THE GLUING PROPERTY

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Abstract. We introduce a new compactness principle which we call the gluing property. For a measurable cardinal $\kappa$ and a cardinal $\lambda$, we say that $\kappa$ has the $\lambda$-gluing property if every sequence of $\lambda$-many $\kappa$-complete ultrafilters on $\kappa$ can be glued into an extender. We show that every $\kappa$-compact cardinal has the $2^\kappa$-gluing property, yet non-necessarily the $(2^\kappa)^+\text{-gluing property}$. Finally, we compute the exact consistency-strength for $\kappa$ to have the $\omega$-gluing property; this being $\omega(\kappa) = \omega_1$.

1. Introduction

Suppose that $\mathcal{M}$ is a mathematical structure all of whose small substructures $\mathcal{N} \subseteq \mathcal{M}$ have a property $\Phi$. Is it true that $\mathcal{M}$ itself satisfies $\Phi$?

An affirmative answer to the above is an instance of a phenomenon called compactness. Historical experience in Mathematics indicates that there is a natural expectation for compactness. Indeed, several classical results, in various areas, indicate that this is the case when small means finite.

For instance, in Graph Theory, a classical theorem of De Bruijn and Erdős [BE51] establishes that if $G$ is an infinite graph such that all of its finite subgraphs $\mathcal{H} \subseteq \mathcal{G}$ have chromatic number $\leq n$ then $\mathcal{G}$ itself has chromatic number $\leq n$. Another example in this vein is Gödel’s Compactness theorem for First Order Logic (FOL). This latter says that an arbitrary set of sentences $T$ is satisfiable provided all of its finite subsets $S \subseteq T$ are satisfiable. This result, in turn, has several applications in other areas of Mathematics, such as Ramsey Theory, Algebra or Topology. To mention just one of these, it is a consequence of the compactness theorem for FOL that every one-to-one polynomial map $f: \mathbb{C}^n \to \mathbb{C}^n$ is in fact a bijection.

The above results witness that the first infinite cardinal $\aleph_0$ is compact. This invites to the following natural question: How about if small means, e.g., countable (i.e., cardinality $< \aleph_1$)? Should we still expect such a garden-variety of examples in favor of compactness of $\aleph_1$? The truth is that this is not the case. To mention one of the various existing counter-examples, the De Bruijn-Erdős’ theorem does not apply in this context. Indeed, a theorem...
of Erdős and Hajnal [EH66] shows that, if the Continuum Hypothesis holds, there is a graph \( G \) of size \( \aleph_2 \) whose chromatic number is \( \aleph_1 \) but all of its subgraphs \( H \) with \( |H| < \aleph_1 \) do have chromatic number \( \leq \aleph_0 \).

The above counter-example is in fact an evidence that “finite” cannot be replaced by “countable” in Gödel’s compactness theorem for FOL. Inspired by this, Keisler and Tarski investigated in [KT64] under what assumptions the infinitary logic \( L_{\kappa,\kappa} \) does satisfy an analogue of Gödel’s theorem, provided \( \kappa \) is an uncountable cardinal. This lead to the discovery of strongly compact cardinals, a cornerstone notion in the theory of large cardinals.

Even though our understanding of the large cardinal hierarchy has vastly improved since the 60’s [Kan09], some natural problems regarding strongly compact cardinals are among the most stubborn open questions in set theory.

Mimicking the tight connection between Gödel’s compactness theorem and the Boolean Prime Ideal theorem, one can characterize the strong compactness of a cardinal \( \kappa \) as follows: every \( \kappa \)-complete filter \( \mathcal{F} \) extends to a \( \kappa \)-complete ultrafilter \( \mathcal{U} \). In [Mit79] Mitchell asked whether the restricted version of this statement —“every \( \kappa \)-complete filter on \( \kappa \) can be extended to a \( \kappa \)-complete ultrafilter”, is witnessed in a model of the form \( L[\bar{U}] \), the canonical inner model for the axiom “\( \forall \kappa (o(\kappa) \leq \kappa^{++}) \)”.

A negative answer to this question was quickly provided by Gitik in [Git93, §2], showing that the consistency strength of this statement is at least that of the existence of a strong cardinal. In his proof, Gitik constructs an extender of arbitrary length using the filter extension property of a cardinal \( \kappa \) that is \( \kappa \)-compact. Unfortunately, Gitik’s argument contains a subtle gap. More specifically, in order to get past the limit steps in his construction of [Git93, Theorem 2.1] a new compactness property is required – we have coined this principle the name of the Gluing property:

**Definition 1.1** (The Gluing Property). Let \( \kappa \) be a measurable cardinal. We say that \( \kappa \) has the \( \lambda \)-gluing property if for every sequence of \( \kappa \)-complete ultrafilters on \( \kappa \), \( \{U_\gamma \mid \gamma < \lambda\} \), there is a \( \kappa \)-complete extender \( E \) and an increasing sequence of ordinals \( \{\eta_\gamma \mid \gamma < \lambda\} \) such that

\[
U_\gamma = \{X \subseteq \kappa^{(\gamma)} \mid \{(j_E(\gamma), \eta_\gamma)\} \in j_E(X)\}.
\]

One may speculate whether it is also possible to glue together various extenders. We will not deal with this issue here.

The gluing property is catching a weakening of the filter extension property. Instead of trying to extend an arbitrary \( \kappa \)-complete filter, we are extending a filter on increasing sequences which has the property that each projection is already a \( \kappa \)-complete ultrafilter.

The adoption of the gluing property is inspired by Gitik’s argument in [Git93, Theorem 2.1]. In fact, Gitik’s argument immediately shows:

**Theorem** (Gitik). If \( \kappa \) has the \( \lambda \)-gluing property for every cardinal \( \lambda \) then there is an inner model with a strong cardinal.
Namely, the assumption “κ is a κ-compact cardinal” in [Git93, §2] should be replaced by “κ has the λ-gluing property for all cardinals λ”.

As our title announces, in this paper we investigate the gluing property. In §3 we show that κ-compact cardinals enjoy the $2^\kappa$-gluing property, yet not more than that. More precisely, we prove the following result:

**Theorem.** If κ has the λ-filter extension property then κ has the $2^\lambda$-gluing property. However, it is consistent that κ is a κ-compact cardinal which does not have the $(2^\kappa)^+\text{-gluing property}.$

Nevertheless we would like to stress that the $2^\kappa$-gluing property (and hence κ-compactness) is sufficiently strong to derive another striking result due to Gitik – the existence of a Prikry type forcing that is universal for all κ-distributive forcings of size κ (see [Git20]). In §2 we provide the reader with a self-contained and complete exposition of this extender-based poset.

Continuing with our study of the gluing property, in §4 we derive the consistency of the ω-gluing property from the existence of a strong cardinal:

**Theorem.** Assume that κ is a strong cardinal. Then, there is a cardinal-preserving generic extension in which κ has the ω-gluing property.

Thus, strongness yields an upper bound for the consistency strength of the ω-gluing property. Following this vein, we conclude the paper with §§5 and 6 where we compute the consistency strength of the ω-gluing property:

**Theorem.** The following statements are equiconsistent:

1. κ has the ω-gluing property.
2. κ is a measurable cardinal with $o(\kappa) = \omega_1$.

The difference between those two results is that in order to prove the second one, we work over a suitable core model, with some anti-large cardinal hypothesis, while the first one is a straight-forward forcing argument.

The set-theoretic notations of this paper are mostly standard. All the new concepts and terminologies will be timely introduced. Nevertheless, we assume our readers to be fluent with the theory of elementary embeddings and Prikry-type forcings. For a comprehensive account in these matters we refer the reader to [Kan09] and [Git10], respectively.

2. **Gitik’s universal extender-based Prikry forcing**

An uncountable cardinal κ is said to be κ-compact if every κ-complete filter on κ can be extended to a κ-complete ultrafilter. In this section we would like to provide a full-detailed account of the following theorem:

**Theorem 2.1** (Gitik). Assume that κ is a κ-compact cardinal. Then, there is a Prikry-type poset $P$ forcing the following: for every poset $Q$ with $|Q| = \kappa$ and $V \models “Q$ is κ-distributive” there is a $Q$-generic filter over $V$. 
Gitik’s result provides a substantial improvement of other universality theorems concerning Prikry-type forcings. For instance, it was well known that if $\kappa$ is $2^\kappa$-supercompact then the usual Supercompact Prikry forcing with respect to a measure on $P_\kappa(2^\kappa)$ projects onto any $\kappa$-distributive forcing of size $\kappa$ (see [Git10, §6.4]). Gitik improved that to a $\kappa$-compact cardinal, while the exact consistency strength is still unclear. More recent developments in this vein have been obtained by Benhamou, Hayut and Gitik in [BGH21].

In this paper, however, our interest in Theorem 2.1 relies on the fact that it hints the gluing property. Succinctly speaking the idea is the following. Firstly, it suffices to establish the theorem for $\kappa$-distributive forcing posets (in $V$) whose underlying set is $\kappa$. So, let $\langle Q_\alpha \mid \kappa \leq \alpha < 2^\kappa \rangle$ be an injective enumeration of all such forcings and $\langle F_\alpha \mid \kappa \leq \alpha < 2^\kappa \rangle$ be the corresponding filters of dense open sets. All of these are $\kappa$-complete filters on $\kappa$. Hence, by $\kappa$-compactness of $\kappa$, each $F_\alpha$ can be extended to a $\kappa$-complete ultrafilter $U_\alpha$. Since $\kappa$-compact cardinals have the $2^\kappa$-gluing property (see Theorem 3.2) we can glue all these measures into a $(\kappa, 2^\kappa)$-extender $E$. Once this is done we shall define an extender-based forcing (Gitik’s Universal Extender-based Prikry forcing) that will simultaneously project onto every poset $Q_\alpha$.

We wish to provide a full-detailed exposition of Gitik’s theorem, which first appeared with fewer details in [Git20]. For this we will use Merimovich’s framework for extender-based forcings. The core concept in Merimovich setup is the notion of object (see e.g., [Mer03, Mer07, Mer11]).

So, put $I := [2^\kappa \setminus \kappa]^{\leq \kappa}$ and invoke the $2^\kappa$-gluing property of $\kappa$ to produce an extender $E = \langle E_a \mid a \in I \rangle$ such that, for each $a \in I$, $E_a$ concentrates on

$$X_a := \{ \nu : a \rightarrow \kappa \mid \nu \text{ is order-preserving and } |\nu| < \kappa \}$$

and $\{ e_a^* X \mid X \in E_{\{a\}} \} = U_\alpha$, where $e_a$ is the map $e_a(\nu) := \nu(\alpha)$.

