Supplementary appendix to the paper Hierarchical cheap talk Not for publication

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1 Monotonicity of the set of pure-strategy equilibria

Here we formalize the statement that the set of equilibrium outcomes is monotonically decreasing in the bias of any intermediator, as a corollary of Proposition 2.

Corollary 1: Let $k \in \{2, ..., n-1\}$ and fix the preferences of all players other than k. Let u^k be a payoff function implying positive (respectively, negative) bias. If v^k is more positively (resp. negatively) biased than u^k, then for every pure strategy PBNE of the indirect communication game in which player k's payoff function is v^k, there is an outcome-equivalent pure-strategy PBNE of the indirect communication game in which player k's payoff function is u^k.

Proof: We will provide a proof for the case of positive biases. The case of negative biases is perfectly symmetric.

Let G^u and G^v stand for the games where the payoff function of player k is u^k and v^k, respectively. Let s^* constitute a PBNE G^v. Then Proposition 1 implies that $\Theta(y)$ is an interval (possibly degenerate) for every $y \in Y$, where Y is the set of actions induced by s^*. Proposition 2 implies that there is an outcome-equivalent PBNE to s^* in G^u iff

$$\int\limits_{\theta\in\Theta(y)} u^k(y,\theta)f(\theta)d\theta \geq \int\limits_{\theta\in\Theta(y)} u^k(y',\theta)f(\theta)d\theta \tag{1}$$

for every $y, y' \in Y$ (recall our convention for the above inequality if $\Theta(y)$ is a singleton). Also by Proposition 2, since s^* constitutes a PBNE G^v, we have:

$$\int\limits_{\theta\in\Theta(y)} v^k(y,\theta)f(\theta)d\theta \geq \int\limits_{\theta\in\Theta(y)} v^k(y',\theta)f(\theta)d\theta \tag{2}$$

for every $y, y' \in Y$.

Fix now $y, y' \in Y$. Since u^k implies positive bias, (1) holds trivially if $y' < y$. Suppose now that $y' > y$.

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Since v^k is more positively biased than u^k, condition (1) in the main text, together with $f(\theta) > 0$ for every $\theta \in \Theta$, implies there exists affine transformations of v^k and u^k, v^{k*} and u^{k*} respectively, such that:

$$\frac{\partial \int_{\theta\in\Theta(y)} v^{k*}(\theta,\widehat{y})f(\theta)d\theta}{\partial\widehat{y}} > \frac{\partial \int_{\theta\in\Theta(y)} u^{k*}(\theta,\widehat{y})f(\theta)d\theta}{\partial\widehat{y}} \tag{3}$$

for all $\widehat{y} \in [y, y']$. This implies $\int\limits_{\theta\in\Theta(y)} v^{k*}(y,\theta)f(\theta)d\theta - \int\limits_{\theta\in\Theta(y)} v^{k*}(y',\theta)f(\theta)d\theta$ is strictly smaller than $\int\limits_{\theta\in\Theta(y)} u^{k*}(y,\theta)f(\theta)d\theta - \int\limits_{\theta\in\Theta(y)} u^{k*}(y',\theta)f(\theta)d\theta$. Then (2) implies (1). ■

2 Complete characterization of 2-action and 3-action single-component equilibria in the uniform-quadratic case

To simplify notation, we will label each message such that

$$m_k^j = E(\theta|m_k = m_k^j).$$

With this notation we have that player 3's strategy is just $y = m_2$, while the set of messages sent by player 1 correspond to the midpoints of the partition cells of the given equilibrium.

We let $\Delta = b^2 - b^1$.

2.1 2-action mixed equilibria

Without loss of generality, assume that $b^2 < 0$ (the case of $b^2 > 0$ is perfectly symmetric). It is convenient to do an analysis with a fixed signed b^2, because the sign of b^2 determines after which message(s) player 2 mixes in a given type of equilibrium. Notice that the example in Figure 1 corresponds to the case of $b^2 > 0$. When we draw Figure 2, we obtained the region with $b^1 > 0$ and $b^2 > 0$ by rotating the region with $b^1 < 0$ and $b^2 < 0$ in a point-symmetric manner with respect to the origin.

