Robust almost fully revealing equilibria in multi-sender cheap talk*

Attila Ambrus† and Shih En Lu‡

This Version: July 2010

Abstract

We show that in multi-sender cheap talk games, if the state space is large enough, then there exist equilibria arbitrarily close to full revelation of the state that are robust to introducing imperfections in the senders' observations. The result implies that even when only equilibria robust to noise are considered, if there are multiple experts to consult with and the state space is large, there is a communication equilibrium that is strictly better for the principal than delegating the decision right to one of the experts.

1 Introduction

In sharp contrast to the predictions of cheap talk models with a single sender (Crawford and Sobel (1982), Green and Stokey (2007)), if a policymaker has the chance to consult multiple experts and the state space is large enough, there exist equilibria in which the policymaker always learns the true state. This observation was first made by Krishna and Morgan (2001), while Battaglini (2002) gives necessary and sufficient conditions for the existence of such fully revealing equilibria.[1]

One important implication of these results is that with a large enough state space, under the best equilibrium for the sender, retaining the decision right and consulting multiple experts is superior to delegating the decision power to one of the informed agents. In contrast, as Dessein (2002) shows, the best outcome from communicating with one expert can be strictly worse than delegating the decision to the expert. In particular, if the expert's bias is small enough, delegation is optimal. In the political science literature, Gilligan and Krehbiel (1987) establishes a similar point in the context of legislative decision-making.

Does this imply that the question of communication versus delegation becomes trivial when there are multiple experts to consult (and the state space is large)? As noted by Krehbiel (2001) and Battaglini (2002), among other papers, the answer depends on how plausible the very informative equilibria of multi-sender cheap talk games are. The concept of plausibility can be approached different ways.[2] One standard theoretical approach is requiring the equilibrium to be robust to small perturbations of the game, in the sense that *there is an equilibrium in the perturbed game close to the original equilibrium.* There are reasons to think that this approach, in particular requiring robustness with respect to introducing a small noise in

*We would like to thank Eduardo Azevedo, Georgy Egorov, Drew Fudenberg, Yuichiro Kamada, Tymofiy Mylovanov, Larry Samuelson and especially Joel Sobel for useful comments.

†Department of Economics, Harvard University, Cambridge, MA 02138, ambrus@fas.harvard.edu

‡Department of Economics, Harvard University, Cambridge, MA 02138, shihenlu@fas.harvard.edu

[1] See also Baron and Meirowitz (2006) and Eső and Fong (2008).

[2] For example, Krehbiel (2001) argues for empirical plausibility, while Battaglini (2002) and Ambrus and Takahashi (2008) propose different restrictions on out-of-equilibrium beliefs.

the experts' information, considerably restricts the set of equilibria in multi-sender cheap talk games. If the state space is large enough and two experts observe the state perfectly (an assumption that is not realistic in most settings), it is easy to create fully revealing equilibria, by requiring the experts to reveal the state truthfully, and rendering an action that is bad for both senders in case of nonmatching reports. The latter is possible because after an out-of-equilibrium profile of messages by the experts, nothing restricts the beliefs of the receiver. Such constructions obviously break down if the experts observe the state with noise, however small, since nonmatching reports then occur along the equilibrium path.

Indeed, Battaglini (2002) provides a result along the above lines that casts doubt on the plausibility of fully revealing equilibria in 1-dimensional state spaces with two oppositely biased senders. He shows that no fully revealing equilibrium is robust to introducing a particular type of noise, which we will refer to as *replacement noise*: with high probability, a sender exactly observes the realization of the state, but with the remaining probability, she receives the realization of an independent random variable (which has full support over the state space). However, the paper does not examine how close robust equilibria can get to full revelation in a 1-dimensional state space. In multidimensional state spaces, Battaglini (2002) only examines unrestricted state spaces and state-independent biases.[3] Moreover, the paper uses a weaker notion of robustness than the one defined in the previous paragraph; hence, the paper's positive results in multi-dimensional settings do not provide an answer to how informative equilibria that are robust according to our stronger definition can be, even in the investigated settings.[4]

Battaglini (2004) shows that if the state space is a multi-dimensional Euclidean space and the distribution of the prior is diffuse, then there exists a fully revealing equilibrium that is robust to introducing another type of noise, which we will refer to as *additive noise*: the senders observe the sum of the state and a noise term that is independent across senders.[5] However, the proof of this result depends crucially on the particular prior distribution. For proper prior distributions, there is no equilibrium proposed in the literature that is robust to additive noise and improves on the receiver communicating to only one of the senders.

To summarize, results in the previous literature reveal how close robust equilibria can be to full revelation only in particular cases. In this paper, we contribute to the literature by showing that if the state space is large, there exist equilibria arbitrarily close to full revelation that are robust to a small amount of noise in the senders' observations. The result is especially stark for replacement noise: if the senders' biases are bounded, then for large enough state spaces, independently of dimensionality, there exist equilibria arbitrarily close to full revelation that can be attained as a limit of equilibria of games with noisy observations, for *any* sequence of games in which the noise goes to zero. For additive noise, we establish a weaker result, namely that *there exists* a sequence of one-dimensional games with the noise going to zero for which the same is true. We first establish these results for large bounded state spaces, and we then extend them to unbounded state spaces. The latter requires the assumption that the tail of the prior distribution is thin enough.

The central idea is to construct an equilibrium in the noiseless limit game where, for any message m_1 sent by sender 1 and any message m_2 sent by sender 2 along the equilibrium path, there is a set of states *with positive measure* where the prescribed message profile is (m_1, m_2). Put simply, any combination of

[3] For relaxing these assumptions, see Ambrus and Takahashi (2008).

[4] The notion of robustness in Battaglini (2002), although it uses the concept of replacement noise, only puts restrictions on beliefs after out-of-equilibrium message pairs. The paper shows that in a class of multi-dimensional settings, there generically exist fully revealing equilibria with no out-of-equilibrium message pairs, which trivially satisfy the above restrictions. However, it does not examine whether these equilibria can be achieved as limits of equilibria of games with replacement noise.

[5] In fact, Battaglini (2004) shows that in the investigated setting, the equilibrium proposed in Battaglini (2002) remains valid for any level of noise, not only small ones. Other papers that investigate multi-sender cheap talk games in which the senders observe the state with nontrivial noise include Austen-Smith (1990a, 1990b, 1993) and Wolinsky (2002). These papers consider discrete state spaces. Ottaviani and Sørensen (2006) investigates a different type of model with imperfectly informed senders.

messages that are used in equilibrium are on the equilibrium path, as in the multi-dimensional construction of Battaglini (2002), and the receiver's action following any of these message profiles can be determined by Bayes' rule. The new feature of our construction is that any pair of equilibrium messages is sent with strictly positive probability. This makes the equilibrium robust to introducing a small amount of noise.

As Battaglini (2002) observed, the above type of construction constitutes an equilibrium whenever the set of states where (m_1', m_2) or (m_1, m_2') is prescribed (for $m_1' \neq m_1$ and $m_2' \neq m_2$) is far away, relative to the senders' biases, from set of states where (m_1, m_2) is prescribed. We show that even when the set of states corresponding to each message pair is small, the above condition can be satisfied when the state space is sufficiently large. Thus, the action taken by the receiver can be made arbitrarily close to the true state with *ex ante* probability arbitrarily close to 1, if Θ is large enough. In such equilibria, which are robust to a small amount of noise in the senders' observations, the receiver's expected utility is arbitrarily close to her expected utility in the case of full revelation of the state.

When the state space is a large bounded interval of the real line, we propose the following construction. The state space is partitioned into n^2 subintervals with equal size, for some large n corresponding to the number of equilibrium messages of each sender. The $n \times n$ different combinations of the equilibrium messages are then assigned to the subintervals in the partition such that any two subintervals in which a sender sends the same message are far away from each other. Essentially, intervals in the partition are labeled with 2-digit numbers in a base-n number system in a particular way, and sender 1 is supposed to report the first digit of the interval from which she received a signal, while sender 2 is supposed to report the second digit of the interval from which she received a signal. The construction relies on the fact that if the state space is large, then n can be taken to be such that $\frac{1}{n^2}$ times the length of the state space (the size of the subintervals) is small, but $\frac{1}{n}$ times the length of the state space (which is roughly the distance between subintervals in which a sender sends the same message) is large.

It is instructive to contrast the above result with the one in Battaglini (2002) establishing the nonexistence of fully revealing equilibria robust to replacement noise in one-dimensional state spaces. We show that despite this negative result, there are equilibria arbitrarily close to fully revealing that are robust to replacement noise. To reconcile these facts, we note that a sequence of equilibrium profiles of the type we construct inducing finer and finer revelation of information does not have a limit. This is because a finer and finer interval partition of the real line does not have a well-defined limit as the sizes of the intervals in the partition go to zero.