For each $\alpha \in [\kappa, 2^\kappa)$, the seed of $U_\alpha$ is denoted by $\sigma_\alpha$; to wit, $\sigma_\alpha := [id]_{U_\alpha}$.

**Definition 2.2.** Let $P^*$ be the poset consisting of maps $f : \text{dom}(f) \rightarrow [\kappa]^{<\omega}$, where $\text{dom}(f) \in I$, $\kappa \in \text{dom}(f)$ and for each $\alpha \in \text{dom}(f)$,

$$f(\alpha) \text{ is both } <\text{-increasing and } \leq_{Q_\alpha}\text{-decreasing.}$$

Given $f, g \in P^*$ we write $f \leq g$ if $f \supseteq g$.

For each $f \in P^*$ and $\alpha \in \text{dom}(f)$, denote $\mu_\alpha^f := \text{max}(f(\alpha))$.

It is clear that $P^*$ is a $\kappa^+$-directed-closed forcing.

**Definition 2.3 (Objects).** Fix $f \in P^*$. A map $\nu : \text{dom}(\nu) \rightarrow \kappa$ is called an $f$-object if it has the following properties:

1. $|\nu| \leq \nu(\kappa)$;
2. $\text{dom}(\nu) \in [\text{dom}(f)]^{<\kappa}$ and $\kappa \in \text{dom}(\nu)$;
3. for all $\alpha \in \text{dom}(\nu)$, $\mu_\alpha \leq \nu(\alpha)$ and $\nu(\alpha) \leq_{Q_\alpha} \mu_\alpha$.

1When $f$ is clear we shall suppress the superscript $f$ and just write $\mu_\alpha$. 

Proof. that \( \sigma_f \) is an \( \alpha \) \( \in \) \( \text{dom}(f) \). 

Given \( \nu, \eta \in \text{OB}(f) \), write \( \nu < \eta \) if \( \text{dom}(\nu) \subseteq \text{dom}(\eta) \) and for every \( \alpha \in \text{dom}(\nu) \), \( \nu(\alpha) < \eta(\kappa) \) and \( \eta(\alpha) \leq \kappa \), \( \nu(\alpha) \).

**Definition 2.9.** \( \text{Suppose } \pi \in \text{OB}(f) \). Define \( \text{Denote by } \text{OB}(f) \) \( \text{Remark } 2.7 \). Given \( \nu, \eta \in \text{OB}(f) \), write \( \nu < \eta \) if \( \text{dom}(\nu) \subseteq \text{dom}(\eta) \) and for every \( \alpha \in \text{dom}(\nu) \), \( \nu(\alpha) < \eta(\kappa) \) and \( \eta(\alpha) \leq \kappa \), \( \nu(\alpha) \).

**Definition 2.4 (Measures).** Given \( f \in \mathbb{P}^* \) define \( E(f) \) as follows:

\[
X \in E(f) \iff X \subseteq \text{OB}(f) \wedge \sigma_f \in j_E(X),
\]

where \( \sigma_f := \{ (j_E(\alpha), \sigma_\eta) \mid \alpha \in \text{dom}(f) \} \) and \( \sigma_\eta \) denotes the seed of the measure \( U_\eta \), where \( \eta \) is the unique ordinal such that \( Q_\alpha/\mu_\alpha = Q_\eta \), where \( Q_\alpha/\mu_\alpha \) is the forcing that consists of all conditions stronger than \( \mu_\alpha \) in \( Q_\alpha \). \(^2\)

It is automatic that \( E(f) \) is a \( \kappa \)-complete measure on \( \text{OB}(f) \).

**Definition 2.5.** Let \( f \in \mathbb{P}^* \) and \( \nu \in \text{OB}(f) \). Then, \( f(\nu) \) denotes the function with domain \( \text{dom}(f) \) such that for all \( \alpha \in \text{dom}(f) \),

\[
f(\nu)(\alpha) := \begin{cases} f(\alpha) \wedge (\nu(\alpha)), & \text{if } \alpha \in \text{dom}(\nu); \\ f(\alpha), & \text{otherwise}. \end{cases}
\]

In general, if \( \nu \) is a finite \( \prec \)-increasing sequence in \( \text{OB}(f) \), the function \( f_\nu \) is recursively defined as \( f_\nu := \{ f_\nu \mid \nu \} \).

**Definition 2.6 (f-Trees).** A set \( T \) of finite \( \prec \)-increasing sequences in \( \text{OB}(f) \) is called an \( f \)-tree if it is a tree with respect to end-extensions\(^3\) and \( \text{Succ}_T(\nu) := \{ \eta \in \text{OB}(f) \mid \max \nu \prec \eta : \text{Succ}_T(\nu) = \eta \} \subset E(f) \),

for all \( \nu \in T \cup \{ \langle \rangle \} \).

For an \( f \)-tree \( T \) and \( \nu \in T \), denote \( T_\nu := \{ \eta \in T \mid \nu \prec \eta \} \).

**Remark 2.7.** For every \( f \)-tree \( T \) and every \( \nu \in T \), \( T_\nu \) is again an \( f_\nu \)-tree.

Given \( f, g \in \mathbb{P}^* \) with \( f \leq g \), denote by \( \pi_{\text{dom}(f), \text{dom}(g)} \) the restriction map from \( \text{OB}(f) \) to \( \text{OB}(g) \); namely, the map \( \nu \mapsto \nu \upharpoonright \text{dom}(f) \).

**Definition 2.8.** Denote by \( \mathbb{P} \) the set of all pairs \( p = \langle f^p, T^p \rangle \) such that \( f^p \in \mathbb{P}^* \) and \( T^p \) is an \( f^p \)-tree. Given \( p, q \in \mathbb{P} \), we write \( p \leq q \) iff \( f^p \supseteq f^q \) and \( \pi_{\text{dom}(f^p), \text{dom}(f^q)} T^p \subseteq T^q \).

**Lemma 2.9.** Suppose \( f, g \in \mathbb{P}^* \) are such \( f \leq g \) and \( T \) is a \( g \)-tree. Then, \( T^* := \{ \langle \nu_0, \ldots, \nu_{n-1} \rangle \in \text{OB}(f)^{<\omega} \mid \langle \nu_0 \upharpoonright \text{dom}(g), \ldots, \nu_{n-1} \upharpoonright \text{dom}(g) \rangle \in T \} \) is an \( f \)-tree such that \( \pi_{\text{dom}(f), \text{dom}(g)} T^* \subseteq T \).

**Proof.** Clearly, \( T^* \) is a tree under end-extension. Fix, \( \nu \in T \cup \{ \langle \rangle \} \).

\( \leadsto \) If \( \nu = \langle \rangle \) then \( \text{Succ}_{T^*}(\langle \rangle) \in E(f) \) is equivalent to \( \sigma_f \in j(T^*) \). Observe that \( \sigma_f \upharpoonright j(\text{dom}(g)) = \sigma_f \upharpoonright j^* \text{dom}(g) = \sigma_g \in j(T) \), hence \( \sigma_f \in j(T^*) \).

\(^2\)This make sense because \( Q_\alpha/\mu_\alpha \) is \( \kappa \)-distributive of size \( \kappa \) and our enumeration is injective.

\(^3\)I.e., if \( \eta \in T \) then \( \eta \upharpoonright n \in T \) for all \( n \leq |\eta| \).
Let $\vec{v} = \langle \nu_0, \ldots, \nu_{n-1} \rangle$. We need to check that $\max_{j} j(\vec{v}) < \sigma_f$ and $\sigma_f \in j(T^\ast)$. The latter has been proved before. About the former, $j(\vec{v}) = \langle j(\nu_0), \ldots, j(\nu_{n-1}) \rangle$, where $\text{dom}(j(\nu_{n-1})) = j^\ast \text{dom}(\nu_{n-1})$ and $j(\nu_{n-1})(j(\alpha)) = \nu_{n-1}(\alpha) < \kappa \leq \sigma_f(j(\alpha))$. Thus, $j(\nu_{n-1}) < \sigma_f$. □

**Definition 2.10** (One point extensions). Let $p \in P$ and $\langle \nu \rangle \in T^p$. We denote by $p^\ast \langle \nu \rangle$ the pair $\langle f_p^\ast, T_p^\ast \rangle$. For every $\vec{v} \in T^p$, $p^\ast \vec{v}$ is defined recursively in the natural way. Also, by convention, $p^\ast \langle \rangle := p$.

**Lemma 2.11.** For each $p \in P$ and $\vec{v} \in T^p \cup \{\langle \rangle\}$, $p^\ast \vec{v} \in P$.

*Proof.* Let us prove the lemma by induction on $n$, the length of $\vec{v}$.

If $n = 0$ the result follows from the very definition of $p^\ast \langle \rangle$. So, suppose the above property holds for all $\vec{v} \in T^p$ with $|\vec{v}| = n$. Choose $\langle \nu_0, \ldots, \nu_n \rangle \in \times T^n$ and set $\vec{v}':= \langle \nu_0, \ldots, \nu_{n-1} \rangle$. By the induction hypothesis, $q := \langle f_{\vec{v}'}^p, T_{\vec{v}'}^p \rangle \in P$. Also, by definition, $\langle \nu_n \rangle \in T^q$. We thus need to show that $q^\ast \langle \nu_n \rangle \in P$.

On one hand, it is clear that $T_{\vec{v}'}^q$ is an $f_{\nu_n}^q$-tree. On the other hand, $f_{\nu_n}^q \in P^\ast$. Indeed, $T^q$ is a $f_{\nu_n}^q$-tree, hence $\nu_n \in \text{OB}(f_{\vec{v}'}^q)$, and thus

$$\nu_n(\alpha) \geq \text{max}(f_{\vec{v}'}^q(\alpha)) = \mu_0^q \text{ and } \nu_n(\alpha) \leq \mu_0^q,$$

for $\alpha \in \text{dom}(\nu)$. Thereby, $f_{\nu_n}^q$ witnesses the clauses of Definition 2.2. □

**Definition 2.12** (The main poset). Gitik’s universal Extender based forcing is the set $P$ endowed with the following order: for two conditions $p, q \in P$, we write $p \leq q$ iff there is $\vec{v} \in T^q \cup \{\langle \rangle\}$ such that $p \leq^* q^\ast \vec{v}$.

