Rationalizing Choice with Multi-Self Models[*]

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Abstract

To facilitate systematic study of multi-self (or group) decision making, this paper proposes a framework that encompasses a variety of models proposed in economics, psychology, and marketing. We model choice as arising from the aggregation of a collection of utility functions. We propose a method for characterizing the extent of irrationality of a choice behavior, and use this measure to provide a lower bound on the set of choice behaviors that can be rationalized with n utility functions. Within a class of models, generically at most five "reasons" are needed for every "mistake."

JEL Codes: D11, D13, D71
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1 Introduction

The classical model of choice endows the decision-maker (DM) with a single preference relation that she uses to select the best element from any set of alternatives. The single implication of this model is context-independent choice, or the *Independence of Irrelevant Alternatives* (IIA), which dictates that an alternative which is deemed optimal in a set remains optimal in any subset.[1] A growing body of evidence suggesting that behavior violates IIA has spurred interest in alternative models. Since the seminal work of May (1954), formal models of multi-self decision-making proposed include Kalai, Rubinstein and Spiegler (2002), Fudenberg and Levine (2006), Manzini and Mariotti (2007), and Green and Hojman (2009) in the economics literature, where selves are often seen as rationales or manifestations of temptation and self control processes; Kivetz, Netzer and Srinivasan (2004) in the marketing literature, where selves are different considerations or criteria for evaluating products; and Tversky (1969), Shafir, Simonson and Tversky (1993) and Tversky and Simonson (1993) in the psychology literature, where the *multiplicity of self*, comprised of different motivational systems, has long been viewed as a normal feature instead of a sign of pathology.[2]

At the same time, empirical evidence on household demand strongly suggests that it also cannot arise from the maximization of a single utility. An extensive literature examines the microeconomic implications of collective choice in households where each member is a utility maximizer. This literature, which includes Browning and Chiappori (1998) and Chiappori and Ekeland (2006), examines such models under the restriction of Pareto-efficient household behavior. One question addressed in this setting is, given a household demand function over N goods, when do there exist n utility functions $\{u_i\}_{i=1}^n$ and a continuously differentiable function μ of prices and income such that the demand arises from the weighted utilitarian

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[1]This also implies transitive choice behavior, which is often violated in experimental settings (e.g., see Tversky (1969) and Lee, Amir and Ariely (2007)).

[2]Even psychologists who prefer a unitary view of the self accept that "the singular self is a hypothetical construct, an umbrella under which experiences are organized along various dimensions or motivational systems" and which "is fluid in that it shifts in different contexts as various motivations are activated" (Lachmann 1996). An expanded shortlist of the multiple-selves or multiple-utility literature includes Benabou and Pycia (2002), Masatlioglu and Ok (2005), Evren and Ok (2007), and Chatterjee and Krishna (forthcoming). This literature is also related to the application of social choice tools in multi-criteria decision problems, as in Arrow and Raynaud (1986), and is related more generally to the theory of multiattribute utility (see Keeney and Raiffa (1993)). Another approach, developed in Bernheim and Rangel (2007) and Salant and Rubinstein (2008), allows for context-dependence by considering extended choice situations where behavior can depend on unspecified *ancillary conditions* or *frames*. While information effects can explain some context dependence (Sen (1993), Kochov (2007), Kamenica (2008)), they cannot explain many systematic violations of IIA (Tversky and Simonson (1993)).

maximization of $\sum_{u \in U} \mu(\text{price,income})u(\cdot)$ given the budget set (i.e., weights and preferences vary independently).

These different strands of the literature all connect empirically observed, "irrational" choice behavior to the the aggregation of different objectives. Despite this commonality, each literature focuses on different questions. The multi-self decision-making literature is often interested in which irrational behaviors are predicted by a fixed model; the household decision-making literature is often interested in finding a model (the number of individuals and aggregation procedure) that can explain observed choice.

We propose a unified framework to examine models of collective decision-making where choice sets may influence how preferences are aggregated. More formally, we model the choice as arising as a collection of utility functions U and an *aggregation rule* f (decision-making method) which combines these utility functions in a possibly context-dependent way. That is, given a choice set A, and the utility functions in U, an aggregator f specifies an aggregate utility for every alternative in A. Each of the utility functions in U may correspond to different selves, as in multi-self decision-making, or to different individuals, as in household or organizational decision-making; permitting both interpretations, we simply refer to each utility function as a *member of the group U*. We describe the group's behavior by a choice function c that specifies the alternative selected from each subset of the grand set of alternatives X. A given model of aggregation f *rationalizes* the choice behavior of the group U if for every choice set A, the choice described by c is the unique maximizer of aggregate utility $f \circ U$. This choice behavior need not satisfy IIA. To characterize the extent to which a choice behavior deviates from rationality, we develop a method of counting IIA violations.

We examine a broad class of aggregators characterized by five properties from social choice theory, and show that many models of multi-self decision-making proposed in the existing literature can be formally translated into an aggregator in our framework. Our results thus provide a meta-analysis of various models proposed in that literature, and offer a new way to characterize their explanatory power. As described below, our models are also slightly different than but closely related to existing models of household choice, and so offer a complementary way to analyze data in that setting. We study the set of choice functions each model in our class can rationalize, both with a fixed number of selves, as well as with no *a priori* restriction. Viewed another way, our framework allows us to find, for any given model of aggregation, a bound on the group size that is needed to rationalize an irrational behavior. For some aggregators, it is straightforward to determine the set of choice functions

that can be rationalized. For example, if the method of aggregating the group of utilities is simple utilitarianism, then the set of choice functions is exactly the set of rational choice functions — regardless of the size of the group. But what if, in analogy to models of relative utilitarianism (e.g., Karni 1998), the weight allotted to each member is normalized by its range of utilities over the choice set? Or if the aggregator is the "normalized contextual concavity model" proposed in Kivetz et al. (2004),

$$\sum_{u \in U} (\max_{a' \in A} \ u(a') - \min_{a' \in A} \ u(a')) \cdot \Big[\frac{u(a) - \min_{a' \in A} \ u(a')}{\max_{a' \in A} \ u(a') - \min_{a' \in A} \ u(a')} \Big]^{\rho} ?$$

We use our measure of IIA violations to characterize a lower bound on the set of choice functions that can be rationalized given a model of aggregation; and show there is a *linear* relationship between the size of the group and the number of IIA violations in the bound. For a given class of aggregators, at five members of a group are needed to justify each "mistake." For another class, only one member is needed per mistake. More generally, there is an aggregator-dependent proportionality constant which can be found simply by looking at behavior over three-element sets. An important implication of our results is that without restricting the number of members, even a very structured model of group decision-making might not have testable restrictions on behavior. It is thus important to fix the number of members *a priori* (e.g., as in a "dual-self" model) in order to restrict the set of behaviors that the model can rationalize.

Our class of models has two prominent features. First, aggregation can depend on cardinal information in the utilities. Many existing models of household and multi-self decision-making do make use of cardinal information embedded in different selves' utility functions. In both household and intrapersonal decision-making, the intensities of preferences should be comparable and may play an important role. Cardinal comparisons are even assumed in expected utility theory: a DM trades off utility across possible states. To motivate such comparisons here, suppose a person choosing where to live cares about his children as well as proximity to work. One possible home is adjacent to his workplace in the city but the school is unsafe; the other home is in a suburb which would be a short commute to work but the school is safer. Without cardinality, it is difficult to argue that it is much more important for the children to be in a safe school than it is to have a short commute to work. On the other hand, it is plausible to assume the person is willing to trade a small enough degree of safety for a substantially reduced commuting time. A second feature, which builds on cardinality, is the possibility of compromise. This is a defining feature of the models of household choice,

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which are interested in efficient outcomes arising from an unmodeled bargaining process. As opposed to the multi-self models provided in Kalai et al. (2002) and Cherepanov, Feddersen and Sandroni (2008), but in accordance with Tversky (1969), Tversky and Kahneman (1991), Kivetz et al. (2004), Fudenberg and Levine (2006), Green and Hojman (2009) and others, all of the selves in our framework are "active" for every possible choice set.[3] However, the weights allocated to different selves by the aggregator can depend on the choice set. Thus, the model can capture behavior as in Fudenberg and Levine (2006), where a long-run self must exert more costly self control when more appealing options are available to a short-run self; or Shafir et al. (1993), where the primary rationales for purchasing may depend on the set of available products.

Our results draw a connection between the complexity of a rationalization and the extent to which the choice behavior in question deviates from rationality, as measured by the number of IIA violations.[4] Hence our results differ from Kalai et al. (2002), who study irrational choice by a DM and examine the required complexity of a rationalization of that choice behavior as a function of the number of alternatives available. Their framework also differs from our own; in their setting, a collection of strict preference relations rationalizes a choice function if the choice from each set is optimal for at least one of the preference relations. In this view, each (ordinal) self serves as a dictator for some subset of choices. In contrast, in our framework it can happen that the choice is not the most preferred alternative according to any of the utility functions, but is the *best compromise*, in the sense that it maximizes aggregate utility.

Our results also complement those in the household choice literature, such as Browning and Chiappori (1998) and Chiappori and Ekeland (2006), from which we differ in a number of ways. Browning and Chiappori (1998) show that if there are n goods, then any demand data can be explained by an $(n-1)$-person household. In addition, to explain a given demand function using n people, it is necessary and sufficient that the rank of a certain matrix in a pseudo-Slutsky matrix decomposition be $n-1$, though without further restrictions there can be a continuum of explanatory n-person models (Chiappori and Ekeland (2006)).[5] We

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[3]Psychologists believe that a fluid form of compromise among selves is necessary for healthy behavior. This is as opposed to disassociated selves (i.e., overly autonomous selves), or a high self-concept differentiation (a lack of interrelatedness of selves across contexts) both of which are connected to pathological or unhealthy behavior; see Power (2007), Donahue, Robins, Roberts and John (1993), and Mitchell (1993).

[4]Measuring the complexity of a DM's rationalization by the number of selves is akin to measuring the complexity of an automata by the number of states (e.g., see Salant (2007) in the context of decision-making).

[5]The pseudo-Slutsky matrix is formally defined in Chiappori and Ekeland (2006); the rank condition they give, $SR(n-1)$, is that this matrix can be decomposed as the sum of a symmetric negative semi-

address the question of rationalization by any fixed aggregator satisfying our properties, while the above papers assume that the modeler does not know the underlying decision rule of the household, only that it belongs to a class of budget-weighted utilitarian aggregation rules. The weight of an individual for a given model of aggregation in our framework does not depend directly on the price information, but does depend on the individual's utilities for the alternatives in the budget set. We also examine choice functions instead of demand functions. However, given that demand data is typically finite, rationalizing the demand data corresponds to rationalizing an *incomplete choice function*, and we show that our results can be extended to arbitrary incomplete choice functions.