In the case of additive noise, a similar construction to the one described above can be shown to be robust to particular noise structures. However, the following complication implies that the construction is not robust to *all* small noise: around the boundaries of intervals, even if the noise is very small, there is a nontrivial chance that players miscoordinate, in the sense of inducing an action far enough from the true state. We show that by varying the level of noise and therefore the precision of signals around boundary points, one can construct games with small positive additive noise that have equilibria close to the ones constructed in the case of replacement noise. In particular, the senders play the same strategies as in the former equilibrium, except possibly in small neighborhoods of boundary points. Since by adjusting the noise structure, the latter neighborhoods can be made arbitrarily small, the result implies that the *ex ante* probability of miscoordination can be made arbitrarily low. This establishes that for large enough state spaces, there exist equilibria of the noiseless limit game arbitrarily close to full revelation that are limits of equilibria of a sequence of games with additive noise where, along the sequence, the noise vanishes. This weaker notion of robustness is in the spirit of trembling-hand perfect equilibrium.

Our construction can be used to create robust equilibria arbitrarily close to the first-best outcome of the receiver for unbounded state spaces. For a fixed bounded state space, our construction does not necessarily yield the best robust equilibrium for the receiver. However, if the state space is large enough relative to the biases, it provides a recipe to construct robust equilibria close to full revelation of information, irrespective of the fine details of the game (prior distribution of the noise, preferences of the senders), for a remarkably large class of games. In particular, the construction allows for state-dependent biases for the senders. Moreover, we do not require either the single-crossing condition on the senders' preferences that is usually assumed in the literature, or the assumption that the *sign* of a sender's bias remains constant over the state space. Thus, our results hold for games outside the Crawford and Sobel framework.

The rest of the paper is organized as follows. In Section 2, we introduce the model and the terminology we use. In Section 3, we establish our main results for one-dimensional state spaces (both bounded and unbounded) in games with replacement noise. In Section 4, we extend these results to the case of additive noise. Finally, in Section 5, we discuss further extensions of the model. In particular, we describe how some of our results extend to multidimensional state spaces, to discrete state spaces, to models in which noise is also introduced at other points of the game, and to models in which the receiver has commitment power.

2 The model

2.1 Players, Strategies and Preferences

The model we consider features two senders, labeled 1 and 2, and one receiver. The game starts with sender 1 observing signal s_1 and sender 2 observing signal s_2 of a random variable $\theta \in \Theta$, which we call the *state*. We refer to Θ as the *state space*, and assume that it is a closed and connected subset (not necessarily proper) of $\mathbb{R}^d$ for $d \geq 1$.[6] The prior distribution of θ is given by F, which we assume exhibits a density function f that is strictly positive and continuous on Θ.

After observing their signals, the senders send messages $m_1 \in M_1$ and $m_2 \in M_2$ to the receiver. We assume that M_1 and M_2 are Borel sets having the cardinality of the continuum. After observing the above messages, the receiver chooses an action $y \in \mathbb{R}^d$, and the game ends.

We assume that the receiver's utility function $v(\theta, y)$ is continuous, strictly concave in y, and that $v(\theta, \cdot)$ attains its maximum value of 0 at $y = \theta$. We also assume that sender i's utility function $u_i(\theta, y)$ is continuous, that it is strictly concave in y, and that $u_i(\theta, \cdot)$ attains its maximum value of 0 at $y = \theta + b_i(\theta)$. We refer to $\theta + b_i(\theta)$ as sender i's ideal point at state θ, and to $b_i(\theta)$ as sender i's bias at state θ. Note that neither the signals or the messages directly enter the players' utility functions.

We also maintain the following two assumptions throughout the paper.

A1: For every $y \in R$, $\int_{\mathbb{R}} f(\theta) v(\theta, y) d\theta$ is finite.

A2: For any $\delta \geq 0$, there exists $K(\delta) > 0$ such that $u_i(\theta, \theta + K(\delta)) < u_i(\theta, \theta - \delta)$ and $u_i(\theta, \theta - K(\delta)) < u^i(\theta, \theta + \delta)$, $\forall\ i = 1, 2$.

A1 requires that the expected utility of the receiver from choosing any action is well-defined under the prior. A2 posits that neither sender becomes infinitely more sensitive to the chosen action being on one side of her ideal point than to it being on the other side. In the case of symmetric loss functions around ideal points, which is assumed in most of the literature, A2 is equivalent to requiring that there is a universal bound

[6] Closedness is assumed for notational convenience only. None of the results of the paper depend on this assumption.

on the magnitude of senders' biases. The assumption automatically holds in the case of state-independent biases assumed, for example, in Battaglini (2002, 2004).

2.2 Signal Structures

We will consider both games in which the senders observe the state perfectly, and games in which senders observe the state with noise, for two types of noise structures.

Definition: A *noiseless limit game* is one where the distributions of s_1 and s_2 conditional on θ put probability mass 1 on θ, for every $\theta \in \Theta$.

Definition: In a *game with replacement noise*, there is a random variable $\tau \in \Theta$ with cdf G that is independent of θ and exhibits a continuous density function g strictly positive on Θ such that conditional on any $\theta \in \Theta$, $s_i = \left\{\begin{smallmatrix} \theta \text{ with probability } p \\ \tau \text{ with probability } 1-p \end{smallmatrix}\right.$ for $i \in \{1, 2\}$, for some $p \in (0, 1)$. Moreover, the conditional distributions of s_1 and s_2 are independent.

Given a fixed distribution G of τ, we say that the replacement noise goes to 0 if $p \to 1$.

Definition: In a *game with additive noise*, conditional on any $\theta \in \Theta$, $s_i = \theta + \omega_i$ for $i \in \{1, 2\}$, where ω_1 and ω_2 are random variables on $\mathbb{R}^d$ satisfying that the joint distribution of $(\theta, \omega_1, \omega_2)$ exhibits a continuous density function that is strictly positive on $\Omega = \{(\theta, \omega_1, \omega_2) | \theta, \theta + \omega_1, \theta + \omega_2 \in \Theta\}$ and zero outside Ω. We refer to ω_i as the noise in sender i's observation.

The assumption imposed on the noise structure implies that the signals of both senders always take values in Θ and that they are not perfectly correlated. The normally distributed independent noise terms considered in Battaglini (2004) is an example of the class of noise structures we allow for, when $\Theta = R^d$.

We say that the additive noise is *smaller than* $\delta > 0$ if for every $\theta \in \Theta$, the conditional distribution of (ω_1, ω_2) is closer than δ to the point $(0, 0)$ according to the Ky Fan metric: $\inf\{\varepsilon > 0 | \Pr(|(\omega_1, \omega_2) - (0, 0)| > \varepsilon) \leq \varepsilon\} < \delta$. A sequence of additive noise structures goes to 0 if for any $\delta > 0$, from some point on, the noises in the sequence are smaller than δ.[7]

We note that both in the case of replacement noise and in the case of additive noise, any pair of signals from $\Theta \times \Theta$ can be observed by the senders, at any state. However, while in the case of replacement noise, the probability with which both senders observe the true state is $p^2 > 0$, in the case of additive noise, the probability that the senders' signals coincide (in particular, that both of them observe the true state) is 0.

2.3 Solution concept

The solution concept we use in this paper, along the lines of Crawford and Sobel (1982), is Bayesian Nash equilibrium. However, we note that in the equilibria we construct, all pairs of messages are sent with positive probability; hence, these profiles constitute perfect Bayesian Nash equilibria according to any reasonable definition of the concept.[8]

Formally, let $h(\theta, s_1, s_2)$ be the joint density function of the state and the sender's signals, and for $i \in \{1, 2\}$, let h_i be the marginal density of sender i's signal (the existence of which is guaranteed by our

[7] Since the Ky Fan metric metrizes the topology defined by convergence in probability, this means that conditional noise terms of both senders converge in probability to 0, uniformly across states.

[8] There is no universally used definition of perfect Bayesian Nash equilibrium in games with continuous strategy spaces, but definitions proposed in the literature only differ with respect to updated beliefs after 0-probability events.

assumptions for each of the three signal structures considered). An *action rule* of the receiver is a function $y : M_1 \times M_2 \to R^d$. For every $i \in \{1, 2\}$, sender i's *signaling strategy* is a function $m_i : \Theta \to M_i$.

Definition: Action rule $\widehat{y}$ and signaling strategies $\widehat{m}_i$ $(i \in \{1, 2\})$ constitute a pure strategy *Bayesian Nash equilibrium* iff:

(1) $\widehat{m}_i$ solves $\max_{m_i \in M_i} \int_{(\theta,s_1,s_2)\in\Theta^3} u_i(\theta, y(m_i(s_i), m_{-i}(s_{-i})))h(\theta, s_1, s_2)d\theta ds_1 ds_2$

and

(2) $\widehat{y}$ solves $\max_{y\in\mathbb{R}} \int_{(\theta,s_1,s_2)\in\Theta^3} v(\theta, y(m_i(s_i), m_{-i}(s_{-i})))h(\theta, s_1, s_2)d\theta ds_1 ds_2$.

We henceforth refer to Bayesian Nash equilibrium simply as equilibrium.

3 Equilibria robust to replacement noise

Throughout this section, we assume that the senders observe the state with replacement noise, as defined in the previous section. We also assume that the state space is a connected subset of the real line. For extending the construction to multidimensional state spaces, see Section 5.