**Notation 2.13.** Let $m < \omega$ and $p, q \in P$. We write $p \leq^m q$ as a shorthand for $p \leq q$ and $|\vec{v}| = m$, where $\vec{v}$ is some sequence witnessing $p \leq^* q^\ast \vec{v}$.

Note that, under this convention, $\leq^0 = \leq^*$.

**Lemma 2.14.** $\langle P, \leq^* \rangle$ is $\kappa$-directed-closed.

*Proof.* Let $D = \{ p_\alpha \mid \alpha < \theta \}$ be a $\leq^*$-directed set of conditions in $P$ with $\theta < \kappa$. Set $f := \bigcup_{\alpha < \theta} f_{p_\alpha}$. Since $D$ is $\leq^*$-directed it is immediate that $f \in P^\ast$. Put, $T := \bigcup_{\alpha < \theta} \pi^{-1}_{\text{dom}(f), \text{dom}(f_{p_\alpha})} T_{p_\alpha}$. Arguing as in Lemma 2.9 one can check that $T$ is a $\text{dom}(f)$-tree. Altogether, $p := \langle f, T \rangle \in P$ yields a $\leq^*$-lower bound for the set $D$. □

**Lemma 2.15.** $P$ has the $\kappa^{++}$-cc.

*Proof.* This is as in [Mer07, Proposition 3.17]. □

The proof of the Strong Prikry Property is essentially the same as that from [Mer03]. We reproduce it just for completeness and for the reader’s benefit. Let us begin with an useful lemma about integration of conditions:

**Lemma 2.16.** Let $p \in P$ and $D : T^p \rightarrow V$ be a function such that, for each $\vec{v} \in T$, $D(\vec{v})$ is a $\leq^*$-dense open set below $p^\ast \vec{v}$. Then, there is $p^* \leq^* p$ such that for each $\vec{m} \in T^{p^*}$, $p^* \vec{m} \in D(\vec{m} \upharpoonright \text{dom}(f^P))$. 
Proof. Fix \( \chi \) a big enough regular cardinal and let \( N \prec H_\chi \) be such that \( |N| = \kappa, N^{<\kappa} \subseteq N \) and \( \{ p, \mathbb{P}, \mathbb{P}^*, \{ D(\vec{\nu}) \mid \vec{\nu} \in T^p \} \} \subseteq N \). Using a properness argument one can show that there is \( f^* \leq f^p \) that is totally \( \langle N, \mathbb{P}^* \rangle \)-generic; namely, for every \( E \subseteq \mathbb{P}^* \) dense open set, \( E \in N \), there is \( h \in N \cap E \) with \( f^* \leq h \). For each \( \vec{\mu} \in \text{OB}(\text{dom}(f^*)) \) define \( D^*(\vec{\mu}) \) as

\[
\{ f \leq f^p \mid \exists q \in \mathbb{P} (f^q = f_{\vec{\mu}} \wedge q \leq^* p^\wedge (\vec{\mu} \restriction \text{dom}(f^p)) \wedge q \in D(\vec{\mu} \restriction \text{dom}(f^p))) \}.
\]

Clearly, \( D^*(\vec{\mu}) \in N \) (because \( \vec{\mu} \in N \)) and it is an open subset of \( \mathbb{P}^* \). In addition, it is not hard to check that \( D^*(\vec{\mu}) \) is dense below \( f^p \). Thereby we have that \( D^*(\vec{\mu}) \cup \{ f \in \mathbb{P}^* \mid f \perp f^p \} \) is a dense subset of \( \mathbb{P}^* \) in \( N \) and so, by properness, there is \( h \in N \) in this set with \( f^* \leq h \). Thus, \( h \in D^*(\vec{\mu}) \).

The above shows that, for each \( \vec{\mu} \in \text{OB}(\text{dom}(f^*)) \), there is a map \( h_{\vec{\mu}} \) in \( D^*(\vec{\mu}) \cap N \), which amounts to saying that there is a condition \( \langle f_{\vec{\mu}}, T_{\vec{\mu}}^\beta \rangle \) in the dense open set \( D(\vec{\mu} \restriction \text{dom}(f^p)) \) such that \( \langle f_{\vec{\mu}}, T_{\vec{\mu}}^\beta \rangle \leq^* p^\wedge (\vec{\mu} \restriction \text{dom}(f^p)) \).

We would like to integrate all these trees \( T_{\vec{\mu}}^\beta \) into a single one \( T^* \) for which \( \langle f^*, T^* \rangle \leq^* \langle f^p, T^p \rangle \) and, for each \( \vec{\mu} \in T^* \), \( \langle f^*, T^* \rangle \wedge T_{\vec{\mu}} \leq^* \langle f_{\vec{\mu}}, T_{\vec{\mu}}^\beta \rangle \).

Let \( T^* := \pi_{\text{dom}(f^*)}^{-1} \), where this latter tree (call it \( \Delta T \)) is defined by induction so as to ensure that, for each \( \vec{\mu} \in \Delta T \), \( \langle T_{\vec{\mu}} \rangle = T_{\vec{\mu}}^\beta \). It should be clear that \( T^* \) is the sought tree.

\begin{lemma}[Strong Prikry property]
For each \( p \in \mathbb{P} \) and \( D \subseteq \mathbb{P} \) dense open there is a condition \( p^* \leq^* p \) and an integer \( n < \omega \) such that, for each \( m \geq n \), if \( q \leq^m p^* \) then \( q \in D \).
\end{lemma}

**Proof.** Let \( p \in \mathbb{P} \) and \( D \subseteq \mathbb{P} \) be as above. The next is the crux of the matter:

\begin{claim}
There is \( p^* \leq^* p \) such that

1. \( \text{“} p^* \wedge \vec{\mu} \in D^* \text{”} \) for all \( \vec{\mu} \in T_{p^*}^\beta \),
2. \( \text{“} \forall q \leq^* p^* \wedge (\vec{\mu} \notin D^*) \text{”} \) for all \( \vec{\mu} \in T_{p^*}^\beta \).
\end{claim}

**Proof of claim.** For each \( \vec{\nu} \in T^p \) put

\[
D^0(\vec{\nu}) := \{ q \leq^* p^\wedge \vec{\nu} \mid q \in D \},
\]

and

\[
D^1(\vec{\nu}) := \{ r \leq^* p^\wedge \vec{\nu} \mid \forall q \leq^* r \ (q \notin D) \}.
\]

For each \( \vec{\nu} \in T^p \), \( D(\vec{\nu}) := D^0(\vec{\nu}) \cup D^1(\vec{\nu}) \) is \( \leq^* \)-dense open below \( p^\wedge \vec{\nu} \).

Applying the previous claim we find \( p^* \leq^* p \) such that, for each \( \vec{\mu} \in T_{p^*}^\beta \), \( p^* \wedge \vec{\mu} \in D(\vec{\mu} \restriction \text{dom}(f^p)) \). Let us define, by induction, a tree \( T^* \subseteq T_{p^*}^\beta \) such that \( \langle f^*, T^* \rangle \) is a condition \( \leq^* \)-below \( p \) witnessing the claim.

Firstly, either the set

\[
\{ \mu \in \text{OB}(\text{dom}(f^p)) \mid p^* \wedge (\mu) \notin D^0(\mu \restriction \text{dom}(f)) \}
\]

is \( E(\text{dom}(f^p)) \)-large or so is its complement,

\[
\{ \mu \in \text{OB}(\text{dom}(f^p)) \mid p^* \wedge (\mu) \in D^1(\mu \restriction \text{dom}(f)) \}.
\]
For each $\alpha \in \mathbb{Q}$ retained in every dense open subset of $\mathbb{Q}$ produces the configuration described in Theorem 2.1.

Proof. Fix a dense open set $Q$ defines a $\mathbb{P}$-tree. Some $q \in Q$ set $A$. Let $(T^1)_{\mu} := T^1_{\mu}$. This defines another $E(\text{dom}(f^*))$-tree $T^1 \subseteq T^*$ which is our first approximation to $T^*$.

The second approximation to $T^*$ is defined as follows: for each object $\mu \in \text{Lev}_1(T^1)$ look at
\[ X_\mu^0 := \{ \nu \in \text{OB}(\text{dom}(f^*)) \mid \mu \prec \nu \land p^* \land (\mu, \nu) \in D^0(\langle \mu, \nu \rangle \upharpoonright \text{dom}(f)) \} \]
and at its complement
\[ X_\mu^1 := \{ \nu \in \text{OB}(\text{dom}(f^*)) \mid \mu \prec \nu \land p^* \land (\mu, \nu) \in D^1(\langle \mu, \nu \rangle \upharpoonright \text{dom}(f)) \} \]

Denote by $X_\mu$ the unique of the above sets that is a $E(\text{dom}(f^*))$-large. Set $\{ \mu \in \text{Lev}_1(T^1) \mid X_\mu = X_\mu^0 \}$ and $\{ \mu \in \text{Lev}_1(T^1) \mid X_\mu = X_\mu^1 \}$. At least one of them is large, call it $X^i$, and stipulate that the first level of $T^2$ to be $X$. For each $\mu \in X$, $(T^2)_{\mu} := X^i_\mu$ and for each $\langle \mu, \nu \rangle \in X^i \times X^i_\mu$, $(T^2)_{\langle \mu, \nu \rangle} := (T^1)_{\langle \mu, \nu \rangle}$. This way we define our second approximation, $T^2$.

Proceeding in this vein we generate an inclusion-decreasing sequence of $E(\text{dom}(f^*))$-trees, $\langle T^\alpha \mid n \geq 1 \rangle$, so that $T^* := \bigcap_{1 \leq n < \omega} T^\alpha$ is the sought tree. $\square$

We are now ready to prove the lemma. Since $D$ is dense there must be some $q \leq p^*$ in $D$. Then Clause (2) above fails and so for every $\bar{\mu} \in [T^*]^n$, $p^* \land \bar{\mu} \in D$. The claim follows right away from this and openness of $D$. $\square$

**Corollary 2.18** (Prikry Property). For each $p \in \mathbb{P}$ and every sentence $\varphi$ in the language of forcing there is $p^* \leq p$ such that $p^*$ decides $\varphi$.

Let $G$ a $\mathbb{P}$-generic filter over $V$. For each $\alpha \in [\kappa, 2^\kappa)$ define
\[ G_\alpha := \bigcup \{ f^p(\alpha) \mid p \in G \land \alpha \in \text{dom}(f^p) \} \]
For each $\alpha \in [\kappa, 2^\kappa)$, $\text{sup}(G_\alpha) = \kappa$ and $\text{otp}(G_\alpha) = \omega$. Also, $G_\alpha \neq G_\beta$, for all $\alpha, \beta \in [\kappa, 2^\kappa)$ with $\beta \neq \alpha$. The following lemma shows that, indeed, $\mathbb{P}$ produces the configuration described in Theorem 2.1.