There are several recent contributions to the literature on multi-self decision-making, which mostly focus on a different set of questions than we do. Of these, the most related is Green and Hojman (2009), who also study a class of aggregation methods. Because they model a DM as a probability distribution over all possible ordinal preference rankings, their framework is difficult to compare to models of multi-self decision-making with a discrete number of cardinal selves, but is related to models in the voting literature (e.g., Saari 1999). Extending results from that literature, they show that if choice is determined by a voting rule satisfying a monotonicity property, then their model can explain any choice behavior.[6] The rest of the paper focuses on welfare analysis. Bernheim and Rangel (2007) and Chambers and Hayashi (2008) also focus on welfare analysis given choices contradicting rational decision-making. Other related work includes Manzini and Mariotti (2007), Masatlioglu and Nakajima (2007) and Cherepanov et al. (2008), who consider sequential application of multiple rationales to eliminate alternatives, a process they show can rationalize certain choice functions. Finally, Fudenberg and Levine (2006) consider a dual-self model of dynamic choice, where the two selves' utilities are aggregated in a menu-dependent way.[7]

2   A framework for group choice

We observe a group's choice behavior on a finite set of alternatives X. Denote by $P(X)$ the set of nonempty subsets of X. The *choice function* $c : P(X) \to X$ identifies the alternative $c(A) \in A$ chosen from each $A \in P(X)$. A *rationalization* of a choice function consists of

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definite matrix and another matrix of rank at most $n - 1$. One intuition for the proof, which relies on exterior differential calculus, is that the Pareto-frontier for n people is $n - 1$ dimensional, and weights and preferences can be varied independently.

[6]Our result on rationalization is independent of this monotonicity theorem.

[7]See also Chatterjee and Krishna (forthcoming) for a model of dual-self decision-making.

two components, a collection of *utility functions* U and an *aggregator* f that combines these utilities into an aggregated utility function, in a way that may possibly depend on the menu. Viewed as a multi-self model, these utility functions represent a DM's conflicting motivations or priorities. The aggregator corresponds to the method of sorting out different priorities to come to a decision. To simplify notation, in the main text we define a simplified framework in which the aggregator treats all utility functions symmetrically. However, in Supplementary Appendix A we allow nonanonymous aggregation and extend our construction and main results to asymmetric aggregators that can treat utility functions differently based on a "type." This feature often arises in models of multi-self decision-making, such as with long run and short run types of selves.

Formally, given a grand set of alternatives X, a utility function $u : X \to \mathbb{R}$ describes the utility level allocated to each alternative $x \in X$.[8] A *group* U is an unordered list of utility functions. We will often refer to a utility function in U as a *member of the group*.[9] By definition of an unordered list, a group can have multiple identical utility functions, and there is no order hierarchy defined over these members. Formally, for a given grand set of alternatives X, a group U is an element of $\mathcal{U}(X) = \cup_{n=1}^{\infty} \mathcal{U}^n(X)$, where $\mathcal{U}^n(X)$ is the set of all unordered lists of utility functions over X of size n. We denote the number of members of a group U by $|U|$, or simply n when no confusion would arise.

An aggregator f specifies an aggregate utility for every alternative a in every choice set A, given any (finite) grand set of alternatives X and any group U defined over these alternatives. Formally, the domain over which f is defined is $\{a, A, X, U | X \in \mathcal{X}, U \in \mathcal{U}(X), A \in P(X), a \in A\}$, where $\mathcal{X}$ is the set of conceivable finite grand sets of alternatives. Since the choice set A is one of the arguments of the function, f aggregates the utilities of the group in a possibly context-dependent way.[10] An aggregation rule may be seen as a particular theory of how members of the group are activated by choice sets: the aggregator determines the weight each member receives on the choice set as a function of its utility levels over the alternatives. The grand set of alternatives X appears as an argument of the aggregator, not only because

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[8]Though aggregation in our framework is cardinal, the model has the "ordinal" feature that there can be many "equivalent" representations of an aggregator in this context. In particular, if f rationalizes the choice function c using the group U, then so does any increasing transformation of f. Similarly, if f rationalizes c using the group U, then $f \circ h^{-1}$ rationalizes c using the group $h \circ U$, where $h : \mathbb{R} \to \mathbb{R}$ is invertible on the appropriate domain. That is, given any representation U and f, one can obtain an equivalent representation by applying a monotone transformation of utilities in U, if a corresponding transformation is applied to the aggregation function f as well.

[9]In combinatorics this object is also referred to as a multiset.

[10]We could permit aggregators with restricted domains: let $\hat{\mathbb{R}}^X$ be a convex subset of $\mathbb{R}^X$ and let $\mathcal{U}^n = \times_{i=1}^{n} \hat{\mathbb{R}}^X$.

the evaluation of an alternative $a \in A$ could potentially depend on alternatives outside the choice set A, but also because this enables a "comparative static": we study how the number of members needed to rationalize a choice rule depends on the size of X.

A choice function reveals the element deemed best in that set. We say that a model of aggregation f rationalizes a choice function if from every choice set the alternative that maximizes the aggregated utility is the one selected by the choice function.[11]

Definition 1. *A group U's choice function $c(\cdot)$ on X is rationalized by an aggregator f if for every $A \in P(X)$, $c(A) = \arg\max_{a \in A} f(a, A, X, U)$.*

3 Counting IIA violations

What kinds of behavior can an aggregator rationalize? Consider one of the simplest types of aggregators, the model of utilitarianism:

$$f(a, A, X, U) = \sum_{u \in U} u(a).$$

The only choice function that utilitarianism can rationalize is rational choice, that is, choice which satisfies the Independence of Irrelevant Alternatives (IIA). IIA requires that if $a \in A \subset B$ and $c(B) = a$ then $c(A) = a$. This says that if an alternative is chosen from a set, then it should be chosen from any subset in which it is contained. It is well known that a choice function can be rationalized as the maximization of a single preference relation if and only if it has no violations of IIA. A non-utilitarian model of aggregation, however, might be able to rationalize a choice function that violates IIA. In order to be able to characterize the set of choice functions that a model of aggregation can rationalize, we now propose a way to measure the extent to which a choice function violates IIA.

The number of IIA violations can be determined straightforwardly for choice functions over three-element sets; e.g., if the choice over pairs is transitive but the second-best element according to the pairs is selected from the triple, there is one violation of IIA. For a larger set of alternatives, there are different plausible ways to define the number of violations. For

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[11]The underlying model f encodes additional information, such as the ranking of unchosen alternatives in each set, that might be observable using a larger data set than that provided by a choice function. However, using only simple revealed preference on the choice from a menu, only the best choice from each set (i.e., the choice function) is elicited in light of the potential menu-dependence of choices.

example, suppose that

$$c(\{a, b, c, d, e, f\}) = d$$
$$c(\{a, b, c, d, e\}) = b$$
$$c(\{a, b, c, d\}) = b$$
$$c(\{b, c, d\}) = c.$$

In light of $c(\{a, b, c, d, e, f\}) = d$, IIA dictates that the last three choices should be d (but they are not). In light of $c(\{a, b, c, d, e\}) = b$, IIA dictates that the choice from $\{b, c, d\}$ should be b (but it is not), and the IIA implication for $\{b, c, d\}$ is again violated in light of $c(\{a, b, c, d\}) = b$. Hence, one way of counting would indicate five IIA violations with respect to the above four choice sets.

However, according to our counting procedure, there are two IIA violations in this example: only the choices from $\{a, b, c, d, e\}$ and $\{b, c, d\}$ are associated with violations. The reason is that while $c(\{a, b, c, d\}) = b$ does contradict $c(\{a, b, c, d, e, f\}) = d$, the intermediate choice $c(\{a, b, c, d, e\}) = b$ itself implies by IIA that $c(\{a, b, c, d\}) = b$. With this motivation in mind, our accounting procedure associates an IIA violation with a choice set if and only if the choice from the set contradicts the choice from some superset *and* there is no choice from a set in between the two that could justify the violation.

Definition 2 (IIA violation). *The set A causes an IIA violation under the choice function $c(\cdot)$ if (1) there exists B such that $A \subset B$ and $c(B) \in A \setminus \{c(A)\}$, and (2) for every A' such that $A \subset A' \subset B$, $c(A') \notin A$.*

The total number of IIA violations is then defined as follows.

Definition 3 (Number of IIA violations). *The total number of IIA violations of a choice function $c(\cdot)$ is given by $\text{IIA}(c) = \#\{A \in P(X) \mid A$ causes an IIA violation$\}$.*

The above definition can yield a large number of IIA violations for choice rules that can be defined relatively easily. Consider, for example, the choice function that arises when one strict preference ordering dictates choice whenever the menu contains some highlighted alternative, while an opposite strict preference ordering dictates choice in the absence of that alternative. In Section 7.4 we provide a construction that collapses IIA violations compatible with each other into a single violation, and show how the construction can be used to sharpen our results.

There are other plausible measures for the number of IIA violations implied by a choice function. One alternative measure would be the minimal number of sets at which the choice function would have to be changed to make it rational. This measure can in general be either larger or smaller than our measure of the number of IIA violations.[12]

4   Additive and scale invariant models

In this section we study additively separable and scale-invariant aggregators that can be written in the following form:

$$f(a, A, X, U) = \sum_{u \in U} g(a, \{u(a')\}_{a' \in A}),$$

where g takes as an argument a single utility function u and evaluates an alternative $a \in A$ based on the set of values that u takes on A. We assume g satisfies the following three properties:

1. (Individual agreement) $g(a, \{u(a')\}_{a' \in A}) \geq g(b, \{u(a')\}_{a' \in A})$ whenever $u(a) \geq u(b)$. This is a minimal consistency requirement: g rates higher the alternative that u prefers.

2. (Neutrality) $g(a, \{u(a')\}_{a' \in A}) = g(\pi \circ a, \{u(\pi \circ a')\}_{a' \in A})$ for any permutation $\pi : X \to X$. Neutrality says that the treatment of alternatives depends on their utilities and not their names.

3. (Scale invariance) $g(a, \{\alpha u(a')\}_{a' \in A}) = \phi(\alpha) g(a, \{u(a')\}_{a' \in A})$ for any $\alpha \in \mathbb{R}$ and some invertible and odd $\phi : \mathbb{R} \to \mathbb{R}$. This says the unit in which preference intensity is measured does not matter: the group $(\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$ is aggregated in an analogous manner as the group $(u_1, u_2, \ldots, u_n)$.

This class of additive aggregators, which we denote $\mathcal{F}^*$, includes various menu-dependent versions of utilitarianism.

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[12]Indeed, suppose that pairwise choices exhibit the transitive ranking a preferred to b preferred to c. Under our measure, there is one violation of IIA if $c(\{a, b, c\}) = b$, which is defeated once in the pair $\{b, c\}$, and two violations of IIA if $c(\{a, b, c\}) = c$, which is defeated twice. The alternative measure counts one violation either way. To see that the alternative measure can also be larger, consider the choice function over $\{a, b, c, d, e\}$ which chooses the alphabetically-lowest alternative in all sets, except that b is chosen in three-element sets in which it is contained as well as from the pair $\{a, b\}$. The alternative measure counts four violations, while ours counts three.

The following result gives a lower bound on the set of behaviors a generic aggregator $f \in \mathcal{F}^*$ can rationalize, where genericity is with respect to a topology over $\mathcal{F}^*$ which is based on the sup metric and formally defined in the appendix.[13] The result provides a linear connection between the complexity of the observed behavior (as measured by the number of IIA violations) and the degree of freedom in the model (as measured by the size of the group). One way to view this result is that at most five "good reasons" are needed for every "mistake" that the DM makes.