3.1 Large bounded state spaces

First, we consider the case when $\Theta = [-T, T]$ for some $T \in \mathbb{R}_{++}$, and show that for every $\varepsilon, \delta > 0$, if T is large enough and the noise parameter is low enough, then there exists an equilibrium of the cheap talk game in which, at every state, the probability that the distance between the induced action and the state is smaller than δ is at least $1 - \varepsilon$.

To establish this result, we consider the following signaling profile for the senders. For any $T \geq K(\delta)$, let $n_{\delta,T}$ be the largest integer such that $\frac{T}{n_{\delta,T}} \geq K(\delta)$. Partition Θ to $n_{\delta,T}$ equal intervals, which we will refer to as *blocks*. Note that the size of each block is $\frac{2T}{n_{\delta,T}}$, which is by construction between $2K(\delta)$ and $4K(\delta)$. Next, further partition each block into $n_{\delta,T}$ equal subintervals, to which we will refer as *cells*. We will use $I_{j,k(i,j)}$ to denote the jth cell in the ith block, where $k(i, j) = \left\{\begin{smallmatrix} i+j-1 \text{ if } i+j-1\leq n \\ i+j-1-n \text{ if } i+j-1>n \end{smallmatrix}\right.$. Thus, block i is partitioned into the following $n_{\delta,T}$ cells: $\{(1, i), (2, i+1), ..., (n_{\delta,T} - i+1, n_{\delta,T}), (n_{\delta,T} - i+2, 1), ..., (n_{\delta,T}, i-1)\}$, and there is a total of $n^2_{\delta,T}$ cells. For completeness, assume that the cells in the partition are closed on the left and open on the right, with the exception of cell $(n_{\delta,T}, n_{\delta,T})$, which is closed at both ends. Define signaling profile $(\mu_1^{\delta,T}, \mu_2^{\delta,T})$ such that for every $j, k \in \{1, ..., n_T\}$, after receiving signal $s_1 \in I_{j,k}$, sender 1 sends message m_1^j, and after receiving signal $s_2 \in I_{j,k}$, sender 2 sends message m_2^k. Figure 1 below illustrates the above signaling profile.

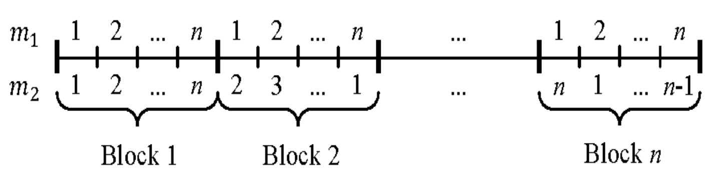

Figure 1: Signaling profile for large bounded intervals

Let $y^{\delta,T}$ be an action rule that maximizes the receiver's expected payoff given $(\mu_1^{\delta,T}, \mu_2^{\delta,T})$. Note that $y(m_1^j, m_2^k)$ is uniquely defined for $j, k \in \{1, ..., n_{\delta,T}\}$ for any noise structure we consider, since the conditional beliefs of the receiver after receiving such message pairs are given by Bayes' rule, and the receiver's utility function is strictly concave. As for out-of-equilibrium messages $m_i \neq m_i^j$ for all $j \in \{1, ..., n_{\delta,T}\}$, assume that the receiver interprets each as having the same meaning as a message sent in equilibrium. No sender will then have an incentive to deviate to an out-of-equilibrium message.

Proposition 1: For every $\delta > 0$, there exists $T(\delta) > 0$ such that if $T > T(\delta)$, then strategy profile $(\mu_1^{\delta,T}, \mu_2^{\delta,T}, y^{\delta,T})$ constitutes an equilibrium in the noiseless limit game, and we have $|y - \theta| < \delta$, for every $\theta \in \Theta$.

Proof: By construction, the receiver plays a best response in the proposed profile, so we only need to check the optimality of the senders' strategies.

Note that $n_{\delta,T} \to \infty$ as $T \to \infty$. Since, by construction, $\frac{2T}{n_{\delta,T}} \leq 4K(\delta)$ for any $T \geq K(\delta)$, the above implies that the cell size, $\frac{2T}{n_{\delta,T}^2}$, goes to 0 as $T \to \infty$. Note that for every $j, k \in \{1, ..., n_{\delta,T}\}$ and $\theta \in I_{j,k}$, the assumptions on v imply that if both senders play according to the prescribed profile, then the action induced at θ lies within $I_{j,k}$.

Also by construction, if the other sender plays the prescribed strategy, all other actions that a sender could induce by sending a different message than prescribed are more than $\frac{n_{\delta,T}-2}{n_{\delta,T}}$ times the block size away. The latter is by construction at least $2K(\delta)$, so if $n_{\delta,T} > 4$, those actions are more than $K(\delta)$ away.

The above imply that there exists $T(\delta) > 0$ such that $|y^{\delta,T}(\mu_1^{\delta,T}(\theta), \mu_2^{\delta,T}(\theta)) - \theta| < \delta$ if $T > T(\delta)$, and any deviation by a sender, given strategy profile $(\mu_1^{\delta,T}, \mu_2^{\delta,T}, y^{\delta,T})$, would induce an action y by the receiver such that $|u - \theta| > K(\delta)$. By the definition of $K(\delta)$, this implies that there is no profitable deviation by either sender. ■

Intuitively, the proposed construction is an equilibrium because the cell associated with message pair (m_1^j, m_2^k) for any $j, k \in \{1, ..., n_{\delta,T}\}$ is far away from any cell in which the prescribed message pair is either (m_1, m_2^k) with $m_1 \neq m_1^j$, or (m_1^j, m_2) with $m_2 \neq m_2^k$. This holds despite the fact that for large T the cell size, hence the distance between states and induced actions, becomes small.

Next, we show that if noise parameter $1 - p$ is small enough, then profile $(\mu_1^{\delta,T}, \mu_2^{\delta,T}, y^{\delta,T})$ remains an equilibrium in a game with replacement noise.

Proposition 2: Suppose $\delta > 0$ and $T > T(\delta)$. Then for any noise distribution G, there exists $\underline{p}(G) < 1$ such that $p > \underline{p}(G)$ implies that in a game with replacement noise structure (G, p) strategy profile $(\mu_1^{\delta,T}, \mu_2^{\delta,T}, y^{\delta,T})$ constitutes an equilibrium.

Proof: Note that since both f and g are continuous and strictly positive on the compact Θ, as $p \to 1$, given signaling strategies $(\mu_1^{\delta,T}, \mu_2^{\delta,T})$, the conditional distribution of θ given message pair (m_1^j, m_2^k) in the game with replacement noise converges weakly to the conditional distribution of θ given message pair (m_1^j, m_2^k) in the noiseless limit game, for every $j, k \in \{1, ..., n_{\delta,T}\}$. Then since the expected payoff of the receiver resulting from choosing some action y after message pair (m_1^j, m_2^k) is continuous with respect to the weak topology in the conditional distribution of θ given (m_1^j, m_2^k), the theorem of the maximum implies that $y^{\delta,T}$ is continuous in p, even at $p = 1$. This implies that the expected payoff of sender i resulting from sending message m_i^l after receiving signal s_i is continuous in p, for every $i \in \{1, 2\}$, $l \in \{1, ..., n_{\delta,T}\}$ and $s_i \in \Theta$, even at $p = 1$. Moreover, in the noiseless limit game, after signals $s_1, s_2 \in I_{j,k}$, sending message m_1^j yields a strictly higher expected payoff for sender 1 than m_1^l for $l \neq j$, and sending message m_2^k yields a strictly higher expected payoff for sender 2 than m_2^l for $l \neq k$. Thus, the same holds for p close enough to 1. This establishes the claim. ■

The intuition behind Proposition 2 is that the receiver's optimal action rule given $(\mu_1^{\delta,T}, \mu_2^{\delta,T})$ is continuous in p, even at $p = 1$. Therefore, the expected payoff of a sender when sending different messages after a certain signal changes continuously in p as well. Since in the noiseless limit game, a sender strictly prefers to send the prescribed message to sending any other equilibrium message, the same holds for noisy games with p high enough.

Note that the above propositions imply that for any $\delta > 0$, if the state space is large enough, then there is an equilibrium of the noiseless limit game, where the action induced at any state is at most δ away from the state, that is robust to replacement noise in a strong sense: it can be obtained as a limit of equilibria of games with vanishing replacement noise, for any noise distribution G.

The propositions also imply the following result.

Corollary 1: Fix payoff functions $v(.,.)$ and $u_i(.,.)$, $i = 1, 2$, defined over $\mathbb{R}^2$ satisfying A1. Take any sequence of games with bounded interval state spaces $[-T_1, T_1]$,$[-T_2, T_2]$,... , state distributions $F_1, F_2, ..$, noise distributions G_1,G_2,... and payoff functions $(v^1, u_1^1, u_2^1), (v^2, u_1^2, u_2^2), ...$ such that v^j, u_1^j and u_2^j are restrictions of v, u_1 and u_2 to $[-T_j, T_j] \times \mathbb{R}$. If $T_i \to \infty$ as $i \to \infty$, then there exists a sequence of noise levels $p_1, p_2, ...$ with $p_i < 1$ for every $i \in Z_{++}$ and $p_i \to 1$ as $i \to \infty$, such that there is a sequence of equilibria of the above games with equilibrium outcomes converging to full revelation in $\mathbb{R}$.