**Lemma 2.19** (Universality). For each $\alpha \in [\kappa, 2^\kappa)$, $G_\alpha$ is eventually contained in every dense open subset of $\mathbb{Q}_\alpha$ lying in $V$. In particular,
\[ \{ q \in \mathbb{Q}_\alpha \mid \exists p \in G_\alpha (p \leq_{\mathbb{Q}_\alpha} q) \} \]
defines a $\mathbb{Q}_\alpha$-generic filter over $V$.

**Proof.** Fix a dense open set $D \subseteq \mathbb{Q}_\alpha$ and set
\[ E := \{ p \in \mathbb{P} \mid \alpha \in \text{dom}(f^p) \land \forall (\nu) \in \text{Succ}_{T^p}(\langle \rangle) (\nu(\alpha) \in D) \} \]
We claim that $E$ is dense: Let $p \in \mathbb{P}$ and assume that $\alpha \in \text{dom}(f^p)$. Since $T^p$ in an $f^p$-tree, $\text{Succ}_{T^p}(\langle \rangle) \subseteq E(f^p)$. Let $\eta \in [\kappa, 2^\kappa)$ be the unique ordinal such that $\mathbb{Q}_\eta = \mathbb{Q}_\alpha / \mu_\alpha$. Note that $U_\eta \leq_{\text{RK}} E(f^p)$, as witnessed by the map $e_\alpha : (\nu) \mapsto \nu(\alpha)$. In particular, $e_\alpha \text{Succ}_{T^p}(\langle \rangle) \subseteq U_\eta$ and thus the set $A := e_\alpha \text{Succ}_{T^p}(\langle \rangle) \cap D$ is $U_\eta$-large. Pulling $A$ back with $e_\alpha$ one obtains a set $A' \subseteq \text{Succ}_{T^p}(\langle \rangle)$ with $A' \subseteq E(f^p)$ such that for all $\langle \nu \rangle \in A$, $\nu(\alpha) \in D$. $\square$
Repeating this process by induction on the height of the tree $T^p$ one gets an $f^p$-tree $T'$ such that $T' \subseteq T^p$ and $\text{Succ}_T(\langle \rangle) = A'$. Clearly, $p' := \langle f^p, T' \rangle$ is a condition in $\mathbb{P}$ with $p' \leq^0 p$ and $p' \in E$.

Let $p \in G \cap E$. Since $\{p^\upharpoonright \nu \mid \nu \in T^p\}$ forms a maximal antichain below $p$ there is some $f^p$-object $\nu$ such that $p^\upharpoonright \nu \in G$, hence

$$\nu(\alpha) = f^p(\nu)(\alpha) \in G_\alpha \cap D.$$  

Say $\nu(\alpha)$ is the $n^\text{th}$-member of $G_\alpha$. By the very definition of the forcing it turns out that all the members of $G_\alpha$ past stage $n$ are $\leq \aleph_\alpha$-stronger than $\nu(\alpha)$ and thus, by openness of $D$, $G_\alpha \subseteq^* D$. \hfill \Box

The above concludes the proof of Gitik’s theorem (Theorem 2.1).

3. (Sub)compactness and the gluing property

3.1. The filter extension property and the gluing property. In this section we analyze the implications between the $\lambda$-filter extension property and the gluing property. In particular, we show that if $\kappa$ is $\kappa$-compact then $\kappa$ has the $2^\kappa$-gluing property. As we will demonstrate this implication is optimal, for there are $\kappa$-compact cardinals without the $(2^\kappa)^+$-gluing property. Recall that a cardinal $\kappa$ is said to have the $\lambda$-filter extension property (for $\lambda \geq \kappa$) if every $\kappa$-complete filter on $\lambda$ extends to a $\kappa$-complete ultrafilter.

We commence with a strengthening of [Hay19, Corollary 6]:

**Lemma 3.1.** Let $\kappa \leq \lambda$ be uncountable cardinals with $\lambda^{< \kappa} = \lambda$. Then the following two are equivalent:

1. $\kappa$ has the $\lambda$-filter-extension property;
2. for every transitive model $M$ with $2^\lambda \subseteq M$, $|M| = 2^\lambda$ and $^\kappa M \subseteq M$ there is a model $\mathcal{N}$ and an elementary embedding $j: M \rightarrow \mathcal{N}$ such that:
   - (a) $\text{crit}(j) = \kappa$;
   - (b) $^\kappa \mathcal{N} \subseteq \mathcal{N}$;
   - (c) $s \in \mathcal{N}$, $j^{= 2^\lambda} \subseteq s$ and $\mathcal{N} \models |s| < j(\kappa)$.

**Proof.** The implication (2) $\Rightarrow$ (1) follows from [Hay19, Corollary 6]. Assume that $\kappa$ has the $\lambda$-filter-extension property and let $M$ be as above. In [Hay19, Corollary 6] it is shown that there is a transitive model $\mathcal{N}$, $\bar{s} \in \mathcal{N}$ and an elementary embedding $i: M \rightarrow \mathcal{N}$ witnessing (a) and (c) above. We shall now slightly modify both $\mathcal{N}$ and $i$ in order to obtain the desired outcome.

Define $\mathcal{X} := \{i(f)(\kappa, \bar{s}) \mid f \in M\}$. Following Hamkins, [Ham97], $\mathcal{X} \prec \mathcal{N}$. In particular, $\mathcal{X}$ is well-founded. Let $\pi: \mathcal{X} \rightarrow \mathcal{N}$ be the Mostowski collapse, and put $j := \pi \circ i$, $s := \pi(\bar{s})$ and $N := \pi^{= \mathcal{X}}$. Note that $\pi(\kappa) = \kappa$ and

$$N = \pi^{= \{i(f)(\kappa, \bar{s}) \mid f \in M\}} = \{j(f)(\kappa, s) \mid f \in M\}.$$  

Clearly, $s \in N$, $j^{= 2^\lambda} \subseteq s$ and $N \models |s| < j(\kappa)$. Next, we check Clause (b).

Let $\bar{x} = \langle x_\alpha \mid \alpha < \kappa \rangle \subseteq N$. For each $\alpha < \kappa$, $x_\alpha = j(f_\alpha)(\kappa, s)$ for some $f_\alpha \in M$. Observe that establishing $\bar{x} \in N$ amounts to verifying that $\langle j(f_\alpha) \mid \alpha < \kappa \rangle \in N$. This is indeed the case: let $h: \kappa \times M \rightarrow M$ be defined as
$(\beta, \bar{s}) \mapsto \langle f_\alpha(\beta) \mid \alpha < \beta \rangle$. Since $^*\mathcal{M} \subseteq \mathcal{M}$, the sequence $\langle f_\alpha \mid \alpha < \kappa \rangle \in \mathcal{M}$ and therefore $h \in \mathcal{M}$. So,
\[ j(h)(\kappa, s) = \langle j(f_\alpha)(\kappa) \mid \alpha < \kappa \rangle = \langle x_\alpha \mid \alpha < \kappa \rangle \in N. \]
\[ \square \]

**Theorem 3.2.** Let $\kappa \leq \lambda$ be uncountable cardinals with $\lambda^{<\kappa} = \lambda$.
If $\kappa$ has the $\lambda$-filter-extension property then it has the $2^\lambda$-gluing property.

**Proof.** Let $\langle U_\alpha \mid \alpha < 2^\lambda \rangle$ be a sequence of measures on $\kappa$ and consider $\mathcal{X} := \langle U_\alpha \mid \alpha < 2^\lambda \rangle \cup \{ A \mid A \in U_\alpha, \alpha < 2^\lambda \}$. By coding $\mathcal{X}$ appropriately we may assume that $\mathcal{X} \subseteq 2^\lambda$. Let $M$ be a transitive model as in the statement of Lemma 3.1. Using the $\lambda$-filter-extension property of $\kappa$ and Lemma 3.1 we find a transitive model $N$ with $^*\mathcal{N} \subseteq \mathcal{N}$ and $s \in \mathcal{N}$, and an elementary embedding $j: M \to N$ with $\text{crit}(j) = \kappa$, $j^{2^\lambda} \subseteq N$ and $N \models |s| < j(\kappa)$. Note that $j^E \mathcal{X} \subseteq j^2 \mathcal{X} \subseteq s$ hence $j^E U_\alpha \subseteq s \cap j(U_\alpha)$ for all $\alpha < 2^\lambda$. Also, $j(U_\alpha) \in N$, so that $s \cap j(U_\alpha) \in N$ is a subset of $j(U_\alpha)$ of cardinality $< j(\kappa)$. Since $j(U_\alpha)$ is a $j(\kappa)$-complete measure in $N$ we conclude that $\bigcap j(U_\alpha) \notin \emptyset$. For each $\alpha < 2^\lambda$ pick $s_\alpha$ be a witness for it. Note that $s_\alpha \in \bigcap j^E U_\alpha$.

Let us show that we can pick $s_\alpha$ to be increasing. Indeed, in $N$, we can apply this procedure by recursion for each measure of the form $j(U)_\zeta$, $\zeta \in s$. Namely, for each $\zeta \in s \cap j(2^\lambda)$, pick $\bar{s}_\alpha \in \bigcup j(U)_\zeta \cap s$, larger than $\sup \{ \bar{s}_\eta \mid \eta \in s \cap \zeta \}$. Since $N \models |s| < j(\kappa)$, this is possible. Finally, we let $s_\alpha = \bar{s}_j(\alpha)$.