By Bayes' rule, m_1 only depends on x_1: we must have $m_1^1 = x_1/2$ and $m_1^2 = \frac{1+x_1}{2}$. Also by Bayes' rule, $m_2^2 = m_1^2 = \frac{1+x_1}{2}$. Because player 2 must be indifferent between her messages after receiving m_1^2, we must also have $m_2^1 = m_2^2 + 2b^2 = \frac{1+x_1}{2} + 2b^2$. Then, for player 1 to be indifferent between both messages in state x_1 we must have

$$x_1 = 1 + 2\Delta.$$

So it must be that $-\frac{1}{2} \le \Delta \le 0$. Substituting this value of x_1 we can solve for the messages $m_1^1 = \Delta + \frac{1}{2}$, $m_2^1 = 1 + \Delta + 2b^2$, and $m_1^2 = m_2^2 = 1 + \Delta$. For the probability $p(m_2^1|m_1^2)$, which we denote simply by p, by Bayes' rule we have

$$p = \frac{1}{8} \cdot \frac{(1+4b^2)(1+2\Delta)}{b^2\Delta}.$$

For this to be feasible, $p \ge 0$, so $-\frac{1}{4} \le b^2$ (as $1 + 2\Delta = x_1$ has to be nonnegative). From $p \le 1$ we get $\Delta \le -2b^2 - \frac{1}{2}$. It is trivial to check that these conditions together with the condition $0 \le x_1 \le 1$ are also sufficient for equilibrium. In terms of b^1 and b^2 the constraints become $\max\{-\frac{1}{4}, b^1 - \frac{1}{2}\} \le b^2 \le \frac{1}{3}b^1 - \frac{1}{6}$.

2.2 3-action mixed equilibria

Without loss of generality, assume that $b^2 > 0$ (the case of $b^2 < 0$ is perfectly symmetric). Notice that the example in Figure 3 corresponds to the case of $b^2 < 0$. Similarly to what we did in Figure 2, when we draw Figure 4, we obtained the region with $b^1 > 0$ and $b^2 < 0$ by rotating the region with $b^1 < 0$ and $b^2 > 0$ in a point-symmetric manner with respect to the origin.

As in the case of 2-action mixed PBNE, the messages sent in equilibrium by player 1 are determined by x_1 and x_2: $m_1^j = (x_{j-1} + x_j)/2$ for every $j \in \{1, 2, 3\}$ where we let $x_0 = 0$ and $x_3 = 1$. By Bayes' rule $m_1^1 = m_2^1 = x_1/2$. Using player 2's indifferences between messages in which she mixes, we get that $m_2^2 = x_1/2 + 2b^2$, $m_2^3 = x_1/2 + x_2$.

Player 1's indifference, when the state is x_2, is equivalent to

$$x_2 = x_1 + 2\Delta.$$

Denote the probabilities $p(m_2^{j+1}|m_1^j)$ by p_j. From Bayes' rule applied to m_2^3, we get

$$p_2 = \frac{(1 - x_2)(1 - x_2 - x_1)}{x_2 \Delta}.$$

And using Bayes' rule for m_2^2 we get

$$p_1 = \frac{1}{4} \frac{\left(x_2 - 4b^2\right)(2x_2 - 1)(1 - x_1)}{x_1 x_2 b^2}.$$

These equations determine the equilibrium in terms of x_1. Now, to actually calculate x_1, it is necessary to work with player 1's indifferences between two nontrivial lotteries.

Assuming that $0 \leq x_1 \leq x_2$ we must have that

- $p_2 \geq 0$ iff $(x_1 + x_2)/2 = x_1 + \Delta \leq 1/2$.
- $p_2 \leq 1$ iff $x_2 = x_1 + 2\Delta \geq 1/2$.

