1 January 2014

Given \((X, Y) = \text{Tr}(XY)\) is negative definite, consider

\[ ds^2 = -\text{Tr}(dgg^{-1}dgg^{-1}). \]

Firstly, \(\frac{dg}{dt}g^{-1} \in L(G)\). To see this, let

\[ g(t + \epsilon) := h(\epsilon)g(t), \quad g, h \in G. \]

\[ \Rightarrow g(t) + \epsilon \frac{dg(t)}{dt} = h(\epsilon)g(t). \]

\[ \Rightarrow h(\epsilon) = I + \epsilon \frac{dg}{dt}g^{-1}. \]

\[ \Rightarrow \frac{dg}{dt}g^{-1} \in L(G). \]

This allows us to view \(\frac{dg}{dt}g^{-1}\) as the \(X\) or \(Y\) in the inner product expression. We write

\[ \left(\frac{ds}{dt}\right)^2 = -\text{Tr}\left(\frac{dg}{dt}g^{-1}\frac{dg}{dt}g^{-1}\right). \]

\[ \Rightarrow ds^2 = -\text{Tr}(dgg^{-1}dgg^{-1}). \]

Hence \(ds^2\) is positive definite. This tells us that \(ds^2\) is a Riemann metric on \(G\). Note that \(ds^2\) is interpreted as the squared length of \(dg\), an infinitesimal tangent element to \(G\) at \(g\), and \(g^{-1}\) is used to map \(dg\) back to the identity.

\(ds^2\) has “large \(G \times G\) invariance”. To see this, note that if under left and right actions,

\[ g \rightarrow g_1gg^{-1}, \quad dg \rightarrow g_1(dg)g^{-1}, \]

then

\[ dgg^{-1} \rightarrow g_1(dg)g^{-1}. \]

Then

\[ ds^2 = -\text{Tr}(dgg^{-1}dgg^{-1}) \rightarrow ds^2, \]

using the cyclic permutation of elements in trace. Hence the metric is highly symmetric.

The Lagrangian for the free motion of a particle on \(G\) can be written as

\[ \mathcal{L} = -\text{Tr}(gg^{-1}gg^{-1}). \]

Let

\[ g(t) \rightarrow g(t) + \delta g(t) = g(t)(I + g^{-1}\delta g(t)) = g(t)(I + \delta X(t)). \]
\[ \Rightarrow \delta S = -2 \int dt \text{Tr}(\delta (\dot{gg}^{-1})gg^{-1}), \]

where

\[ \delta (\dot{gg}^{-1})gg^{-1} = (\delta \dot{g})g^{-1} \dot{gg}^{-1} - \dot{gg}^{-1}(\delta g)g^{-1}\dot{gg}^{-1} \]

\[ = \frac{d}{dt}(\dot{gg}^{-1}) + (\delta g)g^{-1}\dot{gg}^{-1} - (\delta g)g^{-1} \frac{d}{dt}(\dot{gg}^{-1}) - \dot{gg}^{-1}(\delta g)g^{-1}\dot{gg}^{-1}. \]

We can discard the total derivative (first term) and the second and the last terms cancel due to the cyclic permutation of trace elements. Hence the equation of motion is

\[ \frac{d}{dt}(\dot{gg}^{-1}) = 0. \]

We can write \( \dot{g}^{-1} := X_0 \in L(G) \). Then the general solution is

\[ g(t) = g_0 \exp(tX_0). \]

Under the left and right action of \( G \),

\[ g_0 \rightarrow g_1 g_0 g_2^{-1}, \quad X_0 \rightarrow g_1 X_0 g_2^{-1} \]

where \( g_1, g_2 \in G \).

For \( G = SU(2) \), since a general element in \( L(SU(2)) \) can be written in the form \( X = iv \cdot \sigma \), the relation

\[ \exp(iv \cdot \sigma) = \cos |v| I + i\frac{v \cdot \sigma}{|v|} \sin |v| \]

tells us that the exponential map gives periodic solutions always.