This result contrasts with Proposition 2 in Battaglini (2002), which establishes that, if the senders are oppositely biased and the biases are state-independent, then there does not exist a fully revealing equilibrium robust to replacement noise in a one-dimensional state space, no matter how large the state space is. To reconcile the above results, it is useful to observe that although the sequence of outcomes induced by the sequence of equilibria that Corollary 1 refers to converges to full revelation of the state in $\mathbb{R}$, such sequences of equilibrium strategy profiles do not have a well-defined limit in the noiseless limit game with state space $\mathbb{R}$. This is because the limit of a sequence of interval partitions where the size of the intervals go to zero is not well-defined. For a direct analysis of equilibria when the state space is $\mathbb{R}$, see the next subsection.

3.2 Unrestricted state space

In this subsection, we assume that $\Theta = \mathbb{R}$. We show that the equilibrium construction introduced in the previous section can be extended to this case when the prior distribution of states has thin enough tails.

The state space is still partitioned into n^2 cells, and combinations of the n equilibrium messages are allocated to different cells in the same order as before. The difference is that in the case of an unrestricted state space, only the middle $n^2 - 2$ cells can be taken to be small; the extreme cells are infinitely large. Hence, in the equilibria we construct, even with no noise, the implemented action will be far away from the state with nontrivial probability in states in the extreme cells. But if the profile is constructed such that the middle $n^2 - 2$ cells cover interval $[-T, T]$ for large enough T, then for small noise, the *ex ante* probability that the induced action is within a small neighborhood of the realized state can be made close to 1.

The extra assumption needed for this construction guarantees that for large enough block size, even in the extreme cells, the senders prefer inducing the action corresponding to the cell instead of deviating and inducing an action in a different block.

Let $y_{\theta,d,L}$ (respectively, $y_{\theta,d,R}$) be the optimal action for the receiver when her belief about the true state follows density d truncated so the true state is in $(-\infty, \theta]$ (respectively, $[\theta, \infty)$).

A3: There exist $C, Z > 0$ such that for all $\theta < -C$, $|y_{\theta,f,L} - \theta| < Z$, and for all $\theta > C$, $|y_{\theta,f,R} - \theta| < Z$.

A3 requires the tail of the prior distribution to be thin enough: it is satisfied if f converges quickly enough to 0 at $-\infty$ and ∞, relative to how fast the loss functions at moderate states diverge to infinity as the action goes to $-\infty$ or ∞. For example, if v exhibits quadratic loss invariant in θ, a sufficient condition for A3 to hold is that $\lim_{x\to-\infty} \frac{F(x)}{f(x)}$ and $\lim_{x\to\infty} \frac{F(x)}{f(x)}$ exist. This is clearly true if f converges to 0 exponentially fast, so for v quadratic, A3 holds for exponential distributions or any distribution converging to 0 faster, such as the normal.

Proposition 3: If the state space is $\mathbb{R}$ and A3 holds, then for every $\delta, \eta > 0$, there exists $\underline{p} < 1$ such that, in a noisy game with $p > \underline{p}$, $|y - \theta| < \delta$ with *ex ante* probability at least $1 - \eta$.

Proof: Consider the following strategy profile in the noiseless limit game.

Let T be such that $F(T) - F(-T) = 1 - \frac{\eta}{2}$. As in Subsection 3.1, partition $\mathbb{R}$ into n blocks. Blocks 2 through $n-1$ are equally sized and large enough so that each is bigger than $K(\delta) + 2\delta$, and they together cover $[-T, T] \cup [-C, C]$, where C is the corresponding constant in A3.

For each $k \in \{1, ..., n\}$, we will further partition block k into n cells, labeled as in Subsection 3.1. Block 1 minus the leftmost cell and block n minus the rightmost cell are each bigger than $\max\{K(Z), K(\delta) + \delta\}$, where Z is the corresponding constant in A3. We choose n large enough so that each of the middle $n^2 - 2$ cells, which are of equal size, is smaller than δ. For the sake of completeness, let each of the middle $n^2 - 2$ cells be closed on the left and open on the right.

Label the cells as in Subsection 3.1, and consider the following strategy profile:

- when s_1 falls in cell (j, k), sender 1 sends message m_1^j;
- when s_2 falls in cell (j, k), sender 2 sends message m_2^k;
- $y(m_1^j, m_2^k)$ is an optimal response to m_1^j, m_2^k given the above strategies, for every $j, k \in \{1, ..., n\}$;
- the receiver associates any out-of-equilibrium message to a message sent by that player in equilibrium, and after any other message pair, the receiver chooses the corresponding $y(m_1^j, m_2^k)$ for some $j, k \in \{1, ..., n\}$.

This profile constitutes an equilibrium in the noiseless limit game, which has the property that $|y - \theta| < \delta$ with *ex ante* probability at least $1 - \frac{\eta}{2}$. This is because message pairs are allocated to cells in a way that at any state, any action that a sender could induce other than the prescribed one is strictly worse for the sender than the prescribed action.

Analogous arguments as the ones used in the proof of Proposition 2 establish that for large enough p, the above profile still constitutes an equilibrium. Moreover, it is easy to see that for large enough p, conditional on the state being in the middle $n^2 - 2$ cells, the probability that $|y - \theta| < \delta$ is at least $1 - \frac{\eta}{2}$. Then, the *ex ante* probability of $|y - \theta| < \delta$ is at least $1 - \eta$, concluding the proof. ■

Proposition 3 implies that, if A3 holds, in a game in which the state space is the real line, for any $\delta > 0$, there exists an equilibrium robust to small replacement noise in which the distance between any state and the action induced in that state is less than δ with high *ex ante* probability.

4 Additive noise

In this section, we show that the same almost fully revealing equilibria considered in the previous section are robust to continuous additive noise, in the sense that there exists a sequence of additive noise structures going to 0 such that there is a sequence of equilibria in the corresponding noisy games that converges to full revelation. That is, while we cannot show that the equilibria we propose in the noiseless limit game are robust with respect to any noise structure small enough, we establish that there is a sequence of positive noise structures converging to zero with respect to which these equilibria are robust. This notion of robustness is similar in spirit to the notion of perfection captured in trembling-hand perfect equilibrium or sequential equilibrium.

We refer to the collection of conditional distributions of signals (ω_1, ω_2) on θ, for $\theta \in \Theta$, as a *noise structure.*

4.1 Large bounded state spaces

Like in Subsection 3.1, here we assume $\Theta = [-T, T]$ for some $T \in \mathbb{R}_{++}$.

The next result establishes that for any $\delta > 0$, if the state space is large enough, then there exist equilibria in games with small noise in observations (satisfying our assumptions) for which the *ex ante* probability that the induced action is in a δ-neighborhood of the realized state is arbitrarily close to 1.

Proposition 4: For every $\varepsilon, \delta, \eta, > 0$, there is $T(\delta) > 0$ such that if $T > T(\delta)$, then there exists a noise structure with noise smaller than ε that supports an equilibrium for which $|y - \theta| < \delta$ with *ex ante* probability at least $1 - \eta$.

The main difficulty in extending the construction from Section 3 to additive noise comes from the fact that no matter how small the noise is, around the boundary points of cells, senders face a nontrivial uncertainty regarding the message sent by the other sender. This occurs because the noise is continuous, so that a sender receiving a signal on one side of a boundary will believe that the other sender has received a message on the other side with nontrivial probability. Hence, it is likely at states around cell boundary points that the agents "miscoordinate", in the sense of one agent sending a message corresponding to the cell on the left of the boundary point, while the other agent sends a message corresponding to the cell on the right of the boundary point. For general noise, this miscoordination possibility may unravel the equilibrium, both by affecting the senders' incentives near the boundaries and by changing the receiver's actions. Therefore, we need to find a noise structure such that the potential for miscoordination does not unravel the profile. Such a noise structure is described in detail in the proof of the proposition, which can be found in the Appendix. We describe below the issues that arise in its construction.

If the noise is small enough and a sender assumes that the other players follow the profile constructed in the previous section, then at all states far enough from cell boundary points, the sender is almost certain that the other sender will send the message prescribed for that cell in the candidate equilibrium profile, and therefore strictly prefers to follow the equilibrium profile as well. However, close to the boundary points, we need to ensure that senders do not deviate to the message prescribed for the other side. We show that the noise structure - in particular the relative precision of signals around the boundary points - can be adjusted such that when receiving a signal on one side of a boundary point, both senders think that the other sender is likely to receive a signal on the same side of the boundary. This leads them to prefer sending a message prescribed for their actual signal rather than the message prescribed for signals on the other side of the boundary point. In particular, the noise structure is constructed such that at signals corresponding to boundary points, the senders are exactly indifferent between the messages corresponding to the cells on the two sides of the boundary.

An additional complication is that around some of the boundary points (the ones separating blocks), there might be a region around the boundary point where a sender might prefer to send a different message than either of the ones prescribed on the two sides of the boundary. Intuitively, around a boundary point there is a region in which, no matter how one specifies a continuous signal structure, there is nontrivial chance of miscoordination. This might induce a sender at states close to a boundary point to send a third message, even if that message results in miscoordination almost surely, if the likely actions induced by this third message are relatively better for the sender than both of the actions resulting from miscoordination resulting from the messages corresponding to the cells neighboring the boundary.