Notice that (assuming $x_2 \geq 1/2$, which follows from $p_2 \leq 1$)

- $p_1 \geq 0$ iff $x_2 \geq 4b^2$, or $b^2 \leq x_2/4 = x_1/4 + \Delta/2$.
- $p_1 \leq 1$ iff

$$\begin{aligned} 2\,x_1^3 + \left(-3 + 4\,b^2 - 8\,b^1\right) x_1^2 + \left(2\,b^2 + 1 + 10\,b^1 - 8\,b^1\,b^2 + 8\,(b^1)^2\right) x_1 \\ +8\,(b^2)^2 - 2\,b^2 - 2\,b^1 - 8\,(b^1)^2 \geq 0 \end{aligned} \tag{4}$$

These equations are complicated, but we will simplify them below.

The final equation we need is player 1's indifference constraint when her type is x_1. This reduces to

$$\begin{aligned} 6\,x_1^3 + \left(12\,b^2 - 9 - 24\,b^1\right) x_1^2 + \left(-24\,b^1\,b^2 + 3 + 18\,b^1 - 6\,b^2 + 24\,(b^1)^2\right) x_1 \\ +8\,(b^2)^2 - 8\,(b^1)^2 - 2\,b^2 - 2\,b^1 = 0 \end{aligned} \tag{5}$$

Unfortunately, the closed form Cardano solution of this equation is very complicated and not very helpful. But we may use it to simplify the condition that $p_1 \leq 1$ to

$$(3x_1 + 4\Delta - 1)\Sigma \geq 0,$$

where $\Sigma = b^1 + b^2$. Assuming $p_2 \leq 1$, this reduces to

- $p_2 \leq 1$ iff $\Sigma \geq 0$.

Summing up, there is a solution iff we can find x_1 solving equation (5) with:

- $1/2 - 2\Delta \leq x_1 \leq 1/2 - \Delta$ from $0 \leq p_2 \leq 1$.
- $0 \leq 2\Sigma \leq x_1$ from from $p_1 \leq 1$ and $p_1 \geq 0$.

Note that these imply $0 \leq x_1 \leq x_1 + 2\Delta = x_2 \leq 1$.
Now we have to consider two cases.

Case 1: $b^2 \leq 1/8$

In this case, the binding constraints are:

- $1/2 - 2\Delta \leq x_1 \leq 1/2 - \Delta$ from $0 \leq p_2 \leq 1$.
- $0 \leq \Sigma$ from from $p_1 \leq 1$.

Notice that we must have $\Delta \leq 1/4$.
Let $f(x_1)$ be the function defined by the left-hand side of (5). We have

$$f(1/2 - 2\Delta) = (1 - 4\Delta)(2\Delta - b^2)$$

$$f'(1/2 - 2\Delta) = (6 + 24b^2)\Delta - 3/2 \leq 0$$

and

$$f(1/2 - \Delta) = -\frac{1}{2}(1 + 2\Delta)(1 - 6\Delta)\Sigma$$

$$f'(1/2 - \Delta) = -\frac{3}{2}(1 - 2\Delta)^2 \leq 0.$$

First notice that the leading coefficient of the cubic $f(x)$ is positive. Because f' is negative in both endpoints of the interval, there is no solution in the interval if f has the same sign in the extremes. So there are two possible cases:
(i) $2\Delta - b^2 \leq 0$ and $1 - 6\Delta \leq 0$. This leads to a contradiction, as $1/6 \leq \Delta \leq b^2/2 \leq 1/16$.
(ii) The other case is $2\Delta - b^2 \geq 0$ and $1 - 6\Delta \geq 0$. There are equilibria in this case iff:

- $0 \leq \Delta \leq 1/6$
- $b^2 \leq 1/8$
- $\Sigma \geq 0$
- $2\Delta \geq b^2$ or $b^1 \leq b^2/2$ (notice that this implies $0 \leq \Delta$).

Case 2: $b^2 \geq 1/8$

In this case, the binding constraints are:

- $0 \leq 2\Sigma \leq x_1 \leq 1/2 - \Delta$.