Next, let us form the corresponding gluing extender. For each $\alpha \in [2^\lambda]^{<\kappa}$, let $\nu_\alpha := \{ (j(\alpha), s_\alpha) \mid \alpha \in a \}$. Since $^*\mathcal{N} \subseteq \mathcal{N}$, $\nu_\alpha \in N$. Define $X \in E_a \iff X \subseteq X_a \land \nu_\alpha \in j(X)$,
\[ \text{where } X_a := \{ f: a \to \kappa \mid f \text{ increasing } \land |f| < \kappa \}. \]
This yields a $\kappa$-complete ultrafilter on $X_a$ such that $E_a \in \mathcal{M}$.\footnote{Note that here we need to use that $2^\lambda \subseteq \mathcal{M}$ and $^*\mathcal{M} \subseteq \mathcal{M}$.} Put $E := \langle E_a \mid a \in [2^\lambda]^{<\kappa} \rangle$ and observe that for each $\alpha < 2^\lambda$ and $A \in U_\alpha$, $A := \{ \{ (\alpha, \gamma) \} \mid \gamma \in A \}$ is a member of $E_{\alpha}$ as needed. \[ \square \]

Combining Theorem 3.2 with [Hay19, Theorem 11] we get:

**Corollary 3.3.** Let $\kappa \leq \lambda$ be uncountable cardinals with $\lambda^{<\kappa} = \lambda$.
If $\kappa$ is $\lambda^+\Pi_1^1$-subcompact then it has the $2^\lambda$-gluing property.

In the next subsection we show that the above result is (consistently) optimal, by exhibiting a generic extension where $\kappa$ is $\kappa^+\Pi_1^1$-subcompact but it does not have the $(2^\kappa)^+$-gluing property.

### 3.2. $\Pi_1^1$-subcompactness and the gluing property.

The next auxiliary result will be instrumental later on:

**Lemma 3.4.** If $\kappa$ is a $\kappa^+\Pi_1^1$-subcompact cardinal then $o(\kappa) = (2^\kappa)^+$.\footnote{Note that here we need to use that $2^\lambda \subseteq \mathcal{M}$ and $^*\mathcal{M} \subseteq \mathcal{M}$.}

**Proof.** The proof mimics Solovay’s original argument proving the analogous property for supercompact cardinals. The following is the key claim:
Claim 3.4.1. For $X \subseteq \mathcal{P}(\kappa)$ there is a normal measure $U$ on $\kappa$ such that $X \in \text{Ult}(V,U)$.

Proof of claim. Assume towards contradiction that for some $X \subseteq \mathcal{P}(\kappa)$ the above statement is false. Then the formula defined by

$$\Phi(X,\kappa) \equiv \forall U \ (U \text{ normal measure on } \kappa \implies X \notin \text{Ult}(H(\kappa^+), U))$$

is a true $\Pi^1_1$-sentence in $\langle H(\kappa^+), \in, X \rangle$. Indeed, one can formulate the statement $X \notin \text{Ult}(H(\kappa^+), U)$ as a first order statement in the structure $\langle H(\kappa^+), \in, X, U \rangle$ as follows: for every function $f$ with domain $\kappa$ such that $f(\alpha) \subseteq \mathcal{P}(\alpha)$ and for every $y \subseteq \kappa$, $y \in X$ if and only if $\{ \alpha < \kappa \mid y \cap \alpha \in f(\alpha) \} \in U$. Using that $\kappa$ is $\kappa^+\Pi_1^1$-subcompact we can find an ordinal $\varrho < \kappa$, $\check{X} \subseteq H(\varrho^+)$ and an elementary embedding $j: \langle H(\varrho^+), \in, \check{X} \rangle \to \langle H(\kappa^+), \in, X, U \rangle$ with $\text{crit}(j) = \varrho$, $j(\varrho) = \kappa$ and such that

$$(*) \quad \langle H(\varrho^+), \in, \check{X} \rangle \models \Phi(\check{X}, \varrho).$$

Note that $(*)$ entails $H(\kappa^+) \models \exists Y \Phi(Y, \varrho)$, and this is a first order statement.

Let $U_\varrho$ be the measure on $\varrho$ induced by $j$: that is, $U_\varrho := \{ X \subseteq \varrho \mid \varrho \in j(X) \}$. Put $i := j \upharpoonright \langle H(\varrho^+), \in \rangle$ and let $\ell$ be the ultrapower map induced by $U_\varrho$. Put $M := \text{Ult}(H(\varrho^+), U_\varrho)$ and let $k$ be the elementary embedding factoring $i$ through $\ell$. Note that $\text{crit}(k) > |\mathcal{P}(\varrho)^M|$. By elementarity of $k$, $M \models \exists Y \Phi(Y, \varrho)$, hence there is $Y \in M$ such that $M \models \Phi(Y, \varrho)$. Since $Y \subseteq \mathcal{P}(\varrho)^M$ and $\text{crit}(k) > |\mathcal{P}(\varrho)^M|$ we have $k(Y) = k^\kappa Y = Y$. Thus, $\Phi(Y, \varrho)$ holds in $H(\kappa^+)$. However note that $Y \in M = \text{Ult}(H(\varrho^+), U_\varrho)^{H(\kappa^+)}$, which yields the desired contradiction. \[\square\]

Let $\mathcal{U} = \langle U_\alpha \mid \alpha < 2^\kappa \rangle$ be a sequence of measures on $\kappa$. Note that $\mathcal{U}$ can be naturally coded as a subset of $\mathcal{P}(\kappa)$ and hence the above claim yields a measure $U$ such that $\mathcal{U} \subseteq \text{Ult}(H(\kappa^+), U) \subseteq \text{Ult}(V, U)$. Since we can repeat this argument $(2^\kappa)^+$-many times we infer that $o(\kappa) = (2^\kappa)^+$. \[\square\]

Lemma 3.5. Assume that the GCH holds. Suppose that $\kappa$ is a cardinal with the $\kappa^+$-gluing property and $o(\kappa) = \kappa^+$. Then, there is an embedding $j: V \to M$ such that $\text{crit}(j) = \kappa$, $\kappa M \subseteq M$ and $\kappa^+_M = \kappa^+$. \[\square\]

Proof. Let $\mathcal{U} = \langle U_\alpha \mid \alpha < \kappa^+ \rangle$ be a $\prec$-increasing sequence of measures on $\kappa$. Using the $\kappa^+$-gluing property of $\kappa$ we can find a $(\kappa, \kappa^+)$-extender $E = \langle E_\alpha \mid a \in [\kappa^+]^{\leq \kappa} \rangle$ such that for all $\alpha < \kappa^{++}$ and $A \in U_\alpha$ then

$$\{ \langle \alpha, x \rangle \mid x \in A \} \in E_{\langle \alpha \rangle}.$$ 

Let $i_\alpha : V \to N_\alpha \simeq \text{Ult}(V, U_\alpha)$.

Claim 3.5.1. For every $\alpha < \kappa^{++}$, $\kappa^+_{N_\alpha} \geq \alpha$.

Proof of claim. Suppose towards a contradiction that this is false, and let $\alpha < \kappa^{++}$ be minimal such that $\kappa^+_{N_\alpha} < \alpha$. Fix $\beta < \alpha$ arbitrary, and let $i_{\alpha, \beta} : N_\alpha \to \text{Ult}(N_\alpha, U_\beta) \simeq N_{\alpha, \beta}$. Similarly, let $i_{\beta, \alpha} : N_\beta \to \text{Ult}(N_\beta, i_\beta(U_\alpha)) \simeq N_{\beta, \alpha}$. 


Standard arguments show that \( N_{\alpha,\beta} = N_{\beta,\alpha} \) and that \( i_{\alpha,\beta} \circ i_\alpha = i_{\beta,\alpha} \circ i_\beta \). Letting \( N := N_{\alpha,\beta} \) one has in particular that \( i_{\beta,\alpha}(\kappa_{N_{\beta}}^+ +) = \kappa_N^+ \), and since \( \text{crit}(i_{\beta,\alpha}) = i_\beta(\kappa) > \kappa_{N_{\beta}}^+ \) actually \( \kappa_{N_{\beta}}^+ = \kappa_N^+ \). By the minimality assumption, \( \kappa_{N_{\beta}}^+ \geq \beta \) and hence \( \kappa_N^+ \geq \beta \). All in all, we arrive at

\[
\beta \leq \kappa_N^+ < i_{\alpha,\beta}(\kappa) < \kappa_{N_\alpha}^+. \tag{5}
\]

From all of this we conclude that \( \kappa_{N_\alpha}^+ \geq \alpha \), which is impossible. \( \square \)

Let \( j : V \to M \) be the elementary embedding induced by the extender \( E \). By \( \kappa \)-completeness of the extender \( E \) we have that \( {}^*M \subseteq M \).

**Claim 3.5.2.** \( \kappa_M^+ = \kappa^+ \).

**Proof of claim.** Let \( \alpha < \kappa^+ \). By Claim 3.5.1, \( \kappa_{N_\alpha}^+ \geq \alpha \), hence \( \kappa_M^+ \geq \alpha \).

Let \( k_\alpha : M_\alpha \to M \) be the canonical factor map. Then, \( \kappa_M^+ \geq k_\alpha(\alpha) \geq \alpha \).

Since the above is valid for every \( \alpha < \kappa^+ \) we have \( \kappa_M^+ \geq \kappa^+ \). \( \square \)

At this stage we have accomplished the proof of the lemma. \( \square \)

We now use the previous lemma to show that (consistently) there are \( \kappa^+-\aleph_1 \)-subcompact cardinals without the \( (2^\kappa)^+ \)-gluing property.

Let \( S \) be the Easton-support iteration \( \langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \beta \leq \alpha < \kappa \rangle \) defined as follows: If \( \alpha < \kappa \) is an inaccessible cardinal then we stipulate that

\[
\mathbb{I}_{\mathbb{P}_\alpha} \forces_{\mathbb{P}_\alpha} \text{“} \dot{Q}_\alpha \text{ is the forcing to add a } \square_{\alpha^+}, \text{-sequence”}. \notag
\]

Otherwise, \( \mathbb{I}_{\mathbb{P}_\alpha} \forces_{\mathbb{P}_\alpha} \text{“} \dot{Q}_\alpha \text{ is trivial”} \). Let \( G \) a \( V \)-generic filter for \( S \).

**Theorem 3.6.** Assume the GCH holds and that \( \kappa \) is a \( \kappa^+-\aleph_1 \)-subcompact cardinal such that \( \text{Refl}(E_\omega^\kappa^+) \) holds. Then, the following hold in \( V[G] \):

1. \( \kappa \) is \( \kappa^+-\aleph_1 \)-subcompact;
2. There is no embedding \( j : V \to M \) with \( \text{crit}(j) = \kappa \) and \( \kappa_M^+ = \kappa^+ \).

In particular, in \( V[G] \) the cardinal \( \kappa \) does not have the \( \kappa^+ \)-gluing property.