We address this extra complication in the following manner. We note that if regions around boundaries where senders send different messages than those prescribed in the limit equilibrium are small enough, then no matter how the senders deviate within the above regions, the receiver's action after equilibrium message pairs is always included in the corresponding cell. Our noise construction ensures that these regions are indeed sufficiently small. Moreover, we construct the noise structure such that outside the small intervals of possible deviations, senders do not want to deviate from the strategies specified in the limit equilibrium.

The last step of the argument employs a fixed-point theorem to show the existence of an equilibrium, given our construction and noise structure.

4.2 Unrestricted state space

Under A3, modifying the construction in subsection 4.1 in the same way as presented in subsection 3.2 extends Proposition 4 to unrestricted state spaces.

Proposition 5: If the state space is $\mathbb{R}$ and A3 holds, then for every $\varepsilon, \delta, \eta, > 0$, there exists a noise structure with noise smaller than ε that supports an equilibrium for which $|y-\theta| < \delta$ with *ex ante* probability at least $1-\eta$.

Proof: Consider the same equilibrium strategy profile in the noiseless limit game as in the proof of Proposition 3, which has the property that $|y-\theta| < \delta$ with *ex ante* probability at least $1-\frac{\eta}{2}$.

Analogous arguments as the ones used in the proof of Proposition 4 establish that there exists a noise structure with noise smaller than ε and a corresponding equilibrium of the noisy game with the feature that conditional on the state being in the middle n^2-2 cells, the probability that $|y-\theta| < \delta$ is at least $1-\frac{\eta}{2}$. In this equilibrium, the *ex ante* probability of $|y-\theta| < \delta$ is at least $1-\eta$, concluding the proof. ■

5 Extensions and discussion

5.1 Multidimensional state spaces

Our construction can be readily extended to multidimensional state spaces for replacement noise if there are no restrictions on the state space (the state space is the whole Euclidean space). In particular, for any $\delta > 0$, instead of intervals of size δ, the state space can be partitioned to d-dimensional hypercubes with edges of size δ such that cubes are of the form $\{x \in \mathbb{R}^d | x_i \in (k_i\delta, (k_i+1)\delta], \forall\, i \in \{1, ...d\}\}$, for $k_1, ..., k_d \in \mathbb{Z}$. Now take a countably infinite set of messages for each sender such that each of these sets are indexed by $\mathbb{Z}^d$. Hence, a typical message for sender i is labeled as $m^i_{j_1,...,j_d}$ $(j_1, ..., j_d \in \mathbb{Z})$. We can then allocate the messages that sender i sends corresponding to dimension $l \in \{1, ..., d\}$ to sets of cells with the same l-coordinate the same way as in Subsection 3.2. Proceeding like this for all d dimensions results in a strategy profile of the senders such that each pair of possible messages is identified with a unique cell in the above partition. Moreover, the profile is constructed such that at every state, sending a different message than the one corresponding to the cell containing the state results in a message pair identified with a cell far away from the original state, whether the sender deviates in one or more dimensions from the prescribed message. Proving that this profile, together with a strategy of the receiver that is best response to the message profile, is an equilibrium in a game with small enough replacement noise is analogous to the proof of Proposition 3.

For large bounded state spaces, the construction used in subsection 3.1 can be extended in a straightforward manner to d-dimensional hypercubes for $d > 1$. For different types of bounded state spaces in $\mathbb{R}^d$ our construction cannot be applied directly. However, the same qualitative insight still holds. Suppose the state space can be partitioned into n^2 cells with diameter at most δ, and that there is a bijection from $M \equiv \{m^1_1, ..., m^n_1\} \times \{m^1_2, ..., m^n_2\}$ to the cells in the partition such that for any $(m_1, m_2) \in M$ and for any $(m'_1, m'_2) \in M$ with either (i) $m_1 = m'_1$ and $m_2 \neq m'_2$, or (ii) $m_1 \neq m'_1$ and $m_2 = m'_2$, it holds that the distance between the partition cells associated with (m_1, m_2) and with (m'_1, m'_2) are at least $K(\delta)$ away from each other. Then there is an equilibrium of the noiseless limit game which is robust to a small amount of replacement noise or additive noise. We do not investigate this direction further in the current paper.

5.2 Introducing noise at different stages of the game

In this paper, we investigate equilibria robust to perturbations of a multi-sender cheap talk game in the observations of the senders. This is the type of perturbation most discussed in the literature. However, similar perturbations can be introduced at various other stages of the communication game: in the communication phase (the actual message received by the receiver is not always exactly the intended message by a sender) and in the action choice phase (the policy chosen by the sender is not exactly the same as the intended policy choice).[9]

The equilibria we propose in this paper are robust with respect to the above perturbations as well. To see this for the case of small perturbations in the receiver's action choice, note that in the equilibria we construct, senders strictly prefer sending the prescribed message to any other equilibrium message (while out of equilibrium messages lead to the same intended actions as equilibrium ones). Hence both for replacement noise and additive noise, if the noise in the action choice is small enough, senders still strictly prefer sending the prescribed message to any other message that they send along the path of play.

For noise in the communication phase, it is more standard to introduce replacement noise, as in Blume *et al.* (2007), since there typically is no natural metric defined on the message space (messages obtain their meanings endogenously, through the senders' strategies). A small modification of our equilibrium construction makes the profile robust with respect to such noise. In particular, take any equilibrium we constructed in Sections 3 and 4. In these equilibria, the state space is partitioned into connected subsets that we refer to as cells, and each cell is associated with an equilibrium message pair. Only a countable (finite or infinite) number of messages are sent in equilibrium by both senders. However, it is easy to construct outcome-equivalent equilibria in which all messages are sent along the equilibrium path. This can be achieved

[9] For an analysis of noisy communication in one-sender cheap talk, see Blume et *al.* (2007). See also Chen et *al.* (2008) for a one-sender cheap talk game in which both the sender and the receiver are certain behavioral types with small probability, a model resembling one in which there is a small replacement noise in both the communication and the action choice stages.

by partitioning each sender's message space into a number of subsets equal to the number of equilibrium messages, such that each subset has the power of the continuum, and associating each subset with one of the equilibrium messages. We then replace senders' strategies from the previous equilibrium with ones where, after any signal, the sender selects an action randomly (according to a uniform distribution) from the subset of messages associated with the original equilibrium message. The resulting profile remains an equilibrium and induces exactly the same outcome. Moreover, this equilibrium is robust to a small amount of replacement noise, subject to regularity conditions guaranteeing that, when the above strategy profile is played, any message is much more likely to be intended than to be the result of noise.[10]

5.3 Commitment power

If the receiver can credibly commit to an action scheme as a function of messages received, then there exist constructions simpler than the ones we proposed that are robust to small amount of noise and achieve exact truthful revelation of the state. Here, we only discuss the case of replacement noise and one-dimensional state spaces. Mylovanov and Zapechelnyuk (2009) show that a necessary and sufficient condition for the existence of a fully revealing equilibrium in a noiseless two-sender cheap talk game with commitment power and bounded interval state space $[-T, T]$ is the existence of a lottery with support $\{-T, T\}$ with the property that at every $\theta \in [-T, T]$, both senders prefer action θ to the above lottery. The sufficiency of this condition is easy to see: the receiver can commit to an action scheme that triggers the above lottery in case of differing messages from the senders.

We observe that given the above action scheme, truthtelling by the senders remains an equilibrium for small enough replacement noise. This is because if the other sender follows a truthtelling strategy, then after receiving signal θ, sending any other message than θ induces the threat lottery with probability 1, while sending message θ induces θ with high probability. The latter is by construction preferred by the sender if the state is likely to be θ. The above implies that in case of commitment power, there exists a fully revealing equilibrium robust to replacement noise, even if the state space is relatively small. For example, if senders have symmetric and convex loss functions, and are biased in opposite directions, then there exists an equilibrium construction like the one above whenever biases are less than T in absolute value.

5.4 More than two senders

We presented the construction of robust almost fully revealing equilibria in large state spaces for the case of two senders, but obviously the construction remains valid for the case of more than two senders. In particular, it constitutes a robust equilibrium when the receiver ignores messages from all senders but two, and in turn these senders babble, while the remaining two senders and the receiver play the strategies proposed in the previous sections. Hence, the presence of more senders cannot destroy (robust) equilibria. How much benefit additional senders can provide to the receiver for a fixed state space and fixed nontrivial noise level in observations is an open question.

[10] In case of a bounded state space, this is the case if the distribution of the replacement noise has a bounded density function, and the probability of replacement noise is small enough.