Theorem 1. *Given a generic aggregator $f \in \mathcal{F}^*$ (using the topology defined in the appendix), and given any choice function c defined on any finite grand set of alternatives X, no more than $1 + 5 \cdot IIA(c)$ group members are needed to rationalize the choice function.*

Alternatively stated, given a group of size n, a generic aggregator $f \in \mathcal{F}^*$ can rationalize any choice function c, defined on any finite grand set of alternatives X, that exhibits at most $\frac{n-1}{5}$ IIA violations. As can be seen from the proof of this result, it is easy to check whether a given aggregator is of this generic type: if it can generate two particular types of irrational behavior on a triple of alternatives, then it can generate any behavior on any set of alternatives. An immediate implication of Theorem 1 is that in spite of having a structured form, essentially any aggregator $f \in \mathcal{F}^*$ can rationalize any choice function if sufficiently many members are in a group. In other words, if a model of household or multi-self decision-making *does not* restrict the number of individuals or selves, and corresponds to an aggregator satisfying the properties above, then it generates a theory that cannot be refuted. The result therefore points out the importance of putting *a priori* restrictions on the number of group members in such a model - a practice followed in some but not all of the existing literature.

This simple result is a corollary of a more general result, Theorem 1, which is discussed in the ensuing sections. We will soon demonstrate for more general aggregators how to construct a rationalization for a choice function based on its number of IIA violations.

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[13]Somewhat more formally, with respect to a grand set with three alternatives, additive and scale-invariant aggregators can be associated with pairs of operators such that one operator maps from the two-dimensional simplex in $\mathbb{R}^2$ to $\mathbb{R}^2$, and the other one maps from the three-dimensional simplex in $\mathbb{R}^2$ to $\mathbb{R}^2$. We use the topology defined by the supremum distance metric on these pairs of operators.

5 A more general class of models

More generally, consider models of utility aggregation satisfying the following properties. These properties are familiar from the theory of social choice, and are satisfied by several previously proposed multi-self models of decision-making. In the resulting class of models, the aggregation of utilities is cardinal and the framing effect of a choice set operates only through the utility levels of alternatives for different group members.

Formally, we say a group $U \in \mathcal{U}(X)$ is *δ-indifferent with respect to f and X* if

$$\max_{a,b \in A, A \subseteq X} |f(a, A, X, u) - f(b, A, X, u)| < \delta \quad \text{for every } u \in U.$$

For any two groups $U, U' \in \mathcal{U}(X)$, the combined group $(u_1, \ldots, u_{|U|}, u'_1, \ldots, u'_{|U'|}) \in \mathcal{U}(X)$ is denoted by (U, U').

P1 (Neutrality). *For any permutation $\pi : X \to X$, $f(\pi(a), \pi(A), X, U \circ \pi^{-1}) = f(a, A, X, U)$.*

P2 (Individual agreement). *For any $u \in \mathbb{R}^X$, $u(a) \geq u(b)$ if and only if $f(a, A, X, u) \geq f(b, A, X, u)$.*

P3 (Separability). *If $f(a, A, X, U) \geq f(b, A, X, U)$ and $f(a, A, X, \hat{U}) \geq f(b, A, X, \hat{U})$ then $f(a, A, X, (U, \hat{U})) \geq f(b, A, X, (U, \hat{U}))$, with strict inequality if one of the above is strict.*

P4 (Continuity at indifferent members). *If $f(a, A, X, U) > f(b, A, X, U)$ then for any $k \in Z_+$ there is $\delta_k > 0$ such that $f(a, A, X, (U, U')) > f(b, A, X, (U, U'))$ for any δ_k-indifferent $U' \in \mathcal{U}^k(X)$.*

P5 (Profile equivalence). *If $U(a) = U(\hat{a})$ then $f(b, A \cup \{a\}, X, U) = f(b, A \cup \{\hat{a}\}, X, U)$ for all $b \in A$.*

Neutrality and individual agreement were introduced earlier. Separability requires that if two separate groups U and U' each prefer the alternative a to the alternative b, then the combined group also prefers a to b. Individual agreement and separability together imply Pareto-optimality. Continuity at indifferent members requires strict preference orderings implied by the aggregator to be robust to the addition of nearly-indifferent members. This is the axiom that separates the class of aggregators we study from ordinal ones (such as the Borda count, or the model of Kalai et al. (2002)); one member's strict preference ordering is not reversed by adding an arbitrary (finite) number of members, so long as the added

members are sufficiently indifferent over the alternatives. This axiom only has meaning in a cardinal setting, and plays an analogous role in our setting as the Archimedean continuity axiom in expected utility theory. The axiom is weaker than requiring f (or the ordering of the alternatives implied by f) to be continuous in the utilities of members. Finally, profile equivalence says that aggregation is only affected by the utility levels of the alternatives in a given choice set. In particular, choice is not affected by which of two alternatives is adjoined to a set as long as those two alternatives yield exactly the same utility profile to all members. Note that it is not required that both of the elements added to the set A in P5 are from $X \setminus A$. Profile equivalence is related to the notion of *welfarism* in the literature, which requires, for example, that social rankings (Sen 1979) or solutions to bargaining problems (Roemer 1986) depend only on the utility possibilities set.

For ease of exposition, in this section we also restrict attention to aggregators where the aggregate utility of an alternative in a choice set A is independent of alternatives in $X \setminus A$. In Supplementary Appendix D we extend our results to a class of aggregators violating P6.

P6 (Independence of unavailable alternatives). *For any grand sets $X, X' \in \mathcal{X}$ such that $A \subseteq X \cap X'$, and for any group $U^X \in \mathcal{U}(X)$ and $U^{X'} \in \mathcal{U}(X')$ that agree on A (i.e., $U^{X'}(a) = U^X(a)$ for all $a \in A$), the aggregator satisfies $f(\cdot, A, X, U^X) = f(\cdot, A, X', U^{X'})$.*

The following are examples of menu-dependent aggregators satisfying P1-P5, that are equivalent or closely related to models proposed in the existing literature. Two additional aggregators satisfying these properties are studied in Supplementary Appendix B, where we show how to rationalize two simple choice procedures discussed in Kalai et al. (2002): the median procedure and the second-best procedure.[14]

Example 1 (Reference Dependence). *Suppose that the aggregator is given by*

$$f(a, A, X, U) = \sum_{u \in U} (u(a) - mean\ u(A))^\rho,$$

where ρ is an odd integer and mean $u(A)$ is a geometric or arithmetic mean over the set $\{u(a')\}_{a' \in A}$. This is a simple model of reference dependence.

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[14]In particular, Kalai et al. (2002) show that within their framework, the number of selves needed to rationalize these choice procedures becomes unbounded as the alternative space grows large. We show that they can be rationalized within our framework using only two selves, regardless of the size of the alternative space.

Example 2 (Intensity-weighted models). *Suppose there is a strictly monotonic and continuous weighting function $g : \mathbb{R} \to \mathbb{R}$ such that for all $U \in \mathcal{U}$ and choice sets $A \subseteq X$,*

$$f(a, A, X, U) = \sum_{u \in U} g\left(\max_{b \in A} u(b) - \min_{b \in A} u(b) \right) u(a)$$

If $g(\cdot)$ is increasing, members who feel more intensely about the alternatives in the set A receive greater weight in the decision-process, perhaps because they are more vociferous than members who are more or less indifferent among the possibilities. If $g(\cdot)$ is decreasing, the model may be seen as a context-dependent version of the models of relative utilitarianism in Karni (1998), Dhillon and Mertens (1999), and Segal (2000), where a DM's weight in society is normalized by her utility range over the grand set of alternatives. Observe that a is preferred to b in the pair $\{a, b\}$ if and only if

$$\sum_{u \in U} \underbrace{g(|u(a) - u(b)|)(u(a) - u(b))}_{\text{odd function of } u(a) - u(b)} > 0$$

Therefore, for pairwise choices the aggregator is similar to the additive difference model of Tversky (1969), which accounts for potentially intransitive pairwise choice behavior by positing utilities $v_1, v_2, \ldots, v_n$ and an odd $\phi : \mathbb{R} \to \mathbb{R}$ such that $x \succ y$ if and only if $\sum_{i=1}^{n} \phi(v_i(x_i) - v_i(y_i)) > 0$. For larger choice sets, the aggregator can be thought of as a generalization of the additive difference model that permits menu-dependence.

Example 3 (Contextual concavity models from marketing). *Kivetz et al. (2004) (henceforth KNS) considers various models capturing the compromise effect documented in experimental settings. KNS consider goods (e.g., laptops) which have defined attribute levels (e.g., processor speed) and posit utility levels ("partworths") for a given attribute. That is, they consider multiattribute alternatives and predefine the number of "selves" according to their selected good attributes. One type of model considered in KNS is referred to as a contextual concavity model. Using our notation, a symmetric version of the contextual concavity model they propose is given by*

$$f(a, A, X, U) = \sum_{u \in U} (u(a) - \min_{a' \in A} u(a'))^{\rho},$$

where ρ is a concavity parameter.

6 A general result

Recall the intensity-weighted aggregator, which is given by

$$f(a, A, X, U) = \sum_{u \in U} g\left(\max_{b \in A} u(b) - \min_{b \in A} u(b) \right) u(a)$$

where $g(\cdot)$ is strictly increasing. Let us examine how this aggregator behaves on an arbitrary three-element set of alternatives $\{a, b, c\}$. In particular, consider the group $U = (u_1, u_2, u_3, u_4, u_5)$ specified below.[15]

u_1		u_2		u_3		u_4		u_5	
b	2	b	2	c	2	a, c	2	a	2
c	1	a	1	b	1	b	0	b, c	0
a	0	c	0	a	0				

It is easy to verify that the aggregator selects a from the choice set $\{a, b\}$. Suppressing notational dependence on $X = \{a, b, c\}$, observe that $f(a, \{a, b\}, U) = 4g(2) + g(1)$ and $f(b, \{a, b\}, U) = 2g(2) + 3g(1)$, hence $f(a, \{a, b\}, U) > f(b, \{a, b\}, U)$ if and only if $g(2) > g(1)$, which holds since $g(\cdot)$ is strictly increasing. By contrast, the aggregator assigns equal utility to all alternatives in any other menu:

$$f(a, \{a, c\}, U) = f(c, \{a, c\}, U) = 2g(0) + g(1) + 2g(2)$$
$$f(b, \{b, c\}, U) = f(c, \{b, c\}, U) = 3g(1) + 2g(2)$$
$$f(a, \{a, b, c\}, U) = f(b, \{a, b, c\}, U) = f(c, \{a, b, c\}, U) = 5g(2)$$

That is, the group would select a out of the choice set $\{a, b\}$ and otherwise have no strict preference.

We will call such a group defined on $\{a, b, c\}$ a *triple-basis* for this aggregator. Note that in the case of this aggregator, the group above would still be a triple-basis if we were to scale all the utilities by a common constant. Loosely speaking, this means that at any level of δ-indifference, the model can rationalize being indifferent among the alternatives in $\{a, b, c\}$ except when choosing amongst one pair. This is a property we term *triple solvability*, and is formally defined for a general aggregator below.

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[15] In the i-th column, the alternative on the left is assigned the utility number to its right.

Definition 4. *A group* $\hat{U} \in \mathcal{U}(\{a,b,c\})$ *is a triple-basis for* f *with respect to* $\{a,b,c\}$ *if* $f(a,\{a,b\},\{a,b,c\},\hat{U}) > f(b,\{a,b\},\{a,b,c\},\hat{U})$, *and* $f(\cdot,A,\{a,b,c\},\hat{U})$ *is constant for all other* $A \subseteq \{a,b,c\}$. *Aggregator* f *is triple-solvable with* k *members if there exists a triple* $\{a,b,c\}$ *and* $k \in Z_+$ *such that for every* $\delta > 0$ *there is* $U \in \mathcal{U}^k(\{a,b,c\})$ δ*-indifferent with respect to* $\{a,b,c\}$ *constituting a triple-basis for* f *with respect to* $\{a,b,c\}$.