**Proof.** We begin proving the following claim:

**Claim 3.6.1.** In \( V[G] \) the principle \( \text{Refl}(E_\omega^\kappa^+) \) holds.

**Proof of claim.** Let \( \dot{S} \) be a \( S \)-name and \( p \in G \) a condition such that

\[
p \forces_S \text{“} \dot{S} \text{ is stationary in } E_\omega^\kappa^+ \text{”}. \notag
\]

For every \( q \leq p \), define \( S_q := \{ \alpha < \kappa^+ \mid q \forces_S \alpha \in \dot{S} \} \). Since the condition \( p \) forces \( “\bigcup_{q \leq p} S_q = \dot{S}” \) it follows that for some \( q \leq p \) the set \( S_q \) is stationary.

We may assume without loss of generality that \( q \in G \). By our assumption there is \( \delta \) with \( \delta := \text{cf}(\delta) > \omega \) such that \( S_\delta \cap \delta \) is stationary. Put \( S^* := S_\delta \cap \delta \). Clearly, \( \delta^+_S \subseteq S_G \), so it suffices to check that \( q \forces_S \text{“} \dot{S}^* \text{ is stationary”} \).

\(^5\)Here we use that GCH holds in \( N_\alpha \).
If $\theta$ is a Mahlo cardinal then $\mathbb{P}_\theta$ is $\theta$-cc, hence it preserves the stationarity of $S^*$. Also, observe that the tail of the iteration is $(\theta + 1)$-strategically closed, hence it preserves stationary subsets in cofinality $< \kappa$.  

If $\theta$ is a non-Mahlo limit of inaccessibles then $\mathbb{P}_\theta$ is the Easton-support iteration of the previous stages. Assume towards a contradiction that for a $\mathbb{P}_\theta$-name $\check{C}$ and $r \in G_\theta$ we have $r \not\models_{\mathbb{P}_\theta} \check{C}$ is a club $\land \check{C} \cap S^* = \emptyset^\ast$. Since $r = r_\alpha * 1_{(\alpha, \theta)}$ we can construct a $\mathbb{P}_\alpha$-name $\check{C}$ such that $r \not\models_{\mathbb{P}_\alpha} \check{C} = \check{C}^\ast$. This in particular implies that $r_\alpha \not\models_{\mathbb{P}_\alpha} \check{S}^* \text{ is not stationary}$. However, this yields a contradiction because $\mathbb{P}_\alpha$ is a small forcing. Thus, $S^*$ remains stationary after forcing with $\mathbb{P}_\theta$. Yet again, the rest of the forcing is distributive enough to preserve the stationarity of $S^*$ in $V[G]$.

In any other case the argument is easy. In effect, $S$ factors as a two-step iteration of a small forcing and a sufficiently distributive one. \hfill $\square$

The next claim yields Clause (1) of the theorem:

**Claim 3.6.2.** The cardinal $\kappa$ is $\kappa^+ - \Pi^1_1$-subcompact in $V[G]$.

**Proof of claim.** Working in $V[G]$, assume that $\Phi$ is the $\Pi^1_1$-sentence of the form $\forall X, \varphi(X, A)$, where $A \subseteq H(\kappa^+)$. Let us assume that $\langle H(\kappa^+), \in, A \rangle \models \Phi$. Pick $\hat{A}$ be a $\mathbb{S}$-name such that $\check{A}_G = A$. Since GCH holds then $\mathbb{S} \subseteq H(\kappa^+)$, hence $\check{A} \subseteq H(\kappa^+)$. Moreover, every subset $X$ of $H(\kappa^+)$ in the generic extension, has a name $\check{X} \subseteq H(\kappa^+)$ and every element $y$ of $H(\kappa^+)^V[G]$ has a name $\check{y} \in H(\kappa^+)^V$ such that $\check{y}^G = y$.

Let us assume that in $V[G]$,

$$\langle H(\kappa^+), \in, A \rangle \models \forall X \subseteq H(\kappa^+), \varphi(X, A),$$

So the Forcing Theorem yields that for some condition $p$,

$$p \models_{\mathbb{S}} \langle H(\kappa^+), \in, \check{A} \rangle \models \forall X \subseteq \check{H}(\kappa^+), \varphi(\check{X}, \check{A})$$

which means by the above discussion:

$$\langle H(\kappa^+)^V, \in, \check{A}, \models_{\mathbb{S}}, p \rangle \models \forall \check{X}, (\check{X} \text{ is an } \mathbb{S} \text{-name } \Longrightarrow \ "p \models_{\mathbb{S}} \varphi(\check{X}, \check{A})") .$$

Besides, by virtue of the Forcing Definability Theorem, “$p \models_{\mathbb{S}} \varphi(\check{X}, \check{A})$” is a first order sentence in the above-displayed structure. Using that $\kappa$ is $\kappa^+ - \Pi^1_1$-subcompact in $V$ we find $q < \kappa$, $\check{B} \subseteq H(q^+)^{\check{S}_q}$ and $q \in \check{S}_q$, along with an elementary embedding

$$j: \langle H(q^+), \in, \check{B}, \models_{\check{S}_q}, q \rangle \rightarrow \langle H(\kappa^+), \in, \check{A}, \models_{\mathbb{S}}, p \rangle$$

with $\text{crit}(j) = q$, $j(q) = \kappa$ and such that $\langle H(q^+), \in, \check{B}, \models_{\check{S}_q}, q \rangle \models \ "q \models_{\mathbb{S}} \Phi" .$$

Since $\text{crit}(j) = q$ and $\check{S}_q$ is an Easton-support iteration of forcings in $H(q^+)$, it follows that $j \upharpoonright \check{S}_q = \text{id}$. In particular, putting $B := \check{B}_{G_q}$, Silver’s lifting criterion implies that $j$ lifts to

$$j^\ast: \langle H(q^+)^{V[G_q]}, \in, B \rangle \rightarrow \langle H(\kappa^+)^V[G], \in, A \rangle,$$

\[\text{See e.g. [Cum10, Lemma 10.6].}\]
with \( \text{crit}(j^*) = \varrho \) and \( j^*(\varrho) = \kappa \). Also, note that since \( j(q) = p \in G \) and \( j \upharpoonright \mathbb{S}_p = \text{id} \) then \( q \in G \cap \mathbb{S}_p = G_\varrho \).

Finally, since the forcing \( \mathbb{S} \) splits into \( \mathbb{S} \upharpoonright \varrho \) iterated by a \( \varrho^{++} \)-distributive forcing, since
\[
\langle H(q^+) \rangle^V, \varepsilon, \hat{B}, \models_{\mathbb{S}[\varrho][q]} \phi
\]
we conclude that \( q \) actually forces over \( \mathbb{S} \) the same statement.

Hence, \( V[G] \models \langle H(q^+) \rangle^{V[G]} \models \varepsilon, \hat{B} \models \Phi \). \( \square \)

Let us now prove Clause (2). Assume towards contradiction that in \( V[G] \) there is an elementary embedding \( j : V[G] \to M \) such that \( \text{crit}(j) = \kappa \) and \( (\kappa^{++})^M = (\kappa^{++}) \). By elementarity, \( \square_{\kappa^+} \) holds in \( M \). Now, since \( M \) computes correctly \( \kappa^{++} \) we have that \( \square_{\kappa^+} \) holds in \( V[G] \), as well. This indeed forms a contradiction with Claim 3.6.1 above, and thereby Clause (2) is established. Finally, \( \kappa \) cannot have the \( \kappa^{++} \)-gluing property, for otherwise lemmas 3.4 and 3.5 would together imply that Clause (2) fails. \( \square \)

Using that \( \text{Refl}(E^{\omega^{++}}) \) holds for \( \kappa^{++} \)-\( \Pi^1_1 \)-subcompact cardinals one infers the following corollary:

**Corollary 3.7.** Assume that there is a \( \kappa^{++} \)-\( \Pi^1_1 \)-subcompact cardinal.

Then, there is a generic extension where \( \kappa \) is \( \kappa \)-compact but does not have the \( (2\kappa)^+ \)-gluing property.

4. THE \( \omega \)-GLUING PROPERTY FROM A STRONG CARDINAL

In the next sections we deal with the consistency of the \( \omega \)-gluing property, starting with a large cardinal hypothesis weaker than subcompactness. Let us observe first that the \( \omega \)-gluing property is in fact a property of measures and not of extenders.

**Lemma 4.1.** A cardinal \( \kappa \) has the \( \omega \)-gluing property if and only if for every \( \omega \)-sequence of \( \kappa \)-complete ultrafilters \( \langle U_n \mid n < \omega \rangle \) there is a \( \kappa \)-complete ultrafilter \( W \) on \( \kappa^\omega \), concentrating on increasing sequences, such that the projection map \( e_n \) sending \( \eta \in \kappa^\omega \) to \( \langle \eta | n < \omega \rangle \) is a Rudin-Keisler projection from \( W \) to \( U_n \).

**Proof.** The backwards direction is clear, as \( W \) itself can be represented as an extender. For the forwards direction, let \( E \) be a \( \kappa \)-complete extender such that there is a sequence \( \langle \eta_n \mid n < \omega \rangle \), \( U_n = \{ X \subseteq \kappa^{(n)} \mid \langle \langle \eta | n < \omega \rangle \rangle \in j_E(X) \} \).

Let \( W := \{ X \subseteq \kappa^\omega \mid \eta_n \mid n < \omega \} \in j_E(X) \}. \) Then, \( W \) witnesses the validity of the lemma. \( \square \)

Assume the GCH holds and let \( \kappa \) be a strong cardinal. Let \( \ell : \kappa \to V_\kappa \) be a Laver function in the sense of [GS89, §2]. In this section we define a Gitik iteration \( \mathbb{P}_\kappa \) yielding a generic extension where \( \kappa \) has the \( \omega \)-gluing property.

Let \( A \subseteq \kappa \) be the collection of all inaccessible cardinals \( \alpha < \kappa \) such that \( \ell^n \alpha \subseteq V_\alpha \). Let \( A^{\ell} := A \cup \{ \beta + 1 \mid \beta \in A \} \). We define \( \mathbb{P}_\kappa \) by induction on \( \alpha \in A^{\ell} \) as follows. Suppose that \( \mathbb{P}_\beta \) has been defined for all \( \beta \in A^{\ell} \cap \alpha \). If
α is a limit point of $A^\ell$ then $P_\alpha$ is simply the Gitik iteration of the previous stages. Otherwise, suppose that α is a successor point of β in $A^\ell$. If α ≠ β+1 then we stipulate $P_\alpha := P_\beta \ast \dot{Q}_\beta$, where $1_{P_\beta} \Vdash "\dot{Q}_\beta"$ is trivial”. Otherwise we shall distinguish two cases:

(a) If $p \Vdash_{P_\beta} "\ell(\beta) = (U_n^\beta \mid n < \omega)"$ are β-complete uniform measures on $\beta^+$, then $p \Vdash_{P_\beta} "\dot{Q}_\beta = \dot{T}( (U_n^\beta \mid n < \omega) )"$, where this denotes a name for the tree Prikry forcing relative to $(U_n^\beta \mid n < \omega)$ (see [Ben19, §4]).