5.5 Discrete state spaces

The constructions in 3.1 and 3.2 extend in a straightforward manner to large discrete state spaces. Moreover, since for such state spaces, small additive noise and small replacement noise are equivalent, the resulting constructions provide a robust equilibrium for both types of noise. Consider first the case when the state space is a coarse finite grid of a large bounded interval: $\Theta = \{\theta \in [-T, T] | \theta = -T + k \cdot \varepsilon\}$, where $T \in R_+$ is large and $\varepsilon \in R_+$ is small. Define $n_{2\varepsilon,T}$ as in subsection 3.1, and partition $[-T, T]$ to $n^2_{2\varepsilon,T}$ equal-sized subintervals. By construction, each partition contains at least one state from Θ. Then it is easy to see that the same strategy profile as presented in 3.1 gives an equilibrium robust to small amount of noise, in which the supremum of the absolute distance between any possible state and the action induced in that state is at most 2ε. This construction readily extends to other finite one-dimensional state spaces and implies almost full revelation of the state whenever the distance between the two extreme states is large enough, and the maximum distance between two neighboring states is small enough.

5.6 Diffuse prior

Although we analyzed the case of additive noise assuming a proper prior distribution in Section 4, exactly the same construction can be used to establish that our results concerning additive noise remain valid for the case of diffuse prior on $\mathbb{R}^d$ analyzed in Battaglini (2004).

6 References

Ambrus, A. and S. Takahashi (2008): "Multi-sender cheap talk with restricted state spaces," *Theoretical Economics*, 3, 1-27.

Austen-Smith, D. (1990a): "Information transmission in debate," *American Journal of Political Science*, 34, 124-152.

Austen-Smith, D. (1990b): "Credible debate equilibria," *Social Choice and Welfare*, 7, 75-93.

Baron, D. and A. Meirowitz (2002): "Fully-revealing equilibria of multiple-sender signaling and screening models," *Social Choice and Welfare*, 26, 455-470.

Battaglini,M. (2002): "Multiple referrals and multidimensional cheap talk," *Econometrica*, 70, 1379—1401.

Battaglini,M. (2004): "Policy advice with imperfectly informed experts," *Advances in Theoretical Economics*, 4, Article 1.

Blume, A., O. Board, and K. Kawamura (2007): "Noisy Talk", *Theoretical Economics*, 2, 395–440.

Chen, Y., N. Kartik, and J. Sobel (2008): "Selecting cheap talk equilibria," *Econometrica*, 76, 117-136.

Crawford, V. and J. Sobel (1982): "Strategic information transmission," *Econometrica*, 50, 1431-1452.

Dessein, W. (2002): "Authority and communication in organizations," *Review of Economic Studies*, 69, 811-838.

Eső, P. and Y. Fong (2008): "Wait and see," *mimeo, Northwestern University.*

Farrell, J. (1993): "Meaning and credibility in cheap talk games," *Games and Economic Behavior*, 5, 514-531.

Gillighan, T. and K. Krehbiel (1987): "Collective decisionmaking and standing committees: An informational rationale for restrictive amendment procedures," *Journal Of Law, Economics, and Organization*, 3, 287-335.

Gillighan, T. and K. Krehbiel (1989): "Asymmetric information and legislative rules with a heterogeneous committee," *American Journal of Political Science*, 33, 459-490.

Green, J. and N. Stokey (2007): "A two-person game of information transmission," *Journal of Economic Theory*, 135, 90-104.

Krehbiel, K. (2001): "Plausibility of signals by a heterogeneous committee," *American Political Science Review*, 95, 453-457.

Krishna, V. and J. Morgan (2001): "Asymmetric information and legislative rules: Some amendments," *American Political Science Review*, 95, 435—452.

Matthews, S., M. Okuno-Fujiwara and A. Postlewaite (1991): "Refining cheap-talk equilibria," *Journal of Economic Theory*, 55, 247-273.

Mylovanov, T. and A. Zapechelnyuk (2009): "Contracts for experts with opposing interests," *mimeo Penn State University.*

Olszewski, W. (2004): "Informal communication." *Journal of Economic Theory*, 117, 180–200.

Ottaviani, M. and P. Sørensen (2006): "The strategy of professional forecasting," *Journal of Financial Economics*, 81, 441-446.

Rabin, M. (1990): "Communication Between Rational Agents," *Journal of Economic Theory*, 51, 144–170.

Wolinsky, A. (2002): "Eliciting information from multiple experts," *Games and Economic Behavior*, 41, 141-160.

7 Appendix

Proof of Proposition 4: Consider the same construction as in the proof of Proposition 1, with $T(\delta)$ defined analogously. As established in Proposition 1, for $T > T(\delta)$, the above constructed strategy profile constitutes an equilibrium in the noiseless limit game.

We start the construction of the noise structure that establishes the claim in the proposition by making some basic observations regarding the incentives of senders around boundary points if the noise is small and a sender assumes that the other sender plays the limit equilibrium strategy.

Basic considerations regarding senders' incentives around boundaries

In the game with small noise, near a boundary b between two cells, sender i will, in an equilibrium with the noise structure that we exhibit, place a high probability on the other sender sending either the message corresponding to the cell on the left ($L_{-i,b}$) or the one corresponding to the cell on the right ($R_{-i,b}$), but will be uncertain about which of these two messages is sent.

When s_i is close to a boundary b between two cells of the same block, for θ in a neighborhood of b, sending the message $L_{i,b}$ or $R_{i,b}$ is strictly better than sending any other equilibrium message, if the other sender only sends messages $L_{-i,b}$ or $R_{-i,b}$ around b.[11] Thus, around such boundaries, if sender i believes that $-i$ sends $L_{-i,b}$ or $R_{-i,b}$ with sufficiently high probability, sender i will send $L_{i,b}$ or $R_{i,b}$.

This will not necessarily be the case, however, at boundaries between blocks. At s_1 near such a boundary b, suppose that sender 1 puts probability $p(s_1)$ on $m_2 = L_{2,b}$, $q(s_1)$ on $m_2 = R_{2,b}$, and $1 - p(s_1) - q(s_1)$ on m_2 being some other ("third") message. Then the difference between sender 1's utility from sending message m_1 and from sending m_1' is:

$$\begin{aligned}
& p(s_1)\{E[u_1(\theta, y(m_1, L_{2,b}))|s_1] - E[u_1(\theta, y(m_1', L_{2,b}))|s_1] \\
+E[u_1(\theta, y(m_1', m_2))|s_1, m_2 & \neq L_{2,b}, R_{2,b}] - E[u_1(\theta, y(m_1, m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}]\} \\
& +q(s_1)\{E[u_1(\theta, y(m_1, R_{2,b}))|s_1] - E[u_1(\theta, y(m_1', R_{2,b}))|s_1] \\
+E[u_1(\theta, y(m_1', m_2))|s_1, m_2 & \neq L_{2,b}, R_{2,b}] - E[u_1(\theta, y(m_1, m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}]\} \\
+E[u_1(\theta, y(m_1, m_2))|s_1, m_2 & \neq L_{2,b}, R_{2,b}] - E[u_1(\theta, y(m_1', m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}].
\end{aligned}$$

We want the constructed noise structure to be such that this quantity is monotonic in s_1 in a small

[11] This observation is not strictly necessary for the proof, since the argument in the next few paragraphs and the corresponding modifications to the noise structure can be applied to all boundaries. However, we find it instructive to note that only near boundaries between blocks can a third message become optimal.

To see why this is true, we show that at all boundaries between cells in the same block (for ease of notation, we drop the subscripts when referring to a specific boundary),

for all $m_i \neq L_i, R_i$, either (change the order of the arguments of y if $i = 2$):

- $u_i(\theta, y(m_i, R_{-i})) < u_i(\theta, y(R_i, R_{-i}))$ and $u_i(\theta, y(m_i, L_{-i})) < u_i(\theta, y(R_i, L_{-i}))$; or
- $u_i(\theta, y(m_i, R_{-i})) < u_i(\theta, y(L_i, R_{-i}))$ and $u_i(\theta, y(m_i, L_{-i})) < u_i(\theta, y(L_i, L_{-i}))$.

For the discussion below, we will say that either L_i or R_i dominates any other message when the above condition is satisfied, with the understanding that we consider L_{-i} and R_{-i} as the other sender's two only possible messages. Thus, either L_i or R_i will be preferred to all other messages if sender i puts a high enough probability on $m_{-i} = L_{-i}$ or R_{-i}.