Given an aggregator, it is easy to check for the existence of a triple-basis group which would behave this way on a three-alternative set. For scale-invariant aggregators, which satisfy the property that measuring utilities in a different unit does not change the ordering implied by the aggregator, checking the property is particularly simple, since it then suffices to construct one triple-basis which can be scaled as needed.

It turns out that triple solvability holds broadly among the class of aggregators satisfying P1-P5. In particular, it holds for all the aggregators featured in Section 5, and the class of aggregators $\mathcal{F}^*$ generically satisfies this property.[16] The fact that these examples illustrate various models of multi-self decision-making proposed in the literature suggests that this property, which can be checked simply by looking at choice behavior on three-element sets, holds broadly. As our next result shows, this behavioral property has strong implications for the explanatory power of a model.

Theorem 2. *Suppose* f *satisfies P1-P6 and is triple-solvable with* k_f *members. Then, for any choice function* c, *function* c, *defined on any finite grand set of alternatives* X, *no more than* $1 + k_f \cdot IIA(c)$ *group members are needed to rationalize* c. *Alternatively, using* n *members,* f *can rationalize any choice function* c, *defined on any finite grand set of alternatives* X, *that exhibits at most* $\frac{n-1}{k_f}$ *IIA violations.*

We sketch the proof of Theorem 2 below. The result provides a lower bound on the set of rationalizable behaviors for a fixed group size, providing a linear connection between the complexity of the observed behavior (as measured by the number of IIA violations) and the degree of freedom in the model (as measured by the number of members). Again, the result

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[16]Solvability of the reference-dependent aggregator in Example 1 will be shown by the results in the appendix. For the contextual concavity aggregator in Example 3, the following constitutes a triple basis for any $\rho \neq 1$:

u_1		u_2		u_3		u_4		u_5		u_6	
a	4	a	3	b	4	b,c	3	c	3	c	4
b	3	c	2	a	3	a	1	a	2	b	2
c	1	b	1	c	1			b	1	a	1

15

points out the importance of putting *a priori* restrictions on the model in order to generate testable implications.

Note that for each aggregator f, the proportionality constant k_f is independent of the size of the alternative space X, and is calculable using any triple of alternatives (it is the number of group members in a triple basis). This means that the group size required to rationalize a choice function defined on the alternative space X does not increase if the choice function is extended to a larger alternative space $\hat{X}$ in a manner such that no additional IIA violations are created. This formalizes the sense in which the size of the rationalization depends directly on the complexity of the behavior and not the size of the alternative space; the size of the alternative space matters only in the sense that it bounds the number of IIA violations that are possible.

For additive and scale invariant aggregators, we prove that the proportionality constant is uniformly bounded by five. For intuition, notice that checking whether a group constitutes a triple basis for an aggregator requires checking five aggregate utility differences: the aggregate utility difference between any two pairs of alternatives within the set $\{a, b, c\}$, and the aggregate utility difference between the alternatives within each of the three pairs $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$. We prove an intermediate result showing that generically, an aggregator in the class $\mathcal{F}^*$ "stretches" utility differences in a nonlinear, menu-dependent fashion, and that a group of size five thus provides enough degree of freedom to ensure that a triple basis can be constructed.

6.1   Sketch of proof

For simplicity, consider the intensity-weighted aggregator which, as we have shown above, is triple-solvable with five members. Given an arbitrary X and any choice function c defined on X, we can use the triple-basis above to construct a group that rationalize c using the intensity-weighted aggregator f. The procedure works as follows. We examine all possible choice sets in X from smallest to largest, first going through all choice sets of size two, then all choice sets of size three, etc. We ignore any choice set that does not cause an IIA violation. For each choice set A that does cause an IIA violation, the construction creates a group U^A defined on X such that

1. $c(A)$ is selected under $f \circ U^A$ from every subset of A in which it is contained.

2. The group members in U^A cancel each other out under f on every other choice set (that is, on sets not containing $c(A)$ or sets containing some element of $X \setminus A$).

3. The group members in U^A are "indifferent enough" so that their trickle-down effect does not overturn the strict preference of previously constructed group members.

Finally, the construction creates an extra group member u^*, that is indifferent enough never to overturn any of the other members' strict preferences, in the standard way: it allocates the highest utility to $c(X)$, the next highest utility to $X \setminus \{c(X)\}$, and so on. All in all, this procedure constructs a group containing $1 + 5 \cdot \text{IIA}(c)$ members (the one above, and five for each IIA violation).

Using the triple-basis above, it is easy to construct the group U^A associated with a set A that causes an IIA violation. To satisfy the first two properties above, we simply let $c(A)$ play the role of a in the triple-basis, all the elements of $A \setminus \{c(A)\}$ play the role of b, and all the elements of $X \setminus A$ play the role of c. That is, we extend the utilities from $\{a, b, c\}$ to the given X such that: each extended utility function allocates the same utility to $c(A)$ as to a in the triple-basis, the same utility to elements of $A \setminus c(A)$ as to b in the triple-basis, and the same utility to $X \setminus A$ as to c in the triple-basis. Neutrality (P1) and profile equivalence (P5) then imply that the properties of the triple-basis carry over: for each $B \subseteq A$ that contains $c(A)$, $f(c(A), B, X, U^A) > f(y, B, X, U^A)$ for all $y \in B$, and for all other subsets $B' \subseteq X$, $f(x, B'^A) = f(y, B'^A)$ for all $x, y \in B'$. To satisfy the third property above, we can use continuity (P4) and scale all the utilities in the triple-basis by some appropriately chosen $\varepsilon > 0$.

This constructed group rationalizes $c(\cdot)$ under f. The construction ensures that $c(A)$ is selected from any set causing an IIA violation; one need only check that constructed group members do not interfere with choices associated with sets that do not cause IIA violations. To loosely illustrate the idea, consider any nested sequence of choice sets that decreases by one alternative. Given X, or any set which does not cause an IIA violation, all members besides u^* are indifferent, hence by individual agreement (P2) and separability (P3) the preferences of u^* prevail. For the first set of the sequence that contradicts the choice from X, a triple-basis was created with group members who care enough to overrule u^* and guarantee that the c-choice from this set is the f-maximizer (while the other triple-bases created will be indifferent). Similarly, whenever along the sequence there is a set that contradicts the choice of the previous set, another triple-basis was created that overrules the preferences of all members created in association with larger sets.

The above construction implies that if we consider a group with n members, any choice function (on any grand set of alternatives) having fewer than $\frac{n-1}{5}$ IIA violations can be rationalized using this model of aggregation.

7 Extensions

7.1 Weakening solvability: one member per mistake

While triple solvability is a property that is broadly satisfied, it can be seen from our construction that our theorem would still hold under a weaker condition. It suffices that there exist a group which is arbitrarily close to being indifferent on all but one subset $\{a, b\}$ of a triple $\{a, b, c\}$. We formalize this idea in Supplementary Appendix C, where we extend the notion of a triple-basis to an *approximate triple-basis*. For some aggregators, approximate triple-solvability can yield a triple-basis with a drastically smaller group size. Indeed, consider an aggregator of the form

$$f(a, A, X, U) = \sum_{u \in U} h(\max_{a' \in A} u(a'))u(a),$$

where $\lim_{x \to \infty} h(x)x = 0$. Under such an aggregator, the presence of an alternative with very high utility for a group member means that member is given less say in the decision process (a "populist"-type model). This can be used to create a *single-member* approximate triple-basis u: let $u(a)$ and $u(b)$ such that $f(a, \{a, b\}, \{a, b, c\}, u) - f(b, \{a, b\}, \{a, b, c\}, u) = \delta$ (for small enough δ this is always possible), and let $u(c)$ be high enough so that u is ε-indifferent between any two elements given sets containing c. Theorem 6 in Supplementary Appendix C then implies that only one group member is needed to rationalize each "mistake" the group makes (or alternatively, using n members, the aggregator can rationalize all choice functions with no more than $n - 1$ IIA violations).

7.2 Incomplete choice functions

Until now, we have examined choice functions. However, we may be interested in rationalizing demand data, such as in the literature on household choice studied in Browning and Chiappori (1998) and Chiappori and Ekeland (2006), among others. Given that demand data is typically finite, suppose we denote by X the (finite) set of all available allocations,

let each budget set correspond to a subset $A \subset X$, and identify the demand data with a function c that selects the allocation $c(A)$ in the budget set A. Then, rationalizing the demand data corresponds to rationalizing an *incomplete choice function*: c renders a choice to any subset A of X for some collection of subsets $\mathcal{A} \subset 2^X$, but data on choices from sets in $2^X \setminus \mathcal{A}$ is missing. As we show below, our results can easily be extended to arbitrary incomplete choice functions.

Rationalizing an incomplete choice function c with aggregator f implies finding a group $U \in \mathcal{U}(X)$ such that $f(c(A), A, X, U) > f(a, A, X, U)$ for all $a \in A \setminus \{c(A)\}$ and $A \in \mathcal{A}$ (it does not matter what choices f and U imply from sets in $2^X \setminus \mathcal{A}$). To see how our theorems generalize, observe that the only element of the construction that needs to be modified is the number of IIA violations: in this more general context we say that an IIA violation is associated with choice set $A \in \mathcal{A}$ if there is a nested sequence of choice sets $A_1, A_2, ..., A_k$ such that $A_1 = X$, $|A_j| - |A_{j+1}| = 1 \ \forall \ j \in \{1, ..., k-1\}$, and $A_k = A$ for which the choice from A_k contradicts the choice from A_l for some $l < k$, and $A_{l'} \notin \mathcal{A}$ for any $l < l' < k$. It is easy to see that this definition reduces to the original one in case of no missing data. Once the definition of IIA(c) is modified accordingly, it can be shown that Theorem 2 holds (the proof is analogous).[17]

This means that for any aggregator satisfying our conditions, the demand data can be rationalized if there are sufficiently many people in the household. This complements the result obtained in Browning and Chiappori (1998) and Chiappori and Ekeland (2006), where the researcher may not know how the preferences of the household are aggregated, and so the researcher seeks both a collection of utility functions and a model of aggregation to rationalize the data. Our results show that even if the researcher is constrained and knows precisely how preferences in the household are aggregated, if the number of individuals in the (extended) household is large or unknown, then the model still does not imply any testable restrictions on household demand. Our combinatorial approach also permits a simple lower bound on demand data that a household with a known number of individuals can generate, in terms of the number of IIA violations implied by the demand data.

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[17]We note that IIA(c) for an incomplete choice function might be strictly less than IIA(c) for any completion $\widehat{c}$ of c. That is, it can be that any way of specifying choices for sets in $2^X \setminus \mathcal{A}$ creates new IIA violations. Nevertheless, our theorems apply.

7.3 Type-dependent aggregators

The examples of aggregators above all treat group members in the same way. However, many models in the existing literature on multi-self decision-making propose methods of aggregation that treat some selves differently than others. For example, Fudenberg and Levine (2006) propose a dual-self impulse control model with a long-run self exerting costly self-control over a short-run self. One way to generalize this aggregator to any number of selves would be to introduce multiple types of short-term temptations, represented by selves $u_1^{\text{sr}}, u_2^{\text{sr}}..., u_n^{\text{sr}}$, as well as one long-run self u^{lr}. Accommodating such type-dependent models of aggregation in our framework requires an extension of the framework and some extra notation, but no conceptual innovation. In particular, the definition of an aggregator must be extended to include a set of possible types, and the definition of a self must be extended to include a type. For ease of exposition, we restricted ourselves to the simplified framework in the main text and present the extension of the framework in Supplementary Appendix A. Our axioms and main theorem carry through to the extended framework.