Note that the binding ones are $b^2 \geq 1/8$, $b^1 + b^2 \geq 0$ and $3b^2 + b^1 \leq 1/2$. And this implies that $b^2 \leq 1/4$ and $-1/4 \leq b^1 \leq 1/8$. Also, $\Delta \leq 1/2$ and $\Sigma \leq 1/4$. We have

$$f(2(b^1 + b^2)) = 4(1 + 24(b^2)^2 - 10b^2 - 2b^1)\Sigma$$

$$f'(2\Sigma) = 3(1 - 4b^2)(1 - 10b^2 - 6b^1) < 0.$$

If $\Delta \leq 1/6$, then the right endpoint is negative, as we have seen in the analysis of Case 1. So there will be solutions in the area that satisfies the additional constraints

- $\Delta \leq 1/6$
- $1 + 24(b^2)^2 - 10b^2 - 2b^1 \geq 0$

If $\Delta \geq 1/6$, f is positive on the right endpoint. It can be shown that the left endpoint has positive f, so that there can be no solutions.

Summarizing, there are two regions where the equilibria under consideration exist. These are:

- The region corresponding to Case 1:

1. $b^2 \leq b^1 + 1/6$.
2. $b^2 \leq 1/8$.
3. $b^2 \geq -b^1$.
4. $b^2 \geq 2b^1$.

- The region corresponding to Case 2:

1. $\Delta \leq 1/6$.
2. $1 + 24(b^2)^2 - 10b^2 - 2b^1 \geq 0$.
3. $b^2 \geq 1/8$.

These regions are depicted in Figure 4, with the point symmetric rotation discussed at the beginning of this subsection.

3 An additional formal result to Section 4

Proposition 4': If $n = 3$, then for every PBNE there is an outcome-equivalent PBNE, such that Θ can be partitioned into a finite number of intervals $B_1, ..., B_K$, referred below as components, such that for any component B_k the following hold: (i) The interior of B_k can be partitioned into a finite number of intervals $I_1^k \leq ... \leq I_{j_k}^k$ such that player 1 sends message $m_1^{j,k}$ with probability 1 at any $\theta \in int(I_j^k)$ and message $m_1^{j,k}$ is not sent from any state $\theta \notin cl(I_j^k)$; (ii) If the intermediator is positively biased, then for $j \in \{1, ..., j_k - 1\}$ after message $m_1^{j,k}$ he mixes between messages $m_2^{j,k}$ and $m_2^{j+1,k}$, and after message $m_1^{j_k,k}$ he sends message $m_2^{j_k,k}$ with probability 1; (iii) If the intermediator is negatively biased, then for $j \in \{2, ..., j_k\}$ after message $m_1^{j,k}$ he mixes between messages $m_2^{j,k}$ and $m_2^{j-1,k}$, and after message $m_1^{1,k}$ he sends message $m_2^{1,k}$ with probability 1; (iv) The receiver chooses a different action after every message sent in equilibrium.

Proof: Given a PBNE, construct an outcome-equivalent PBNE as we did in the proof of Proposition 4. This allows us to have a partition of Θ, $P = (P_1, \ldots, P_J)$. Parts (i) and (iv) are immediate from the proof of Proposition 4 and from Proposition 3, respectively. For now, set $m_1 = j$ if $\theta \in P_j$, and $m_2 = y$ if $p^3(y|m_2) > 0$ (so y is given probability 1), without loss of generality.

Let partition B be such that each cell B_k is minimal with the property that, if $P_j \subseteq B_k$ and $p^2(y|j) > 0$ where y is a PBNE action taken by player 3, then $P_{j'} \subseteq B_k$ for all j' with $p^2(y|j') > 0$. Fixing k, consider B_k and the partition of B_k, $I^k = (I_1^k, \ldots, I_{j_k}^k)$, whose cells are also the cells of the original partition, $P = (P_1, \ldots, P_J)$.

Now we prove part (iii). The proof for (ii) is perfectly symmetric, so we provide a proof only for part (iii). Let player 1 send a message m_1^j conditional on the state lying in I_j^k (for natational simplicity, in this proof, we suppress the superscripts k on messages). Also, let $y^1, \ldots, y^{j_k}(, y^{j_k+1})$ be the PBNE actions induced by states in B_k (We already know that there are j_k (or $j_k + 1$) actions induced), with $y^j < y^{j'}$ if $j < j'$. Set $m_2^j = y^j$.