Note that (we assume $i = 1$ here; the analysis is analogous for $i = 2$):

- Between cells within the same block $2, ..., n-1$: any message leads to two possibilities (for the two possible messages of sender 2) that are in adjacent blocks, except for one deviation (call it message D) that leads to an action in block 1 if sender 2 says R_2, and in block n if sender 2 says L_2. Clearly, by single-peakedness and the size of blocks, either action L_1 or action R_1 dominates any message other than D. Now if L_1 does not dominate D, it must be that sender 1 prefers the action in block 1 that D leads to over the action roughly one block to the right of the boundary that L_1 leads to, when sender 2 says R_2. But in that case, sender 1 must also prefer the action roughly one block to the left of the boundary that R_1 leads to when sender 2 says L_2 over the action in block n that D leads to. Thus R_1 would dominate D.
- Between cells within block 1 (respectively n): here, saying L_1 (respectively R_1) clearly dominates all messages other than R_1 (respectively L_1).

neighborhood of b, so that sender 1's optimal strategy within that neighborhood can be derived from a finite number of variables (namely the $\frac{n(n-1)}{2}$ indifference points between pairs of messages). This finiteness will later allow us to apply a fixed-point argument. Monotonicity holds if:

(i) $p(s_1)$ decreases sufficiently fast relative to the rates of change of $E[u_1(\theta, y(m_1, L_{2,b}))|s_1] - E[u_1(\theta, y(m_1', L_{2,b}))|s_1]$ and $E[u_1(\theta, y(m_1', m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}] - E[u_1(\theta, y(m_1, m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}]$;

(ii) $q(s_1)$ increases sufficiently fast relative to the rates of change of $E[u_1(\theta, y(m_1, R_{2,b}))|s_1] - E[u_1(s_1, y(m_1', R_{2,b}))|s_1]$ and $E[u_1(\theta, y(m_1', m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}] - E[u_1(\theta, y(m_1, m_2))|s_1, m_2 \neq L_{2,b}, R_{2,b}]$;

(iii) $\frac{p(s_1) - p(s_1')}{q(s_1) - q(s_1')}$ is sufficiently close to -1 for all s_1, s_1' in the neighborhood;

(iv) $u_1(b, y(m_1, L_{2,b})) - u_1(b, y(m_1', L_{2,b}) - [u_1(b, y(m_1, R_{2,b})) - u_1(b, y(m_1', R_{2,b})] \neq 0$.

We will not worry about (iv) because it generically holds (if it does not, we can slightly modify our noise structure or the cell/block construction). We will construct a noise structure that makes (i)-(iii) hold.

We now construct an initial noise structure with small enough noise. We then consider, around the $n^2 - 1$ cell boundaries, a family of perturbations to this initial noise structure that make senders indifferent at each boundary between the messages corresponding to the two adjacent cells. We then show that these perturbations can also be made to satisfy (i)-(iii), which will allow us to conclude the argument with a fixed-point argument. These considerations will be addressed in the *Incentives of senders around boundaries, with respect to "third" messages* section below.

Initial noise structure

Let $h(z, \gamma, \lambda) = \left\{ \begin{smallmatrix} (\frac{1-\gamma}{\lambda} - \gamma) - \frac{\frac{1-\gamma}{\lambda} - 2\gamma}{\lambda}|z| \text{ if } |z| < \lambda \\ \gamma\xi(|z| - \lambda) \text{ if } |z| \geq \lambda \end{smallmatrix} \right.$, where $\gamma, \lambda > 0$ are small numbers such that $\frac{1-\gamma}{\lambda} - \gamma \gg 0$. Assume that $\xi(.)$ is decreasing, Lipschitz continuous, satisfies $\xi(z) = 1 - \Omega z$ for some $\Omega \gg 0$ and $z \in [0, \frac{1-\nu}{\Omega}]$ (for some small ν), and $\int_0^\infty \xi(z)dz = \frac{1}{2}$. We also specify ξ so that as $z \to \infty$, $\xi(z) \to 0$ much faster than $f(z), f(-z) \to 0$. (So ξ decreases very fast around 0 and at ∞, but is of order ν in between, as required to achieve an integral of $\frac{1}{2}$.) Then, for any $y \in \mathbb{R}$, $\int_0^\infty \xi(z)v(z, y)dz$ is finite, since by assumption, $\int_{\mathbb{R}} f(\theta)v(\theta, y)d\theta$ is finite for all y. Note that h is a continuous probability distribution function, symmetric around 0, and with mass $1 - \gamma$ between $-\lambda$ and λ.

Let $H(z, \Lambda) = \left\{ \begin{smallmatrix} \frac{1}{1+2\Lambda} \text{ if } |z| < \Lambda \\ \frac{1}{1+2\Lambda}\xi(|z| - \lambda) \text{ if } |z| \geq \Lambda \end{smallmatrix} \right.$, where $\Lambda > 0$. Note that H is also a continuous probability distribution function, symmetric around 0, and with mass $\frac{2\Lambda}{1+2\Lambda}$ between $-\Lambda$ and Λ. Below, we will be using making this distribution very diffuse by taking large Λ.

Fix some $\overline{\gamma}, \overline{\lambda}$, let $\omega_i|\theta$ have pdf $h(\omega_i, \overline{\gamma}, \overline{\lambda})$, and assume that the senders play according to the above profile. Note that the receiver maximizes $\int_{\mathbb{R}} f_{m_1,m_2}(\theta)v(\theta, y)d\theta$, where f_{m_1,m_2} is her posterior on the state given m_1, m_2, so

$$f_{m_1,m_2}(\theta) = \frac{\int_{z_2 \in (i,j):j=m_2} \int_{z_1 \in (i,j):i=m_1} f(\theta)h_1(z_1 - \theta, \overline{\gamma}, \overline{\lambda})h_2(z_2 - \theta, \overline{\gamma}, \overline{\lambda})dz_1 dz_2}{\int_{\mathbb{R}} \int_{z_2 \in (i,j):j=m_2} \int_{z_1 \in (i,j):i=m_1} f(\theta)h_1(z_1 - \theta, \overline{\gamma}, \overline{\lambda})h_2(z_2 - \theta, \overline{\gamma}, \overline{\lambda})dz_1 dz_2 d\theta'}.$$

For each y, the maximand is continuous in $\overline{\lambda}$ (even at $\overline{\lambda} = 0$, since $v(\theta, a)$ is continuous) and $\overline{\gamma}$ (since $\int_0^\infty \xi(z)v(z, y)dy$ is finite for all y). Therefore, for any $\upsilon > 0$, we can pick $\overline{\gamma}$ and $\overline{\lambda}$ small enough relative to the size of each cell so that, after each pair of messages, the receiver's optimal action with the noise structure is within υ of the optimal action without noise. By picking υ small enough, we ensure that the optimal action with noise always lies in the interior of cell (m_1, m_2).

A family of perturbations of the initial noise structure

We will now perturb the noise structure around the cell boundaries so that each sender prefers sending the message corresponding to the left cell over sending the one corresponding to the right cell when her signal is to the left of the boundary, and vice-versa. At some boundaries, the senders may, within (a small multiple of) $\overline{\lambda}$ of the boundary, actually send another message. We pick $\overline{\gamma}$ and $\overline{\lambda}$ small enough so that after any perturbation in the class described below, the receiver's actions are still less than than υ away from their actions without noise.

The family of perturbations is as follows, for each boundary b:

- For θ at a distance $\zeta < \overline{\lambda}$ to the left of the boundary, $\omega_i|\theta$ has pdf $h(\omega_i, \gamma_{iLb}, \zeta)$ if $\zeta > x\overline{\lambda}$, for some small $x \in (0, 1]$, $\gamma_{iLb} \in (0, \overline{\gamma}]$. Thus, for such ζ, the sender's signal becomes more precise.

- For ζ within $x\overline{\lambda}$ of the boundary, we have $\omega_i|\theta$ following pdf $H(\omega_i, \overline{\Lambda})$, for some very large $\overline{\Lambda} \gg \max_{i=1,2;j=L,R}\{\frac{1-\gamma_{ijb}}{2\gamma_{ijb}}\}$. For such ζ, the signal becomes very imprecise.[12] Since the area can be made arbitrarily small relatively to $\overline{\lambda}$, we ignore it below.

- Finally, for ζ at a distance slightly greater than $\overline{\lambda}$ from the boundary, $\omega_i|\theta$ has pdf $h(\omega_i, \gamma_i(\theta), \overline{\lambda})$, where, moving away from the boundary, $\gamma_i(\theta)$ increases in a Lipschitz continuous way (with Lipschitz constant large, but $\ll \Omega$) from γ_{iLb} to $\overline{\gamma}$.

Define perturbations on the right analogously.

Next, we make a series of observations about the senders' incentives in games with noise structures constructed above, when they assume that the other sender plays a strategy close to the limit equilibrium one.

Incentives of senders around boundaries, with respect to the messages corresponding to adjacent cells

Note that for any $\varphi > 0$, it is possible to choose $\overline{\Lambda}$ large enough and $\overline{\lambda}$ small enough such that given any signal s_i, player i puts probability puts probability less than φ on the state being within $x\overline{\lambda}$ of any boundary. ($\overline{\lambda}$ small is needed for extreme signals, more than $\overline{\Lambda}$ to the left of the leftmost boundary or to the right of the rightmost boundary.) This implies that for any $\rho < 1$, we can pick $\overline{\gamma}$ small enough such that in a vicinity of any boundary, each sender puts probability at least ρ of the other sender receiving a signal in one of the cells adjacent to the boundary.

Consider a boundary b between two cells. Each sender is supposed to send the message L if her signal is to the left of b, and R if it is to the right (though as mentioned above, we will allow senders to send other messages in a small neighborhood of b). Let p_b be the probability that sender 1 puts on s_2 being to the right of b, conditional on s_2 being in an interval adjacent to b, when s_1 is at b. Similarly, let q_b be the probability that sender 2 puts on s_1 being to the left of b, conditional on s_1 being in an interval adjacent to b, when s_2 is at b. Given the other sender's strategy and fixing the receiver's action for each of the four possible pairs of messages $(y_{L,L},\ y_{R,R},\ y_{R,L},\ y_{L,R})$, there are values $p_b^*, q_b^* \in (0, 1)$ such that the senders are indifferent between saying L and saying R.