7.4 Systematic IIA violations

Our construction allocates a different triple-basis (or approximate triple-basis) for every IIA violation. However, there can be IIA violations that are "in the same direction" (that do not contradict each other). In this case, parts of the associated triple-bases in our construction can be combined (or *collapsed*) together to yield tighter bounds.

For example, recall the triple-basis for the intensity-weighted aggregator, and fix some alternative a. Every time the choice of a from some set causes an IIA violation, the triple-basis constructed has a member u_5 in which a is preferred to $X \setminus \{a\}$, all elements of which are indifferent to each other. Under the intensity-weighted aggregator, all of the u_5-members constructed when the choice was a can be collapsed into a single member. More generally, the following is an immediate corollary to Theorem 2.

Corollary 3. *Suppose f satisfies P1-P6 and is triple-solvable with k_f selves. For an arbitrary choice function c, defined on any finite grand set of alternatives X, let*

$$\mathrm{D}(c) = \#\{a \in X \mid c(A) = a \text{ for some } A \subseteq X \text{ causing an IIA violation}\}$$

be the number of distinct elements whose choice is associated with an IIA violation. Then,

there are ℓ, m with $\ell + m \leq k_f$, such that the number of selves needed to rationalize the choice function c is at most $1 + \ell \cdot \mathrm{D}(c) + m \cdot \mathrm{IIA}(c)$.

This effect is particularly pronounced when the triple-basis has only one member, as in the approximately triple-solvable aggregators introduced above. To illustrate this, consider the following example: let $x^* \in X$, and let $\succ_1$ and $\succ_2$ be strict orderings on X such that $x \succ_1 x^*$ and $x \succ_2 x^*$ for every $x \in X \setminus \{x^*\}$, and $y \succ_1 x$ for $x, y \in X \setminus \{x^*\}$ if and only if $x \succ_2 y$. Consider a decision-maker who from choice sets not containing x^* selects the best element according to $\succ_1$, but from choice sets containing x^* selects the best element according to $\succ_2$. This behavior describes, for example, a customer in a restaurant who chooses the tastiest item from a menu if the menu does not contain onion rings, while choosing the healthiest item in the presence of onion rings, because they are so greasy as to make the customer feel guilty about his eating habits. The above simple behavior generates a large number of IIA violations if X is large.[18] However, these IIA violations do not contradict each other: if choice from set B contradicts the choice from $A \supset B$, then there is no $B' \subset B$ such that the choice from B' contradicts the choice from B. As we show below, this can be used to merge all collections of selves into a single collection, drastically reducing the number of selves required to rationalize the customer's choice function.

Consider the aggregator introduced in the previous subsection, which was shown to be approximately triple-solvable with a single group member. Our construction calls for (i) creating one member whose utility function is in line with $\succ_2$; and (ii) creating another member for all sets associated with an IIA violation, such that the utility of x^* is sufficiently high that the member becomes close enough to indifferent in the presence of x^*, and among the other alternatives allocates the highest utility to the choice from the given set. The latter utility functions can all be collapsed into a single member, such that the utility function of that member is in line with $\succ_1$ over $X \setminus \{x^*\}$ (while keeping the utility of x^* at a level that makes the member nearly indifferent in the presence of x^*). Our construction then implies that the above choice function can be rationalized with two group members. This is clearly a tight bound.

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[18]The number of IIA violations is $2^{n-1} - n - 1$: the choice from every set B having at least two elements and not containing x^* contradicts the choice from $B \cup \{x^*\}$.

8   Discussion

The framework we propose in this paper provides a flexible environment for axiomatic investigation of multi-self and household decision-making models. Many of the models proposed in the existing multi-self literature can be translated into our framework such that the resulting aggregators satisfy the basic axioms we posited. However, there are other classes of aggregators that might be of interest, such as ordinal ones, which do not satisfy all our axioms. Our framework can still be useful to examine these aggregators; some of our axioms would need to be replaced by axioms that reflect the characteristics of the aggregators at hand. It may also be feasible to incorporate choice correspondences into our results on rationalizability by extending our definition of IIA violations for choice functions to count both violations of Sen's α and Sen's β (axioms that, when taken together, are equivalent to rational choice behavior for correspondences). Furthermore, our set of axioms can also be supplemented with additional ones, leading to more specific classes of aggregators instead of the broad class of aggregation rules investigated in this paper, and hence to sharper predictions on implied choice with a fixed group size. We leave these directions, as well as extending our framework to dynamic settings, to future research.

Appendix

Proof of Theorem 2

For an arbitrary choice function c we will construct a collection of $1 + k \cdot \mathrm{IIA}(c)$ members which will be shown to rationalize c. This implies the claim in the theorem. In particular, we will construct k members for each set with which an IIA violation is associated, and an extra member for X.

Let $I_1 = \{A_1^1, ..., A_{i_1}^1\}$ be the subsets of X such that there is an IIA violation associated with the set, but there is no proper subset of the set with which an IIA violation is associated. For $j \geq 2$, let $I_j = \{A_1^1, ..., A_{i_{j+1}}^1\}$ be the subsets of X such that there is an IIA violation associated with the set, but there is no proper subset of the set outside $\bigcup_{l=1}^{j-1} I_l$ with which an IIA violation is associated. Let j^* be the largest j such that $I_j \neq \emptyset$.

We will now iteratively construct a group of k members for each set associated with an IIA violation, starting with sets in I_1. Consider any group of k members $\bar{U}^1 = (\bar{u}_1^1, \ldots \bar{u}_k^1)$

that solves the triple $\{a, b, c\}$ (the existence of such a triple follows from triple-solvability). For every $A \subset I_1$, construct now the following group $U^A = (u_1^A, \ldots u_k^A)$:

$$u_i^A(x) = \begin{cases} \bar{u}_i^1(a) & \text{if } x = c(A) \\ \bar{u}_i^1(b) & \text{if } x \in A, \ x \neq c(A) \\ \bar{u}_i^1(c) & \text{if } x \notin A \end{cases}$$

for every $i = 1, ..., k$.

Suppose now that U^A is defined for every $A \in \bigcup_{k=1}^{j} I_k$ for some $j \geq 1$. Let U_k be the group $U_k = (U^{A_1^k}, ..., U^{A_{i_k}^k})$, for $k = 1, ..., j$. Let $\widehat{U}_j = (U_1, ..., U_j)$. By P4, there exists $\delta > 0$ such that for any δ-indifferent group of k members U',

$$f(a, A, X, \widehat{U}_j) > f(b, A, X, \widehat{U}_j) \text{ implies } f(a, A, X, (\widehat{U}_j, U')) > f(b, A, X, (\widehat{U}_j, U')).$$

Then by P3 and P6, we know

$$f(a, A, X, \widehat{U}_j, \widetilde{U}_1, ..., \widetilde{U}_m) > f(b, A, X, \widehat{U}_j, \widetilde{U}_1, ..., \widetilde{U}_m) \text{ implies}$$
$$f(a, A, X, (\widehat{U}_j, \widetilde{U}_1, ..., \widetilde{U}_m, U')) > f(b, A, X, (\widehat{U}_j, \widetilde{U}_1, ..., \widetilde{U}_m, U'))$$

for any $\widetilde{U}_1, ..., \widetilde{U}_m$ group of (exactly) indifferent members.

Let now $I_{j+1} = \{A_1^1, ..., A_{i_{j+1}}^1\}$ be the subsets of X such that there is an IIA violation associated with the set, but there is no proper subset of the set outside I_j with which an IIA violation is associated. By triple-solvability with k members, there is a δ-indifferent group of k members $\bar{U}^{j+1} = (\bar{u}_1^{j+1}, \ldots \bar{u}_k^{j+1})$ that solves the triple $\{a, b, c\}$. For every $A \subset I_{j+1}$, construct now the following group $U^A = (u_1^A, \ldots u_k^A)$:

$$u_i^A(x) = \begin{cases} \bar{u}_i^{j+1}(a) & \text{if } x = c(A) \\ \bar{u}_i^{j+1}(b) & \text{if } x \in A, \ x \neq c(A) \\ \bar{u}_i^{j+1}(c) & \text{if } x \notin A \end{cases}$$

for every $i = 1, ..., k$. Let U_{j+1} be the group $(U_j, U^{A_1^1}, ..., U^{A_{i_{j+1}}^1})$.

The above procedure generates a group $k \cdot \text{IIA}(c)$ members in j^* steps. Then by P3 and P4 there is $\delta_{j^*} > 0$ such that for any δ_{j^*}-indifferent u, $f(a, A, X, U_{j^*}) > f(b, A, X, U_{j^*})$ implies $f(a, A, X, (U_{j^*}, u)) > f(b, A, X, (U_{j^*}, u))$. Finally, construct one more member the following way: let $a_1 = c(X)$ and $a_k = c(X \setminus \{a_1, a_2, \ldots a_{k-1}\})$ for $2 \leq k \leq n$. Construct

23

$u^* : X \to \mathbb{R}$ such that $u^*(a_1) > u^*(a_2) > \cdots > u^*(a_n)$ and u^* is δ_{j^*}-indifferent.

We show the group $U_c \equiv (U_{j^*}, u^*)$ rationalize c with aggregator f.

Observation 1. *For any set A with which there is an IIA violation associated, by the construction of U^A and by P1 and P5, $f(a, B, X, U^A) = f(b, B, X, U^A) \ \forall \ a, b \in B$ and B such that either $B \setminus A \neq \emptyset$ or $c(A) \notin B$, and $f(c(A), B, X, U^A) > f(b, B, X, U^A) = f(b', B, X, U^A)$ $\forall \ b, b' \in B \setminus \{c(A)\}$ and B such that $B \setminus A = \emptyset$ and $c(A) \in B$.*

We will now show that the choice induced by f from any choice set is equal to the choice implied by c. First, note that this holds for X, since by Observation 1, $f(a, X, X, U^A) = f(b, X, X, U^A)$ for every $a, b \in X$ and every A with which there is an IIA violation associated. Moreover, $f(c(X), X, X, u^*) > f(a, X, X, u^*) \ \forall \ a \in X \setminus \{c(X)\}$ by P2. Then repeated application of P3 implies $f(c(X), X, X, U_c) > f(a, X, X, U_c) \ \forall \ a \in X \setminus \{c(X)\}$.

Next, consider any $A \subsetneq X$ which causes an IIA violation. Suppose $A \in I_j$. Observation 1 implies that for any $B \in (\bigcup\limits_{l=1}^{j} I_l) \setminus A$, $f(a, A, U^B) = f(a', A, U^B) \ \forall \ a, a' \in A$, and $f(c(A), A, X, U^A) > f(a, A, X, U^A) \ \forall \ a \in A$. Then repeated implication of P3 implies $f(c(A), A, X, U_j) > f(a, A, X, U_j) \ \forall \ a \in A$. By construction then $f(c(A), A, X, U_c) > f(a, A, X, U_c) \ \forall \ a \in A$.

There are three cases to check for a set A that does not cause an IIA violation.