(b) Otherwise, $1_{P_\beta} \Vdash "\dot{Q}_\beta"$ is trivial”.

Theorem 4.2. Assume the GCH holds that there is a strong cardinal κ.

Then, forcing with $P_\kappa$ yields a model for “κ has the ω-gluing property”.

Proof. Fix $G \subseteq P_\kappa$ a $V$-generic filter. Working in $V[G]$, let $(U_n \mid n < \omega)$ be an ω-sequence of κ-complete uniform measures on κ. Let $p \in G$, and $(\dot{U}_n \mid n < \omega)$ be an ω-sequence of $P_\kappa$-nice-names such that $p$ forces these to have the above property. Note that since $P_\kappa$ has the κ-cc, and $P_\kappa \subseteq V_\kappa$, we have $\dot{U}_n \in V_{\kappa+2}$ for all $n < \omega$. By [GS89, Lemma 2.1] we can find a $(\kappa, \kappa + 2)$-strong embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = (\dot{U}_n \mid n < \omega)$. We will use this to form a measure $V$ on $\kappa^\omega$ with $U_n \leq_{\text{RK}} V$ for all $n < \omega$.

By our GCH assumption, we can enumerate all the $P_\kappa$-nice-names for subsets of $\kappa^\omega$ as $\langle \dot{X}_\alpha \mid \alpha < \kappa^+ \rangle$. Look now at $j(P_\kappa)$. As usual, this decomposes as an iteration of the form $P_\kappa \ast \dot{Q}_\kappa \ast \check{R}$, where

$$1_{P_\kappa \ast \dot{Q}_\kappa} \Vdash_{P_\kappa \ast \dot{Q}_\kappa} "(\check{R}, \leq^*)" \text{ is } \kappa^+\text{-closed}.$$ In addition, since $p \Vdash_N "j(\ell)(\kappa)"$ is an ω-sequence of measures”, this latter condition forces $\dot{Q}_\kappa$ to be the tree Prikry forcing with respect to the sequence $\langle \dot{U}_n \mid n < \omega \rangle$. Note that this definition is absolute between $M[G]$ and $V[G]$.

Define by induction on $\alpha < \kappa^+$ a sequence of $P_\kappa \ast \dot{Q}_\kappa$-names $\langle \dot{r}_\alpha \mid \alpha < \kappa^+ \rangle$ for conditions in $\check{R}$ such that:

- $\langle \dot{r}_\alpha \mid \alpha < \kappa^+ \rangle$ is $\leq^*$-decreasing;
- for all $\alpha < \kappa^+$, $\dot{r}_\alpha \Vdash_{\check{R}} \dot{b}_\kappa \in j(\dot{X}_\alpha)$, where $\dot{b}_\kappa$ denotes the standard name for the generic object added by $\dot{Q}_\kappa$.

Define $U$ as follows: $X \in U$ iff there is $p \in G$ and $\alpha < \kappa^+$ such that $p \cup \{ (\kappa, (\emptyset, \dot{T})) \} \cup r_\alpha \Vdash_{j(P_\kappa)} \dot{b}_\kappa \in j(\dot{X}_\alpha)$, where $\dot{T}$ is a $P_\kappa$-name for a $\langle \dot{U}_n \mid n < \omega \rangle$-tree.

Standard arguments prove that this is a $\kappa$-complete measure on (the increasing sequences in) $\kappa^\omega$, in $V[G]$. For more details, see [Git10, p. 1439].

Claim 4.2.1. $V[G] \models "\forall n < \omega (U_n \leq_{\text{RK}} U)"$. Moreover, for each $n < \omega$, this is witnessed by $e_n : \kappa^\omega \rightarrow \kappa$, the evaluation map $f \mapsto f(n)$.

Proof of claim. We claim that $e_n "X \in U_n"$ for all $X \in U$. Suppose otherwise, and let $p \in G$ be such that $p \Vdash_{P_\kappa} "\dot{X} \in U \land \dot{Y} \notin U_p"$, where $\dot{Y}$ is a $P_\kappa$-nice name for a subset of $\kappa$ such that $\dot{Y}_G = e_n "X"$. Since $\dot{Y}$ is $P_\kappa$-nice, $\dot{Y} \subseteq V_\kappa$.
By definition of $\mathcal{U}$ there is $\alpha < \kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot{T}$ such that
\[
p \cup \{\langle \kappa, (\emptyset, \dot{T}) \rangle \} \cup r_\alpha \models^M_{j(\mathbb{P}_\kappa)} \dot{b}_\kappa \in j(\dot{X}).
\]
In particular,
\[
p \cup \{\langle \kappa, (\emptyset, \dot{T}) \rangle \} \cup r_\alpha \models^M_{j(\mathbb{P}_\kappa)} \dot{b}_\kappa(n) \in j(\dot{Y}) \cap \kappa.
\]
By our choice of $\dot{Y}$, it follows that $j(\dot{Y}) \cap \kappa = \dot{Y}$. So, all in all, the above condition forces “$\dot{b}_\kappa(n) \in \dot{Y}$”. On another front, since $p \models_{\mathbb{P}_\kappa} \dot{Y} \notin U_n$, there is $q \leq p$ in $G$ such that $q \models_{\mathbb{P}_\kappa} (\emptyset, \dot{T}') \leq^* (\emptyset, \dot{T})$, where $\dot{T}'$ is a $\mathbb{P}_\kappa$-name for a tree whose $n^{th}$-level is disjoint from $\dot{Y}$. Then,
\[
q \cup \{\langle \kappa, (\emptyset, \dot{T}') \rangle \} \cup r_\alpha \leq p \cup \{\langle \kappa, (\emptyset, \dot{T}) \rangle \} \cup r_\alpha
\]
and, clearly, the stronger forces “$\dot{b}_\kappa(n) \notin \dot{Y}$”. Note that this conflicts with our previous comments, hence producing the sought contradiction. \hfill \Box

By Lemma 4.1, this completes the proof of the theorem. \hfill \Box

5. Improving the Upper Bound for the $\omega$-Gluing Property

In this section we would like to improve the large cardinal hypothesis employed in Theorem 4.2. In particular we prove the following result:

**Theorem.** It is consistent, relative to the existence of a measurable with Mitchell’s order $\omega_1$, that there is a cardinal $\kappa$ with the $\omega$-gluing property.

Our final verification of the $\omega$-gluing property will follow the steps of §4. However, we will need to implement some non-trivial modifications to the above argument in order to have a better control upon the $\kappa$-complete ultrafilters that will appear in the intermediate models of the final generic extension (see Lemma 5.8 and Theorem 5.19). To be able to identify these $\kappa$-complete ultrafilters we will rely on the assumption that “$V = \mathcal{K}$” and suppose that there is no inner model of $\exists \alpha, o(\alpha) = \alpha^{++}$.

On one hand, the first of these assumptions (yet natural) is probably unnecessary, as demonstrated by Gitik and Kaplan in [GK23]. On the other hand, our second demand will ensure, by a result of Mitchell [Mit84], (which was extended by Schindler [Sch06]), that every iteration of ultrapowers is normal and that it uses only normal measures.

5.1. Coding $\kappa$-Complete Measures after Adding a Fast Function. The first technical problem that we must address is the lack of a Laver function. Formerly, the existence of such object was granted by our departing large-cardinal assumption; namely, strongness. Instead, this time we would like to force some sort of fast function in a way that we can still keep track over the amount of possible measures. For this, we will consider the following non-stationarily-supported variation of Woodin’s fast function forcing.

Let $\mathbb{S}$ be the poset consisting on partial functions $s: \kappa \to H_\kappa$ such that $\text{dom } s \subseteq \text{Inacc}$ and the following two hold:

- $(\text{dom } s) \cap \beta \in \text{NS}_\beta$ for all $\beta \in \text{Inacc} \cap \kappa$,
• and \( s(\alpha) \in H(\alpha^+) \) for all \( \alpha \in \text{dom} \ s \).

The order of \( \mathcal{S} \) is reverse-inclusion. It is easy to show that forcing with \( \mathcal{S} \) does preserve cardinals (see [Ham00]) and that the generic function introduced generates the entire generic extension.

Normal iterations are going to play a significant role in our arguments so we begin recalling the definition and some basic facts about them.

**Definition 5.1.** A system of elementary embeddings \( \{ \iota_{\alpha,\beta} \mid \alpha \leq \beta \leq \delta \} \) between transitive models is called a linear iteration if the following hold:

1. \( \iota_{\alpha,\beta} : M_\alpha \to M_\beta \),
2. for all \( \alpha \leq \delta \), \( \iota_{\alpha,\alpha} = \text{id} \),
3. for all \( \alpha \leq \beta \leq \gamma \leq \delta \), \( \iota_{\beta,\gamma} \circ \iota_{\alpha,\beta} = \iota_{\alpha,\gamma} \), and
4. for every limit ordinal \( \gamma \leq \delta \),

\[ \langle M_\gamma, \{ \iota_{\alpha,\gamma} \mid \alpha < \gamma \} \rangle \text{ is the direct limit of } \langle \iota_{\alpha,\beta} \mid \alpha \leq \beta < \gamma \rangle. \]

A linear iteration is a normal iteration, using normal (internal) measures if:

5. \( \iota_{\alpha,\alpha+1} \) is an ultrapower embedding by a measure \( U_\alpha \in M_\alpha \) on \( \mu_\alpha \),
6. \( \{ \mu_\alpha \mid \alpha < \delta \} \) is a strictly increasing sequence.

**Lemma 5.2.** Let \( \{ \iota_{\alpha,\beta} \mid \alpha \leq \beta \leq \delta \} \) be a normal iteration using normal measures, with critical points \( \{ \mu_\alpha \mid \alpha < \delta \} \). Then, for every set \( x \in M_\delta \) there is a finite normal iteration \( \iota : M_0 \to M \) together with a factor embedding \( k : M \to M_\delta \) such that \( k \circ \iota = \iota_\delta \) and \( x \in \text{range}(k) \).