For \( G = SU(3) \), there are periodic solutions since \( SU(3) \) has \( SU(2) \) subgroups. For example,

\[ g = g_0 \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \in SU(2) \]

is periodic. However, it is not always periodic.

The inner product \( \langle X, Y \rangle \) being negative definite implies that the \( L(G) \) is of compact type. \( G \) is then a compact group as a manifold. For example, consider the negative Killing form \( K_{ij} \) on \( L(SU(2)) \), which is of compact type. In this case we can find an adapted basis \( \{ T_i \} \) such that \( K_{ij} = -\mu \delta_{ij} \). Calculations then show that the structure constants of \( L(SU(2)) \) are totally antisymmetric.

**2 January 2013**

Let \( G \) be a matrix group which is non-abelian and simple.

The Lie algebra \( L(G) \) of a Lie group is the tangent space to the Lie group \( G \) at the identity. In its own right, a Lie algebra is a bilinear map \( L(G) \times L(G) \rightarrow L(G) \) equipped with the Lie bracket structure, \( [ , ] \) such that antisymmetry \( [X,Y] = -[Y,X] \) and the Jacobi identity \( [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \) are obeyed.

To say that \( L(G) \) is of compact type means that the quadratic form on it is negative definite.

Now consider

\[ S = -\int \text{Tr}(U^{-1}\partial_{\mu}UU^{-1}\partial^\mu U)d^4x, \]
where $U(x) \in G$. See the previous section for symmetries of this action. It is important that $L(G)$ is of compact type because only then we will have a negative definite inner product on $L(G)$, thus making the trace gauge invariant and the kinetic energy terms positive.

$S$ is invariant under the gauge transformation

$$U(x) \to g(x)U(x)g(x)^{-1}, \quad g(x) \in G.$$  

Introduce a gauge potential $A_\mu(x)$ and a covariant derivative

$$D_\mu U = \partial_\mu U + \alpha A_\mu U + \beta U A_\mu.$$  

In order for

$$U^{-1}D_\mu U \to g(U^{-1}D_\mu U)g^{-1},$$

we need to impose conditions on real constants $\alpha, \beta$ and on how $A_\mu(x)$ transforms.

Under the above gauge transformation,

$$U^{-1} \to gU^{-1}g^{-1},$$

and

$$D_\mu U \to \partial_\mu (gUg^{-1}) + \alpha A_\mu gUg^{-1} + \beta gUg^{-1}A_\mu$$

$$= (\partial_\mu g)Ug^{-1} + g(\partial_\mu U)g^{-1} - gU^{-1}(\partial_\mu g)g^{-1} + \alpha A_\mu gUg^{-1} + \beta gUg^{-1}A_\mu.$$  

Hence,

$$U^{-1}D_\mu U \to gU^{-1}g^{-1}(\partial_\mu g)Ug^{-1} + gU^{-1}(\partial_\mu U)g^{-1} - (\partial_\mu g)g^{-1} + \alpha gU^{-1}A_\mu gUg^{-1} + \beta A_\mu.$$  

Our aim is

$$U^{-1}D_\mu U \to g(U^{-1}D_\mu U)g^{-1} = gU^{-1}(\partial_\mu U)g^{-1} + \alpha gU^{-1}A_\mu gUg^{-1} + \beta gA_\mu g^{-1}.$$  

If we compare the two expressions, it can be easily seen that if we have

$$A_\mu \to gA_\mu g^{-1} + g^{-1}\partial_\mu g$$

with $\alpha = 1, \beta = -1$, then they are equivalent. Alternatively, we can have

$$A_\mu \to gA_\mu g^{-1} - g^{-1}\partial_\mu g$$

with $\alpha = -1, \beta = 1$. Hence

$$S \to - \int \text{Tr}(gU^{-1}\partial_\mu U g^{-1}gU^{-1}\partial_\mu U g^{-1}) = S$$

by the cyclic permutation of elements in trace.

$A_\mu$ is not dynamical. The gauge invariant term that we can add to the action to make $A_\mu$ dynamical is $\text{Tr}(F_{\mu\nu}F^{\mu\nu})$. 