Case 1: For all $a \in A$, there is no $B \supset A$ such that $a = c(B)$. Then by construction $u^*(c(B)) > u^*(b) \ \forall \ b \in B \setminus \{c(B)\}$. Moreover, by Observation 1, $f(b, B, X, U^A) = f(b, B, X, U^A) \ \forall \ b, b' \in B$ and A with which an IIA violation is associated. Repeated use of P3, together with P2, implies $f(c(B), B, X, U_c) > f(b, B, X, U_c) \ \forall \ b \in B$.

Case 2: There is a unique $a \in A$ such that for some $B \supset A$, $c(B) = a$. First we note that $a = c(A)$ is necessary, otherwise A would have caused an IIA violation. There are two subcases:

Case 2a: For every B such that $B \supset A$ and $c(B) = a$, B did not cause an IIA violation. This means that for all $B \supset A$, $c(B) \notin A \setminus \{c(A)\}$. So just like in Case 1, $u^*(c(B)) > u^*(b)$ $\forall \ b \in B \setminus \{c(B)\}$, and $f(b, B, X, U^A) = f(b, B, X, U^A) \ \forall \ b, b' \in B$ and A with which an IIA violation is associated. Hence, $f(c(B), B, X, U_c) > f(b, B, X, U_c) \ \forall \ b \in B$.

Case 2b: There is $B \supset A$ with $c(B) = a$ such that B caused an IIA violation. Consider any smallest such B, and suppose $B \in I_j$. By Observation 1, for any $A \in \bigcup\limits_{l=1}^{j} I_l$ either $f(c(B), B, X, U^A) > f(b, B, X, U^A) \ \forall \ b \in B$, or $f(b, B, X, U^A) = f(b', B, X, U^A) \ \forall \ b, b' \in B$.

24

But then repeated application of P3 implies that $f(c(B), B, X, U_j) > f(b, B, X, U_j) \ \forall \ b \in B$. By construction, $f(c(B), B, X, U_c) > f(b, B, X, U_c) \ \forall \ b \in B$.

Case 3: There exist at least two elements in A that have each been chosen in some superset. First, note that one of those elements must be $a = c(A)$, otherwise A would have caused an IIA violation. Let $\{b_i\}_i$ be the set of elements other than a such that $b_i \in A$ and $b_i = c(B_i)$ for some $B_i \supset A$. Drop any b_i's such that $B_i \subset B_m$ for some m and call the remaining set $\{b_j\}$. Because A did not cause an IIA violation by assumption, it must be that for each b_j there is A'_j such that $A \subset A'_j \subset B_j$ and $c(A'_j) \in A$. Because B_j does not contain any B_k, we know $c(A'_j) = a$. For each j there may be multiple such A'_j's; consider only the maximal A'_j with respect to the minimal B_j. Now by maximality, for any A'' such that $A'_j \subset A'' \subset B_j$, $c(A'') \notin A$. If there is A'' such that $c(A'') \in A'_j$, since $c(A'') \neq a$, by definition A'_j caused an IIA violation with respect to the first such A''. If for every A'' it is the case that $c(A'') \notin A'_j$, then once again A'_j caused an IIA violation with respect to B. Either way, since $c(A'_j) = a$, we added members to ensure this choice for every j. This means that a should be the choice from A unless for some set B' between the smallest-sized A'_j and A we have $c(B') \in A \setminus \{a\}$ and members were added. But such a set cannot exist by minimality of the B_j's. $\blacksquare$

Proof of Theorem 1

Let $X = \{a, b, c\}$. For compactness, we use the notation

$$
\begin{aligned}
x_1 &= f(a, \{a, b, c\}, X, U) - f(b, \{a, b, c\}, X, U), \\
x_2 &= f(b, \{a, b, c\}, X, U) - f(c, \{a, b, c\}, X, U), \\
x_3 &= f(a, \{a, c\}, X, U) - f(c, \{a, c\}, X, U), \\
x_4 &= f(b, \{b, c\}, X, U) - f(c, \{b, c\}, X, U), \\
x_5 &= f(a, \{a, b\}, X, U) - f(b, \{a, b\}, X, U).
\end{aligned}
$$

Lemma 1. *If $x_3 \neq x_4 + x_5$, and if any one of the three equations $2x_1 + x_2 - x_3 - x_5 = 0$, $x_1 + 2x_2 - x_3 - x_4 = 0$, or $x_1 - x_2 + x_4 - x_5 = 0$ fails, then the aggregator is triple-solvable (with k_f at most $2 + 3|U|$).*

Proof. The first column in the table lists the aggregate values for the group U. But by neutrality, we know that if we can generate the values in column 1, we can also generate the values in the 2nd column using the permutation $(bc)(a)$ over the alternatives, generate the

25

values in the 3rd column using the permutation $(ab)(c)$ over the alternatives, and so on. By using profile equivalence to evaluate each of the values $f \circ u$ and $f \circ u'$ each generated by a single member u and u', with the rankings given in the 6th and 7th headers, respectively, we can also generate the values in those respective columns.

$1: U$	$2: (bc)(a)$	$3: (ab)(c)$	$4: (abc)$	$5: (acb)$	$6: a \sim b \succ c$	$7: a \succ b \sim c$
x_1	$x_1 + x_2$	$-x_1$	x_2	$-x_1 - x_2$	0	x_1
x_2	$-x_2$	$x_1 + x_2$	$-x_1 - x_2$	x_1	x_1	0
x_3	x_5	x_4	$-x_5$	$-x_4$	x_1	x_1
x_4	$-x_4$	x_3	$-x_3$	x_5	x_1	0
x_5	x_3	$-x_5$	x_4	$-x_3$	0	x_1

Then, determinants of three possible 5×5 matrices, each composed of five of the columns above, may be calculated to obtain:

$$\mathrm{Det}(1|3|5|6|7) = x_1^2(x_1 + 2x_2 - x_3 - x_4)(2x_1 + x_2 - x_3 - x_5)(x_3 - x_4 - x_5),$$
$$\mathrm{Det}(1|2|5|6|7) = x_1^2(2x_1 + x_2 - x_3 - x_5)(x_3 - x_4 - x_5)(x_1 - x_2 + x_4 - x_5),$$
$$\mathrm{Det}(2|3|4|6|7) = -x_1^2(x_1 + 2x_2 - x_3 - x_4)(x_3 - x_4 - x_5)(x_1 - x_2 + x_4 - x_5).$$

To prove the result, it suffices to show that there exists U such that defining $x_1, x_2, \ldots, x_5$ as above, one of the determinants above must be nonzero. If one of those determinants is nonzero, then we have find a vector $(c_1, c_2, c_3, c_4, c_5)$ such that the nonsingular matrix times $(c_1, c_2, c_3, c_4, c_5)$ is equal to $(0, 0, 0, 0, \beta)$ for some $\beta \neq 0$. Using scaling, each c_i can be pulled in so that the U corresponding to the i-th column is multiplied by c_i. The resulting group is a triple-basis (and therefore we can get triple solvability through scaling that triple-basis).

The proof is completed in light of the linear dependence of the equations $2x_1 + x_2 - x_3 - x_5 = 0$, $x_1 + 2x_2 - x_3 - x_4 = 0$, and $x_1 - x_2 + x_4 - x_5 = 0$: if any one of these fails, there must be a second which fails too. ∎

Say that $f \in \mathcal{F}^*$ is *non-degenerate* if for some utility function u on $X = \{a, b, c\}$, we have $x_3 \neq x_4 + x_5$ and $2x_1 + x2 \neq x_3 + x_5$ using $U = \{u\}$. We formally establish that for any fixed scaling function $\phi(\alpha)$ the property that an additive, neutral and scale-invariant aggregator $f \in \mathcal{F}^*$ is not degenerate holds generically. In order to define a topology on $\mathcal{F}^*$, we transform the latter set of aggregators to a convenient representation. Note that for a fixed scaling function, specifying the aggregated utilities of n alternatives for members in the

n-dimensional simplex determines the aggregated utilities of n alternatives for all possible members over n alternatives, since any member is a scalar multiple of exactly one member from the simplex. Hence, with respect to a grand set of alternatives with three elements, there is a natural bijection β between additive and scale-invariant aggregators, and the set of pairs of operators

$$\Omega = (O_1, O_2 | O_1 : \Delta_2 \to \mathbb{R}^2; O_2 : \Delta_3 \to \mathbb{R}^3),$$

where O_1 determines how a member's utilities get aggregated in pairs, and O_2 determines how a member's utilities get aggregated in the triple. Define metric d on Ω such that the distance between (O_1, O_2) and (O_1', O_2') is defined as $\max_{i=1,2} \sup_{x \in \mathbb{R}^i} |O_i(x) - O_i'(x)|$.

Lemma 2. *Given the topology induced by d, the pairs of operators in Ω that are associated with non-degenerate aggregators in $\mathcal{F}^*$ is open and dense relative to Ω.*

Proof. For ease of exposition, let

$$\Gamma_1^l(f, v) = f(a, \{a, c\}, v) - f(c, \{a, c\}, v),$$

$$\Gamma_2^l(f, v) = f(a, \{a, b\}, v) - f(b, \{a, b\}, v) + f(b, \{b, c\}, v) - f(c, \{b, c\}, v),$$

$$\Gamma_2^l(f, v) = f(a, \{a, b, c\}, v) - f(b, \{a, b, c\}, v)] + [f(a, \{a, b, c\}, v) - f(c, \{a, b, c\}, v),$$

$$\Gamma_2^l(f, v) = [f(a, \{a, b\}, v) - f(b, \{a, b\}, v)] + [f(a, \{a, c\}, v) - f(c, \{a, c\}, v)],$$

for every $v \in \mathcal{F}^*$. Note that $\Gamma_i^j(v)$ stands for side j of the equation in condition i in the definition of a degenerate aggregator, given aggregator f and a member v.

1. Openness. Suppose that for aggregator f there is a member u over a triple such that neither of the equalities in the definition of a degenerate aggregator hold with equality. Note that u cannot be an indifferent member. Let $\varepsilon_i = \Gamma_i^l(f, v) - \Gamma_i^r(f, v)$ for $i \in \{1, 2\}$, and let $\varepsilon = \max(|\varepsilon_1|, |\varepsilon_2|)$. Next, for every $i, j \in \{a, b, c\}$ such that $i \neq j$, let α^{ij} be such that $\alpha^{ij}(u(i), u(j)) \in \Delta^2$. Note that the terms α^{ij} are uniquely defined. Similarly, let α^{abc} be such that $\alpha^{abc}(u(a), u(b), u(c)) \in \Delta^3$. Let $\alpha = \max(|\alpha^{ab}|, |\alpha^{ac}|, |\alpha^{bc}|, |\alpha^{abc}|)$. Since u is not an indifferent member, $\alpha > 0$. Then for $\delta < \frac{\varepsilon}{8\alpha}$ it holds that $\Gamma_i^l(f', v) \neq \Gamma_i^r(f', v)$ for $i \in \{1, 2\}$ for every f' such that $|\beta(f) - \beta(f')| < \delta$, since each term given f' in the above inequalities can differ from the corresponding term given f by at most $\frac{\varepsilon}{8}$.

