.
\]

What are the energy \( E \) and the potential \( U(z) \) in terms of the constants of motion?

The constants of motion along the geodesic are
\[
\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad z^{-1} \ddot{t} := c_1 = \text{const.};
\]
\[
\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad z^2 \dot{x} := c_2 = \text{const.};
\]
\[
\frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad z^2 \dot{y} := c_3 = \text{const.}
\]
and \( L = 1 \) (for timelike metric), i.e.,
\[
z^{-1} \dot{t}^2 - z^2 (\dot{x}^2 + \dot{y}^2) - \dot{z}^2 = 1.
\]

If we substitute the first three constants into the fourth one, then
\[
z^{-1} c_1^2 z^2 - z^2 (c_2^2 z^{-4} + c_3^2 z^{-4}) - \dot{z}^2 = 1.
\]
\[
\Rightarrow \quad \dot{z}^2 + z^{-1} + c_2^2 z^{-3} + c_3^2 z^{-3} = c_1^2,
\]
i.e., \( \dot{z}^2 + U(z) = E \) with
\[
U(z) = z^{-1} + c_2^2 z^{-3} + c_3^2 z^{-3}, \quad E = c_1^2.
\]

(e) Discuss the qualitative behaviour of a massless (null) particle falling in from large but finite \( z_0 \) with positive energy and an initially non-vanishing velocity component in the \( x \) or \( y \) direction, i.e., initial velocity components \( \dot{x}_0, \dot{y}_0, \dot{z}_0 \) such that \( \dot{x}_0^2 + \dot{y}_0^2 > 0 \) and \( \dot{z}_0 < 0 \).

For the massless case, \( L = 0 \) (for null metric), i.e.,
\[
z^2 + c_2^2 z^{-3} + c_3^2 z^{-3} = c_1^2.
\]
Then
\[
U(z) = c_2^2 z^{-3} + c_3^2 z^{-3} := cz^{-3}, \quad c > 0.
\]

(f) Calculate the geodesic for a massless particle with \( \dot{x}_0 = \dot{y}_0 = 0 \) initially (but \( \dot{z}_0 < 0 \)) and contrast the behaviour with the one discussed in (e).

If \( \dot{x}_0 = \dot{y}_0 = 0 \), then initially \( c_2 = c_3 = 0 \), i.e., \( U = 0 \).