Now suppose the contrary, i.e. that after message m_1^1 player 2 sends two messages, m_2^1 and m_2^2 with positive probabilities. (We know from the proof of Proposition 4 that player 2 mixes over at most two messages, that there is no message induced in equilibrium between these two messages, and that m_2^1 must have positive probability after m_1^1 by the construction of B; hence it suffices to rule out the case in consideration.) For this to be an equilibrium, conditional on m_1^1, player 2 has to be indifferent between m_2^1 and m_2^2:

$$\int_{I_1^k} u^2(\theta, m_2^1) f(\theta) d\theta = \int_{I_1^k} u^2(\theta, m_2^2) f(\theta) d\theta. \tag{6}$$

Next, note that player 3 is maximizing his payoff at m_2^1, which is induced only by m_1^1, so we have:

$$m_2^1 = \arg\max_y \int_{I_1^k} u^3(\theta, y) f(\theta) d\theta.$$

Since player 2 is negatively biased, this implies:

$$m_2^1 > \arg\max_y \int_{I_1^k} u^2(\theta, y) f(\theta) d\theta.$$

Hence, the first-order condition, the strict positiveness of f, and the strict concavity of u^2 implies:

$$\int_{I_1^k} \frac{\partial u^2(\theta, m_2^1)}{\partial y} f(\theta) d\theta < 0.$$

Now, recall again that u^2 is strictly concave. This implies:

$$\int_{I_1^k} \frac{\partial u^2(\theta, \bar{y})}{\partial y} f(\theta) d\theta < 0 \quad \text{for all} \quad \bar{y} \in [m_2^1, m_2^2],$$

which implies:

$$\int_{I_1^k} u^2(\theta, m_2^1) f(\theta) d\theta > \int_{I_1^k} u^2(\theta, m_2^2) f(\theta) d\theta.$$

This contradicts Equation (6), which completes the proof. ■

4 An additional example to Section 3

This example shows that in general networks a pure strategy PBNE outcome of the direct communication game can be an equilibrium outcome in an indirect communication game even if the condition in Proposition 2 is violated for all intermediators.

Consider the uniform-quadratic specification of the model with 4 players, such that the biases for players 1-3 are: $b_1 = 0.1$, $b_2 = 0.8$, and $b_3 = 0.8$. The direct communication game has a 2-cell partition equilibrium with cutoff point 0.3, implying equilibrium actions 0.15 and 0.65. The resulting outcome cannot be supported in the indirect communication games we consider, namely when communication happens along a line sequence. In fact, it could not be supported even if only one of the above intermediators were in the chain, as the bias of both of them are large enough so that they would prefer to induce action 0.65 instead of action 0.15 when the expectation of the state is 0.15. However, suppose now that instead of communicating only to one intermediator, player 1 can send a message, simultaneously, to both players 2 and 3. After observing these messages, both players 2 and 3 can send a message to player 4, who then takes an action. In short, the sender communicates to the intermediator through two parallel channels. In this game there is an outcome-equivalent PBNE to the two-cell partition equilibrium above. The reason is that when the receiver receives inconsistent messages from the two intermediators talking to him, his (out-of-equilibrium) beliefs can be specified in a way that hurts both of the intermediators, for example by specifying a point belief at state 0, leading to action 0.

5 A note on purification

This note addresses two concerns about our model. The first is that the mixed equilibria we describe may sound unappealing, or too complicated to be actually played. In principle it is not clear why the intermediators mix with the exact probablity specified by the equilibrium strategy profile, as they are indifferent between both messages. So it is important to provide a better motivation for why these equilibria would be played.

The second, related concern is that our assumption of perfect information about other player's preferences may not be realistic. While the model assumes that the biases of each player are perfectly known in advance, this need not be so in practice. It would be much more reasonable to assume that there is at least some degree of uncertainty over the biases of other players.

Here, we argue that mixed equilibria can be motivated as equilibria of a game with small amounts of uncertainty over preferences, in the spirit of Harsanyi's purification result. That is, the mixed equilibria of our game are close to pure strategy equilibria of a large class of perturbed games with a small amount of uncertainty over the biases of other players.