2. Denseness. Let $\delta > 0$. Consider a member $u \in \Delta_3$ over $\{a, b, c\}$ such that $u(a) > u(b) >$

$u(c)$. For every $i, j \in \{a, b, c\}$ such that $i \neq j$, let α^{ij} be such that $\alpha^{ij}(u(i), u(j)) \in \Delta^2$. Let $\alpha = \max(|\alpha^{ab}|, |\alpha^{ac}|, |\alpha^{bc}|)$. If for an aggregator f neither of the equalities in the definition of a degenerate aggregator hold, then the aggregator is by definition non-degenerate, hence there is trivially a point in the δ-neighborhood of $\beta(f)$ that corresponds to a non-degenerate aggregator. Otherwise let $\varepsilon \in (0, \frac{\delta}{\alpha})$ be such that $\varepsilon \neq |\Gamma_i^l(f, v) - \Gamma_i^r(f, v)|$ for $i \in \{1, 2\}$.

Consider now any $f' \in \mathcal{F}^*$ for which (i) for triples, f' is equivalent to f; and (ii) for a pair $\{x, y\}$, given any utility function v over $\{x, y\}$ for which $v(x) \geq v(y)$, $f'(x, \{x, y\}, v) = f(x, \{x, y\}, v)$ and $f'(y, \{x, y\}, v) = f(y, \{x, y\}, v)$ if $v(x) - v(y) < u(a) - u(c)$, but $f'(x, \{x, y\}, v) = f(x, \{x, y\}, v) + \varepsilon$ and $f'(y, \{x, y\}, v) = f(y, \{x, y\}, v)$ if $v(x) - v(y) \geq u(a) - u(c)$. In words, with respect to members for which the utility difference between the elements of the pair is at least $u(a) - u(c)$ the aggregated utility is $\varepsilon > 0$ higher than what f yields for the preferred alternative (while it is the same for the other alternative) - otherwise f' is equivalent to f. By construction, $|\beta(f') - \beta(f)| < \delta$. Also note that $\Gamma_1^l(f', v) = \Gamma_1^l(f, v) + \varepsilon$, $\Gamma_1^r(f', v) = \Gamma_1^r(f, v)$, $\Gamma_2^l(f', v) = \Gamma_2^l(f, v)$, and $\Gamma_2^r(f', v) = \Gamma_2^r(f, v) + \varepsilon$. Then $\varepsilon \neq |\Gamma_i^l(f, v) - \Gamma_i^r(f, v)|$ for $i \in \{1, 2\}$ implies that $\Gamma_i^l(f', v) \neq \Gamma_i^r(f', v)$ for $i \in \{1, 2\}$. Hence, f' is non-degenerate. ∎

Theorem 1 follows as a corollary of Theorem 2, Lemma 1 and Lemma 2. ∎

Remark 1. *Fix an aggregator f and suppose there is $U \in \mathcal{U}(\{a, b, c\})$ such that $f \circ U$ can rationalize third-place choice, and that there is $U' \in \mathcal{U}(\{a, b, c\})$ such that $f \circ U'$ can rationalize intransitive behavior. We claim the aggregator is nondegenerate. To prove this, let us assume for the moment that there exists $U \in \mathcal{U}(\{a, b, c\})$ such that $x_3 \neq x_4 + x_5$ and $f \circ U$ rationalizes choice where the worst element in the transitive pairwise ranking is best in the triple. Then we claim either $2x_1 + x_2 \neq x_3 + x_5$ or $x_1 + 2x_2 \neq x_3 + x_4$. By neutrality and symmetry of the condition $x_3 - x_4 - x_5 \neq 0$, note there are two choice behaviors to consider: Case 1: $a \succ_P b \succ_P c$ on pairs, and $c \succ_T b \succeq_T a$ on the triple. Thus $x_3, x_4, x_5 > 0$, with $x_1 \leq 0$ and $x_2 < 0$. Then $2x_1 + x_2 \neq x_3 + x_5$, as LHS $< 0 <$ RHS. Case 2: $a \succ_P b \succ_P c$ on the pairs, and $c \succ_T a \succeq_T b$ on the triple. That is, $x_3, x_4, x_5 > 0$, with $x_1 \geq 0$, $x_2 < 0$. If we can find U such that $f \circ U$ rationalizes this behavior, then observe that $x_1 + 2x_2$ is negative. Hence $x_1 + 2x_2 \neq x_3 + x_4$ because the RHS is positive.*

Now, if $f \circ U$ satisfies $x_3 \neq x_4 + x_5$ then we have shown the claim. Otherwise, consider the group $(U, \varepsilon \cdot U')$, where $\varepsilon > 0$ is a positive scalar, and small enough that by P4 we still rationalize third-place choice with $f \circ (U, \varepsilon \cdot U')$. At the same time, note that $f \circ U'$ must satisfy $x_3 \neq x_4 + x_5$ (for this group) since it rationalizes intransitive behavior. Hence, $f \circ (U, \varepsilon \cdot U')$ also satisfies $x_3 \neq x_4 + x_5$.

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Supplementary Appendices, Not for Publication

This document contains supplementary appendices to "Rationalizing Choice with Multi-Self Models" by Ambrus and Rozen. The main paper is referenced throughout as AR.

A   Non-anonymous aggregators

We extend our framework to incorporate aggregators that treat different group members in a non-anonymous manner, and show how our main result extends to this more general class of aggregators. The description of a member is extended by an abstract type, and the definition of an aggregator is extended to include a set of possible types. The abstract set of types could include, for example, "long-run" and "short-run" selves, or selves caring about different types of objectives, such as the "parental" and "work" selves mentioned in Section 1.

An aggregator $F = (T, f)$ specifies a set of possible types T and a function f that specifies the aggregate utility for every alternative a in every choice set A, given any (finite) grand set of alternatives X and any collection of selves S defined over X and T. A single member s is given by a pair (u, t). For each positive integer n, we denote by $\mathcal{S}^n(X, T)$ the set of all collections of members (unordered lists) defined with respect to X and T, and let $\mathcal{S}(X, T) = \cup_{n=1}^{\infty} \mathcal{S}^n(X, T)$. We will denote a particular collection of members by S, and refer to the members in the group as $s_1, ..., s_n$. To denote the number of members in S, we use the notation $|S|$ or simply n when no confusion would arise.

This extension allows us to consider asymmetric aggregators.

Example 4 (Asymmetric contextual concavity model). *Interpret each member as corresponding to a product attribute, for which the preference belongs to a certain type. The class of preferences is parametrized by a concavity index. The contextual concavity aggregator in Kivetz et al. (2004) is given by*

$$f(a, A, X, S) = \sum_{s \in S}(u(a) - \min_{a' \in A}\ u(a'))^{\rho(t)},$$

where $\rho : T \to \mathbb{R}$ gives the concavity parameter for a type-t member.

Since collections of selves are still defined as unordered lists, by construction aggregators in this framework treat selves of the same type symmetrically. Hence, asymmetries can enter only through different specified types. In particular, the framework constructed in the main text can be viewed as a special case of the extended framework proposed above, when the set of possible types is a singleton. Axioms P1-P6 can be generalized in a straightforward manner to the extended setting. Since the only changes required in the generalization are notational (all statements applying previously to selves now apply to the extended notion of a member), we omit restating the axioms in the extended framework. The main theorem is unchanged. The definition of a triple-basis is unchanged, as is the theorem:

Theorem 4. *Suppose f satisfies P1-P6 and is triple-solvable with k_f selves. Then, using n selves, f can rationalize any choice function c, defined on any finite grand set of alternatives X, that exhibits at most $\frac{n-1}{k_f}$ IIA violations.*

Consider a different type of example.

Example 5 (Costly self-control aggregators). *Fudenberg and Levine (2006) propose a dual-self impulse control model with a long-run self exerting costly self-control over a short-run self. The reduced-form model they derive has an analogous representation in our framework, with two selves: the long-run self, with utility given by u^{lr} (the expected present value of the utility stream induced by the choice in the present), and the short-run self, with utility function u^{sr} (the present period consumption utility).[19] Using our terminology, the reduced form representation of their model assigns to alternative a the aggregate utility $u^{\mathrm{lr}}(a) - C(a)$, where term $C(a)$ depends on the attainable utility levels for the short-run self and is labeled as the cost of self-control. For example, using Fudenberg and Levine (2006)'s parametrization, $C(a) = \gamma[\max_{a' \in A} u^{\mathrm{sr}}(a') - u^{\mathrm{sr}}(a)]^\psi$.*

One way to generalize this aggregator to any number of selves would be to introduce multiple short-term temptations, represented by selves $u_1^{\mathrm{sr}}, ..., u_n^{\mathrm{sr}}$, and to define the aggregator

$$f(a, A, X, S) = u^{\mathrm{lr}}(a) - \sum_{s \in S} \gamma[\max_{a' \in A} u^{\mathrm{sr}}(a') - u^{\mathrm{sr}}(a)]^\psi.$$

Here, the long-run self is treated differently than the rest.

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[19]The long-run self's utility is equal to the short-run self's utility plus the expected continuation value induced by the choice. If the latter can take any value, then u^{lr} is not restricted by the short-run utility u^{sr}. If continuation values cannot be arbitrary (for example they have to be nonnegative) then u^{sr} restricts the possible values of u^{lr}, hence U has a restricted domain. In Fudenberg and Levine (2006) the utility functions also depend on a state variable y. Here we suppress this variable, instead make the choice set explicit.

As in the above generalization of Fudenberg and Levine (2006), it may be the case that a multi-self model places restrictions on how many selves of each type can appear. If types are restricted, the description of the model should also include a set of possible collections of types $\mathcal{C}$, given by a subset of the set of all possible unordered n-long lists of elements of T, for every $n \in Z_+$. The aggregator f need only specify the aggregate utility arising for any collection of selves S defined over X and T for which the implied collection of types is in $\mathcal{C}$.

Our results can be extended in a variety of ways to accommodate such restrictions. The most straightforward one imposes an assumption on the set $\mathcal{C}$ (which is satisfied in Example 5). Assume the existence of a type t and a collection of types $\widehat{T}$ such that appending any number of t-types to $\widehat{T}$ results in a collection of types in $\mathcal{C}$. In the generalized costly self-control aggregator above, the short-run type being t and the singleton set of a long-run type as $\widehat{T}$ satisfy this requirement. Let T_t^n denote the collection of n t-types. An aggregator f is *expandable* with $t \in T$ from $\widehat{T} \in \mathcal{C}$ if $(\widehat{T}, T_t^n) \in \mathcal{C}$ for every $n \in Z_+$. For an aggregator that is expandable with t from $\widehat{T}$ we can define triple-solvability with k type-t selves from $\widehat{T}$ as the existence of a collection of selves consisting of $|\widehat{T}|$ exactly indifferent selves over the triple whose type-composition is as in $\widehat{T}$ and k δ-indifferent selves of type t, such that the above collection of types constitutes a triple-basis for every $\delta > 0$.

Given the above definitions, the following result is obtained.

Theorem 5. *Suppose f is triple-solvable with k type-t selves from $\widehat{T}$. Then, using n selves, f can rationalize any choice function c, defined on any finite grand set of alternatives X, that exhibits at most $\frac{n-1-|\widehat{T}|}{k}$ IIA violations.*

Because the aggregation term for a short-run self is the negative of the symmetric contextual concavity aggregation, it is immediate that the generalized costly self-control aggregator defined above is triple-solvable according to the extended definition.