We wish to show that our family of perturbations can yield these probabilities. For simplicity, in the following, we approximate the prior near the boundary to be uniform, $\overline{\Lambda}$ to be infinite, and assume $\xi(0) = 1$, and $\xi(z) = 0$ for all $z > 0$ (as an approximation for ξ decreasing quickly to very small ν). Then at the boundary, $p = \frac{\gamma_{1Rb}}{\gamma_{1Rb}+\gamma_{1Lb}}$, and $q = \frac{\gamma_{2Lb}}{\gamma_{2Rb}+\gamma_{2Lb}}$, so we can achieve any $(p, q) \in (0, 1)^2$. For any $\nu > 0$, the same will be true for $(\nu, 1-\nu)^2$ with almost uniform prior (note that we pick $\overline{\lambda}$ to be small, and the prior pdf is Lipschitz continuous), continuous ξ, and γ small.

[12] More specifically, there is a small area at the junction of those two segments where the distribution first follows $h(\cdot, \gamma_{iLb}, \lambda)$, where λ increases rapidly and continuously from z to $\frac{1-\gamma_{iLb}}{2\gamma_{iLb}}$, and the distribution then follows $H(\cdot, \Lambda)$ (note that $h(\cdot, \gamma, \frac{1-\gamma}{2\gamma})$ is equal to $H(\cdot, \frac{1-\gamma}{2\gamma})$), where Λ increases rapidly and continuously from $\frac{1-\gamma_{iLb}}{2\gamma_{iLb}}$ to $\overline{\Lambda}$.

Note also that for large Ω, s_i being slightly to the left of the boundary drastically decreases the probability that sender i puts on the true state being (more than $x\overline{\lambda}$) to the right. This in turn drastically increases the probability that s_{-i} is to the left of the boundary. For any $\alpha < 1$, $\beta > 0$, we can specify Ω large enough and ν small enough so that when s_i is at least β away from the boundary, sender i believes that s_{-i} has probability at least α of being on the same side. Note that within β of the boundary, while a sender never sends a message corresponding to the cell on the other side of the boundary due to our noise structure, she may decide to send some third message corresponding to neither cell adjoining the boundary. For any β, if the senders deviate only within β of each boundary, the receiver's action corresponding to each message pair is within a bounded subset, irrespective of what these deviations are. We pick $\beta \ll x\overline{\lambda}$ so that these bounded sets are each within the appropriate cell. We also pick α high enough so that each sender will indeed report the message corresponding to the cell when her signal is at least β away from the boundary (since β is very small, each sender i puts a very low probability m_{-i} being neither message corresponding to an adjacent cell).

In summary, our parameters are: $\gamma_{ijb}, \overline{\gamma}, \overline{\lambda}, \nu, x$ small; β small relative to $x\overline{\lambda}$; α close to 1; $\overline{\Lambda}, \Omega$ large. The key properties of the noise structure are:

- The noise structure is precise enough ($\overline{\gamma}, \overline{\lambda}, x$ are small enough) and the area around each boundary where the senders may deviate is small enough (β is small enough) that each of the receiver's actions $y(m_1, m_2)$ is within a compact subset B_{m_1,m_2} of the appropriate cell, regardless of how the senders deviate within β of boundaries.

- Given all B_{m_1,m_2}, large enough $\overline{\Lambda}, \Omega$, and low enough ν, we can find α high enough such that senders do not deviate more than β away from boundaries. (We may need to make β even smaller than in the previous step so that senders put a high enough probability on the other sender sending the appropriate message. This does not pose a problem, since reducing β (weakly) shrinks the sets B_{m_1,m_2}, which also (weakly) reduces the required α.)

- By selecting the appropriate γ_{ijb}, we make each sender, when her signal is exactly equal to a boundary, indifferent between the messages corresponding to the right and the left cells. Note that due to the bounds on the receiver's actions and the finite number of boundaries, we can place lower and upper bounds strictly within $(0, 1)$ on the requisite relative probabilities for indifference. This implies that given $\overline{\gamma}$, we can impose a lower bound $\underline{\gamma}$ on γ_{ijb}. We also require that the larger of γ_{iLb} and γ_{iRb} be equal to $\overline{\gamma}$, so that $\gamma_{ib} \equiv \gamma_{iLb} - \gamma_{iRb}$ is well-defined and has a one-to-one correspondence with p or q.

Incentives of senders around boundaries, with respect to "third" messages

We can further choose our parameters to satisfy conditions (i)-(iii) (for both senders), within β of each boundary. (i) and (ii) are met by selecting Ω large enough and $\beta < \frac{1-\nu}{\Omega}$ (so that ξ changes at rate Ω throughout β), since the receiver's actions are bounded, and due to the concavity of u_i with respect to θ, u_i is Lipschitz continuous within any bounded set. For (iii), we need to slightly modify our noise structure by adjusting the slopes of the pdf of $\omega_i|\theta$ within β to the left of each boundary, for all θ at a distance between $x\overline{\lambda}$ and $\overline{\lambda}$ of the boundary, and similarly on the right side. The slopes need to be equal to Ω (with a slight adjustment for the prior density) such that $p(\theta) + q(\theta)$ stays roughly constant within β of each boundary.

It then follows that the difference between sender i's utility from sending message m_i and from sending m_i' is monotonic within β small enough of any boundary b (again due to Lipschitz continuity of u_i with respect to θ, the quantity in condition (iv) is guaranteed to have the same sign for all θ within a small enough distance of b). There is therefore exactly one indifference point for each pair of messages at each boundary. Furthermore, with small enough noise, we also know the message in each pair that is preferred under $p = 1$ and under $q = 1$. Thus, the message that is preferred to each side of the indifference point (or, if there is no such point, the dominating message over the β-ball around b) is predetermined.

We have previously shown that near boundaries that are not between blocks, for the appropriate γ_{ijb}, senders will send the same message as in the game with no noise. However, near boundaries between blocks, although with the appropriate γ_{ijb}, senders have the same preference over the messages corresponding to the adjacent cells as under no noise, they may prefer sending a third message. Their strategies near these boundaries can therefore be derived from the location of the $n(n-1)/2$ indifference points between pairs of messages. Specifically, near each boundary, each sender's strategy can be uniquely determined (up to a finite number of points) from a vector in $[-\beta, \beta]^{n(n-1)/2}$, where each entry corresponds to the location, relative to the boundary, of the indifference point for a pair of messages, and where if no such indifference point exists within β of the boundary, the entry corresponds to the endpoint, $-\beta$ or β, where the two messages yield the closer utilities. Therefore, a strategy profile of the form that we have constructed, in a game with our noise structure, can be determined from a vector $\overrightarrow{x} \in [0, x]^{n(n-1)^2}$, corresponding to the concatenation of the $n(n-1)/2$ indifference points per boundary-sender, for both senders, at all $n-1$ boundaries between blocks.

It remains to be shown that an equilibrium exists with the requisite $\overrightarrow{\gamma} = (\gamma_{ib})_{i=1,2; b\in\{-T+\frac{k}{n^2}2T|k\in\{1,\ldots,n^2-1\}\}} \in [\underline{\gamma} - \overline{\gamma}, \overline{\gamma} - \underline{\gamma}]^{2(n^2-1)}$. The proof below uses a fixed point argument to do so.

Existence of an equilibrium profile under the noise structure

We use a fixed point argument to show the existence of equilibrium under the construction that we proposed. The receiver's strategy is represented by the vector $\overrightarrow{y} = (y(m_1, m_2))_{m_1,m_2=1,\ldots,n} \in \times_{m_1,m_2=1,\ldots,n} B_{m_1,m_2}$. The senders' strategies can be inferred from a vector $\overrightarrow{x} \in [0, x]^{n(n-1)^2}$, and our construction also requires a vector $\overrightarrow{\gamma} \in [\underline{\gamma} - \overline{\gamma}, \overline{\gamma} - \underline{\gamma}]^{2(n^2-1)}$ that makes the senders indifferent at each boundary between the actions corresponding to the left and the right cell.

Note that $\overrightarrow{\gamma}$ is continuous in itself, as it continuously affects the sender's posterior beliefs. It is also continuous in $\overrightarrow{y}$ and $\overrightarrow{x}$ because the senders' utility functions are continuous.

Also, $\overrightarrow{y}$ is continuous in $\overrightarrow{x}$ because the receiver's utility function is continuous and strictly concave. It is continuous in $\overrightarrow{\gamma}$, which continuously affects the receiver's posterior beliefs. It does not depend on itself.

Moreover, $\overrightarrow{x}$ is continuous in $\overrightarrow{y}$ and $\overrightarrow{x}$ (the other sender's portion; it obviously doesn't depend on the own sender's portion) because the senders' utility functions are continuous. It is continuous in $\overrightarrow{\gamma}$, which continuously affects the sender's posterior beliefs.

Finally, in each of these cases, $\overrightarrow{\gamma}$, $\overrightarrow{y}$ and $\overrightarrow{x}$ simultaneously changing also continuously affects these variables. Thus, we can apply Brouwer's fixed point theorem, so an equilibrium exists. ■