Because the action spaces in our model are not finite, the classic theorem in Harsányi (1973) cannot be applied directly. But notice that the intermediators in our model receive and send messages in a finite set. So tediously following the reasoning in Harsányi (1973), it can be seen that mixed equilibria in our game are approximated by pure strategy equilibria of games in which there is a small amount of uncertainty over the intermediators biases.

Instead of pursuing the general case of this result, we give an example to illustrate the idea. The example makes it clear how a general result would work, and also makes it intuitive how very general specifications of uncertainty over the intermediator's bias can generate the mixed equilibria.

Consider again the case of a 2 message mixed equilibria, characterized in section 4.1.1. Assume that $b^2 > 0$, and that the biases are in the range where a 2 message mixed equilium exists, in which the intermediator mixes with some probability $0 < p < 1$ when receiving the low message m_1^1.

Assume now that there is a small, but arbitrary amount of uncertainty over the intermediator's bias. That is, ex-ante all that the sender and the receiver knows is that the intermediator's bias is $\tilde{b}^2 = b^2 + \epsilon\nu$, where $\epsilon > 0$ is a small real number and ν has an arbitrary distribution F with a continuous density on $[-1, 1]$.

We now argue that, in the spirit of the purification theorem, as $\epsilon \to 0$, the perturbed game has a pure strategy equilibrium that is very close to the 2 message mixed equilbria.

This is easy to see if we use the characterization obtained in section 4.1.1. Consider a candidate equilibrium in which the intermediator always sends m_2^2 when he receives m_1^2, but what he sends when he receives the low message m_1^1 depends on his type. That is, there is some value $b^{*2} = b^2 + \epsilon\nu^*$ such that he sends the low message m_1^1 if his type is lower than this threshold and send the high message m_2^2 if his type is higher. We will show that for small ϵ it is always possible to find b^{*2} that is ϵ close to b^2 and such that this constitutes an equilibrium. So from now on assume that ϵ is small enough such that there are still 2 message mixing equilibria for intermediator's biases in the range $[b^2 - \epsilon, b^2 + \epsilon]$.

In this equilibrium, after receiving message m_1^1, type b^{*2} must be indifferent between both messages (assume that this type is close to b^2). But then, following the argument in section 4.1.1, this completely pins down what the equilibrium messages are. They are the same as those calculated in section 4.1.1 for biases b^1 and b^{*2}. And the reasoning in that section pins down not just the equilibrium messages, but also the mixing probability, which we denote p^*. We must have that

$$p^* = \frac{1}{8}\frac{\left(1 - 4b^{*2}\right)\left(1 - 2\Delta^*\right)}{\Delta^* b^{*2}},$$

where $\Delta^* = b^{*2} - b^1$. But for this to be an equilibrium, we must also have that p^* is the probability that the intermediator's type exceeds b^{*2}. That is

$$p^* = 1 - F(\frac{b^{*2} - b^2}{\epsilon}).$$

Moreover, because all other incentive constraints are satisfied if the messages are given as in section 4.1.1, we have that this is an equilibrium iff this consistency requirement for p^* is satisfied, that is

$$\frac{1}{8}\frac{\left(1 - 4b^{*2}\right)\left(1 - 2\Delta^*\right)}{\Delta^* b^{*2}} = 1 - F(\frac{b^{*2} - b^2}{\epsilon}).$$

Now, note that the left side is always between 0 and 1 for b^{*2} in the interval $[b^2 - \epsilon, b^2 + \epsilon]$, for small ϵ. But the right side equals 1 for $b^{*2} = b^2 - \epsilon$ and 0 for $b^{*2} = b^2 + \epsilon$. So by the intermediate value theorem there is b^{*2} in $[b^2 - \epsilon, b^2 + \epsilon]$ that satisfies this equation, and for which our construction is indeed an equilibrium.

Moreover, by the characterization in section 4.1.1, we know that equilibrium messages and the mixing probability depend continuously on the intermediator's bias. So for small ϵ this equilibrium is close to the mixed equilibrium described in section 4.1.1. This example illustrates that a way to interpret and motivate the mixed equilibria is that they correspond to pure strategy equilibria of the game perturbed with a small amount of uncertainty over the intermediator's bias.