B   Examples rationalizing common choice procedures

Example 6 (The Median Procedure). *The median procedure is a simple choice rule defined in Kalai et al. (2002). There is a strict ordering $\succ$ defined over elements of X, and the DM always chooses the median element of each $A \subseteq X$ according to $\succ$ (choosing the right-hand side element among the medians from choice sets with even number of alternatives).*

To rationalize this behavior, we consider the following aggregator.

$$f(a, A, X, U) = \prod_{u \in U} (u(a) + \max_{a' \in X} u(a') - \underset{a' \in A}{med}\, u(a')),$$

where $\underset{a' \in A}{med}\, u(a')$ is the median element of the set $\{u(a')\}_{a' \in A}$, with the convention that in sets with an even number of distinct utility levels, the median is the smaller of the two median utility levels. The geometric aggregation implies that in case of selves having exactly the opposite preferences, the aggregated utility of an alternative from a given choice set is maximized when it is closest to the median element of the utility levels from the choice set.

Indeed, we claim that with the above aggregator, two selves can be used to rationalize the median procedure. Let $a_1, a_2, ..., a_N$ stand for the increasing ordering of alternatives in X according to $\succ$, and define $u_1(a_i) = i + \varepsilon$ and $u_2(a_i) = N + 1 - i$ for all $i \in \{1, ..., N\}$. It is easy to see that for small enough $\varepsilon > 0$ it is indeed one of the median elements of any choice set that maximizes f, since the sum of $u_1(a) + \max_{a' \in X} u_1(a') - \underset{a' \in A}{med}\, u_1(a')$ and $u_2(a) + \max_{a' \in X} u_2(a') - \underset{a' \in A}{med}\, u_2(a')$ is constant across all elements of X, and the aggregated utility is defined to be the product of the two terms.

This rationalization is relatively simple and intuitive: the above selves are defined such that the DM is torn between two motivations, one in line with ordering $\succ$, and one going in exactly the opposite direction. Moreover, the geometric aggregation of these preferences drives the DM to choose the most central element of any choice set.

There are many variants of the above aggregator that given two selves with diametrically opposed interests do not select exactly the median from every choice set, but have a tendency to induce the choice of a centrally located element from any choice set. In general, if f is menu-dependent and aggregates the utilities of selves through a concave function, the choice induced by f exhibits a *compromise effect* or *extremeness aversion*, as in the experiments of Simonson (1989): given two opposing motivations, an alternative is more likely to be selected the more centrally it is located. If, on the other hand, f is menu-dependent and convex, then it can give rise to a *polarization effect*, as in the experiments of Simonson and Tversky (1992): the induced choice is likely to be in one of the extremes of the choice set. Hence, our model can be used to reinterpret experimental choice data in different contexts, in terms of properties of the aggregator function.

Another simple procedure Kalai et al. (2002) study is Sen (1993)'s second-best procedure.

Example 7 (Choosing the second best). *Consider the following procedure: there is some strict ordering $\succ$ defined over elements of X, and the DM always chooses the second best element of any choice set, according to $\succ$. We will show that there is an aggregator that can rationalize the choice function given by the above procedure no matter how large X is, using only two selves. For any self u on X, and any $A \subset X$, let $l(u, A)$ be the lowest utility level attainable from A according to u. Moreover, let $g : X \times P(X) \times \mathcal{X} \times R^X \to \mathbb{R}$ be such that*

$$g(a, A, X, u) = \begin{cases} u(a) - \max_{b \in X} u(b) & \text{if } u(a) = l(u, A) \\ u(a) & \text{otherwise.} \end{cases}$$

That is, g penalizes the worst elements of a given choice set, by an amount that corresponds to the best attainable utility in X. Define now the following aggregator: for any $U = \{u_1, ..., u_n\} \in \mathcal{U}(X)$, let $f(a, A, X, U) = \sum_{i=1}^{n} g(a, A, X, u_i)$. That is, f is a utilitarian aggregation, with large disutility associated with alternatives that are worst for some selves in the choice set. We claim that the following two selves rationalize the second-best procedure with f. Let $a_1, a_2, ..., a_N$ stand for the increasing ordering of alternatives in X according to $\succ$, and define $u_1(a_j) = j$ and $u_2(a_j) = N + \frac{N+1-j}{2N}$ for all $j \in \{1, ..., N\}$. Note that the incremental utilities of u_1 when choosing a higher $\succ$-ranked element are larger than the incremental disutilities of u_2. Hence this self determines the preference ordering implied by the aggregated utility, with the exception of the choice between the best alternative and the second-best alternative for u_1 in the choice set. This is because the best alternative for u_1 is the worst one for u_2, and the extra disutility associated with this worst choice for u_2 overcomes the incremental utility for u_1. This rationalization has the simple interpretation of a conflict between a greedy self and an altruistic self.

In contrast, Kalai et al. (2002) show that in their framework, in which exactly one self is responsible for any decision, as the size of X increases, the number of selves required to rationalize either of the above procedures goes to infinity. Kalai et al. (2002) also discuss the idea that when multiple rationalizations are behavior, one with the minimal number of selves is most appealing. While dictator-type aggregators do not provide an intuitively appealing explanation for the median procedure, aggregators in our framework can rationalize the above procedures in simple and intuitive ways.

Note that the aggregators and selves in these examples together rationalize very specific types of behavior. However, a given aggregator might act differently on a different collection of selves. For example, if the two selves did not have exactly opposing preferences in the

example rationalizing the median procedure, the aggregator might not choose a centrally located alternative in every choice set. Hence AR studies the *set* of behaviors that an aggregator can rationalize (with different possible selves).

C   Approximate triple-solvability

For some aggregators a tighter upper bound can be given for the minimum group size needed to rationalize a choice function, by weakening the triple-solvability requirement. It suffices for triple-solvability to hold only *approximately*, which can yield a triple-basis with a smaller group size. For ease of exposition, we state this property for additively separable aggregators.

Definition 5. *We say $\hat{U} \in \mathcal{U}(\{a, b, c\})$ is a (δ, ε)-approximate triple-basis for f with respect to $\{a, b, c\}$ if $f(a, \{a, b\}, \{a, b, c\}, \hat{U}) = f(b, \{a, b\}, \{a, b, c\}, \hat{U}) + \delta$ and $|f(x, A, \{a, b, c\}, \hat{U}) - f(y, A, \{a, b, c\}, \hat{U})| < \varepsilon$ for all other $A \subseteq \{a, b, c\}$ and $x, y \in A$.*

That is, a group U is a (δ, ε)-approximate triple basis for f if given choice set $\{a, b\}$ the aggregated utility of U for a is exactly δ higher than the aggregated utility of b, while U is ε-indifferent among all alternatives given every other choice set.

We say that an aggregator f is *approximately triple-solvable with k members* if there is $\bar{\delta} > 0$ such that exists a (δ, ε)-approximate triple-basis with k members for every $\delta < \bar{\delta}$ and $\varepsilon > 0$. That is, for approximate triple-solvability we do not require that the group in the triple basis is exactly indifferent between all elements in choice sets other than $\{a, b\}$, only that they can be arbitrarily close to being indifferent. Theorem 2 can then be modified as follows.

Theorem 6. *Suppose f satisfies P1-P6 and P9, and is approximately triple-solvable with k_f members. Then, for any finite set of alternatives X, and any choice function $c : P(X) \to X$ that exhibits at most $\frac{n-1}{k_f}$ IIA-violations, f can rationalize c with n members.*

Proof. The only difference compared to the proof of Theorem 2 is in the construction of the rationalizing group. Recall the definition of $(I_j)_{j=1,...,j^*}$ from the proof of Theorem 2. Let $\delta_1 \in (0, \bar{\delta})$. Define iteratively δ_j for $j \in \{2, ..., j^* + 1\}$ such that $\delta_j \in (0, \frac{\delta_{j-1}}{IIA(c)+1})$. Define a member u^X such that u^X is δ_{j^*+1}-indifferent and the preference ordering of the self is $c(X) \succ c(X \setminus \{c(X)\}) \succ ...$ Let

$$\delta^{**} = \min_{x \neq y \in X, \ A \ni x, y} |f(x, A, X, u^X)| - |f(y, A, X, u^X)|.$$

Finally, let $\varepsilon \in (0, \frac{\delta^{**}}{|X|})$. Then for every $j \in \{1, ..., j^*\}$ and $A \in I_j$ construct a group $U^A \in \mathcal{U}(X)$ the following way: take a (δ_j, ε)-approximate triple-basis U, and define U^A by defining, for each $u_i \in U$, a member $u_i^A \in U^A$ by

$$
u_i^A(x) = \begin{cases} u_i(a) & x = c(A) \\ u_i(b) & x \in A \setminus \{c(A)\} \\ u_i(c) & x \in X \setminus A. \end{cases}
$$

Proving the group consisting of u^X and U^A for each $A \in \bigcup_{j=1}^{j^*} I_j$ rationalizes c is analogous to the proof in Theorem 2. ∎

D Relaxing P6

Our main results can be extended to aggregators violating P6, that is, to aggregators that depend in a nontrivial way on alternatives unavailable in a given choice set. However, the appropriate definition of triple-solvability is more complicated.

The main complication arising in the absence of P6 is that triple-solvability needs to be defined on a general X, as opposed to just a triple $\{a, b, c\}$. It is convenient to introduce the following notation: for any triple $\{a, b, c\}$, any basic set of alternatives $X \supset \{a, b, c\}$, and any self u defined on $\{a, b, c\}$, define the set $E(u, X) = \{\widehat{u} : X \to \{u(a), u(b), u(c)\}|\widehat{u}(x) = u(x) \ \forall \ x \in \{a, b, c\}\}$. In words, $E(u, X)$ is the set of extensions of u from $\{a, b, c\}$ to X for which each element in $X/\{a, b, c\}$ receives the same utility as either a or b or c. Similarly, for any $U = (u_1, ..., u_m) \in \mathcal{U}(\{a, b, c\})$, let $E(U, X) = \{(\widehat{u}_1, ..., \widehat{u}_m)|\widehat{u}_i \in E(u_i, X)$ for all $i \in \{1, ..., m\}\}$.

Definition 6. *We say $U \in \mathcal{U}(\{a, b, c\})$ is a universal triple-basis for f if for any $X \supset \{a, b, c\}$ the following holds: for all $\hat{U} \in E(U, X)$, $f(a, \{a, b\}, X, \hat{U}) > f(b, \{a, b\}, X, \hat{U})$, and $f(\cdot, A, X, \hat{U})$ is constant for all other $A \subseteq \{a, b, c\}$.*

A universal triple-basis solves the triple $\{a, b, c\}$ whenever the utilities of unattainable elements don't differ from utilities of elements in $\{a, b, c\}$, for all members in the triple-basis. An aggregator f is *universally triple-solvable* if the following condition is satisfied.

Condition (Universal triple-solvability of f) There exists a triple $\{a, b, c\}$ and $k \in Z_+$ such that for every $\delta > 0$ there is a δ-indifferent $U \in \mathcal{U}^k(\{a, b, c\})$ constituting a universal triple-basis for f with respect to $\{a, b, c\}$.

It is easy to see that for aggregators satisfying P6, universal triple-solvability is equivalent to triple-solvability. If f satisfying P1-P5 is universally triple-solvable with k members, then the same construction can be applied as in the proof of Theorem 2 to obtain an analogous lower bound on the set of choice functions that f can rationalize with a given group size. The proof of this result is analogous to the proof of Theorem 2 and hence omitted.

Theorem 7. *Suppose f satisfies P1-P5 and is universally triple-solvable wrt to X with k_f members. Then, using n group members, f can rationalize any choice function, on any grand set of alternatives X, that exhibits at most $\frac{n-1}{k_f}$ IIA-violations.*