**Proof.** A straightforward induction on \( \delta \) shows that, for every \( x \in M_\gamma \) (\( \gamma \leq \delta \)) there is \( f \in M_0 \) and ordinals \( \alpha_0 < \cdots < \alpha_{n-1} < \gamma \) such that

\[ \iota_\gamma(f)(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{n-1}}) = x. \]

For every \( \alpha < \delta \), let us pick a representation of \( U_\alpha \); namely

\[ U_\alpha = \iota_\alpha(g_\alpha)(\mu_{\beta^\alpha_0}, \ldots, \mu_{\beta^\alpha_{m-1}}), \]

where each \( \beta^\alpha_i \) is below \( \alpha \).

Let us say that an increasing sequence of ordinals \( \bar{a} \in [\delta]^{<\omega} \) is saturated if \( \{ \beta^\alpha_0, \ldots, \beta^\alpha_{m-1} \} \subseteq \bar{a} \) for all \( \alpha \in \bar{a} \). Clearly, every set \( b \in [\delta]^{<\omega} \) is a subset of a finite saturated set \( \bar{a} \); this is because the generators corresponding to each \( \alpha \in \bar{a} \) are strictly below \( \alpha \). In particular, given \( x \in M_\delta \) there is \( \bar{a} = \{ \alpha_0, \ldots, \alpha_{n-1} \} \) a saturated set and a function \( f \in M_0 \) such that

\[ \iota_\delta(f)(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{n-1}}) = x. \]

Let us construct a finite iteration \( \langle \iota_{k,m} \mid k \leq m \leq n \rangle \) witnessing the validity of the lemma. For each relevant \( i \), denote by \( \tilde{\mu}_i \) the critical point of \( \iota_{i,i+1} \). We will construct by induction elementary embeddings \( \tilde{k}_i \) such that \( \tilde{k}_i \circ \tilde{i}_i = \iota_{\alpha_i} \) and moreover \( \tilde{k}_i(\tilde{\mu}_j) = \mu_{\alpha_j} \), for all \( j < i \).

\[ ^7 \text{The restriction of } s(\alpha) \text{ to } H(\alpha^+) \text{ is inessential at this part.} \]

\[ ^8 \text{As customary, } \iota_\alpha \text{ will be a shorthand for } \iota_{0,\alpha}. \]
Since $\tilde{a}$ is saturated, for every $i < n$, the measure $U_{\alpha_i}$ is represented by generators whose indexes belong to $\tilde{a} \cap \alpha_i$. Let $b_i$ be this set of indexes. Let $\bar{\mu} = \langle \bar{\mu}_0, \ldots, \bar{\mu}_{n-1} \rangle$ (note that each time only an initial segment of this sequence is used in order to define the next step). For simplicity of notations, let us assume that $\delta = \alpha_n$. For $i = 0$, let $\tilde{k}_0 = \iota_{a_0}$. Let us assume that $\tilde{k}_i, \iota_i$ and $\bar{\mu} \upharpoonright i$ are defined. Let $\tilde{U}_i = \tilde{i}_i(g_{\alpha_i})(\bar{\mu} \upharpoonright b_i)$. By the inductive hypothesis,

$$k_i(\tilde{U}_i) = \iota_{\alpha_i}(g_{\alpha_i})(\mu_{\alpha_0}^{\alpha_i}, \ldots, \mu_{\alpha_{i-1}}^{\alpha_i}) = U_{\alpha_i}.$$

In particular, $\tilde{U}_i$ is a normal measure on $\bar{\mu}_i$ such that $k_i(\bar{\mu}_i) = \mu_{\alpha_i}$.

Let

$$\tilde{k}_{i+1}(\tilde{U}_i) := \iota_{\alpha_i+1, \alpha_i+1}(\iota_{\alpha_i, \alpha_i+1}(k_i(r))(\mu_{\alpha_i})).$$

Since $\tilde{k}_i(\tilde{U}_i) = U_{\alpha_i}$, it is standard to verify that $k_{i+1}$ is well defined and elementary. Moreover, $\iota_{\alpha_i+1, \alpha_i+1} \circ \bar{i}_{i+1} = \iota_{\alpha_{i+1}}$. We use the assumption that the iteration is normal, in order to know that $\iota_{\alpha_{i+1}, \alpha_{i+1}}$ does not move $\mu_{\alpha_i}$.

Finally, $\tilde{k}_n \circ \bar{i}_n = \iota_\delta$, so we set $k = \tilde{k}_n$.

Let us show that $x \in \text{range } k$. Indeed, without loss of generality,

$$x = \iota_\delta(f)(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{n-1}}) = k \circ \bar{i}_\delta(k(\bar{\mu})) = k((\iota(f)(\bar{\mu}))).$$

In the forthcoming Lemma 5.6 we will characterize the measures in $\mathcal{K}[S]$ where $S \subseteq \mathbb{S}$ is a $\mathcal{K}$-generic. The next general lemma will be instrumental for this purpose:

**Lemma 5.3.** Suppose that $\iota : \mathcal{M} \to \mathcal{N}$ is the direct limit of a normal iteration $\langle \iota_{\alpha, \beta} \mid \alpha \leq \beta < \delta \rangle$ of measures with critical points $\langle \mu_\alpha \mid \alpha < \delta \rangle$, and let $\mathbb{P} \in \mathcal{M}$ be a forcing notion with the following properties:

1. Every condition $p \in \mathbb{P}$ is a partial function $p : \mu_0 \to \mathcal{M}$;
2. If $p \in \mathbb{P}$ then for every $\mathcal{M}$-inaccessible $\beta \leq \mu_0$, $(\text{dom } p) \cap \beta \in \text{NS}^\mathcal{M}_\beta$;
3. For a condition $p \in \mathbb{P}$ and a nowhere stationary set $T \in \text{NS}^\mathcal{M}_{\mu_0}$ there is $q \leq p$ such that $T \subseteq \text{dom } q$.

Then, for every $\mathcal{M}$-generic filter $G \subseteq \mathbb{P}$ we have that

$$\bigcup_{p \in G} \iota(\text{dom}(p)) = \iota(\mu_0) \setminus \langle \mu_\alpha \mid \alpha < \delta \rangle.$$

**Proof.** Denote by $G_i$ the left-hand-side of the above-displayed equation. Here we just deal with the case where $\iota$ is a finite linear iteration with critical points $\langle \mu_0, \ldots, \mu_k \rangle$. The general case is handled similarly, using that any linear iteration is a directed limit of finite iterations, as demonstrated in Lemma 5.2.

The crux of the matter is the following claim:

**Claim 5.3.1.** For every $\alpha \in \iota(\mu_0) \setminus \{\mu_0, \ldots, \mu_k\}$ there is a nowhere stationary set $T \subseteq \mu_0$ with $\alpha \in \iota(T)$.

Using this latter fact it is easy to derive the desired conclusion:

**Claim 5.3.2.** $\text{dom } G_i = \iota(\mu_0) \setminus \{\mu_0, \ldots, \mu_k\}$. 
Proof of claim. Fix $\alpha \in \iota(\mu_0) \setminus \{\mu_0, \ldots, \mu_k\}$ and put
\[ D_\alpha := \{p \in \mathbb{P} \mid \alpha \in \iota(\text{dom}(p))\}. \]
We show that $D_\alpha$ is dense in $\mathbb{P}$. Indeed, let $p \notin D_\alpha$ and use Claim 5.3.1 in order to find a nowhere stationary set $T \subseteq \mu_0$ such that $\alpha \in \iota(T)$. Using Clause (3) above we find $q \leq p$ with $T \subseteq \text{dom} \ q$. Thus, $\iota(\mu_0) \setminus \{\mu_0, \ldots, \mu_k\} \subseteq G_i$.

Conversely, we claim that $G_i \cap \{\mu_0, \ldots, \mu_k\} = \emptyset$. For this fix $i \leq k$ and suppose towards a contradiction that $\mu_i \in G_i$. Let $p \in G$ such that $\mu_i \in \iota(T)$ where $T := \text{dom} \ p$. Notice that $\mu_i \notin \iota_{i+1}(T)$: By elementarity, $\iota_i(T)$ is nowhere stationary, hence $\iota_i(T) \cap \mu_i \in \text{NS}_{\mu_i}^\kappa$. Also, if $\mu_i \in \iota_{i+1}(T)$ then normality of the iteration would imply that $\iota_i(T) \cap \mu_i$ is stationary in $\mu_i$.

Therefore $\mu_i \notin \iota(T)$, yielding the sought contradiction.

So, let us prove Claim 5.3.1 above. Fix $\alpha < \iota(\mu_0)$ and let us find a nowhere stationary set $T \subseteq \mu_0$ such that $\alpha \in \iota(T)$. To avoid trivialities let us further assume that $\alpha \geq \mu_0$. First, $\alpha = \iota(g)(\mu_0, \ldots, \mu_{n-1})$ for some $g: \mu_0^\kappa_0 \rightarrow \mu_0$ and $n \leq k$. Assume in addition that $n$ is the minimal such index.\footnote{In particular, $\mu_{n-1} < \alpha$.}

Define $B := \{\bar{x} \in \mu_0^n \mid \max(\bar{x}) < g(\bar{x})\}$ and $T := g^\ast B$. Clearly, $(\mu_0, \ldots, \mu_{n-1}) \in \iota(B)$. Thus, $\alpha = \iota(g)(\mu_0, \ldots, \mu_{n-1}) \in \iota(T)$.

We claim that $T$ is nowhere stationary. Suppose towards a contradiction that $\rho < \mu_0$ is a regular cardinal such that $T \cap \rho$ is stationary. Then every $\alpha \in T \cap \rho$ is of the form $g(\bar{x})$ for some $\bar{x} \in B \cap \rho^n$. Let $B_\rho \subseteq B \cap \rho^n$ such that $T \cap \rho = g^\ast B_\rho$. Next, let $h_\rho: T \cap \rho \rightarrow \rho$ be given by
\[ \beta \mapsto \min\{\max(\bar{x}) \mid g(\bar{x}) = \beta\}. \]
This map is well-defined and regressive, by the definition of $B$. Thus, there is $T_\rho \subseteq T \cap \rho$ stationary and $\eta < \rho$ with $h_\rho^\ast T_\rho = \{\eta\}$. This latter fact outright implies $T_\rho \subseteq g^\ast \eta$, which is not impossible in that $|g^\ast \eta| = |\eta| < \rho$. This completes the proof of Lemma 5.3.

Remark 5.4. The proof of Claim 5.3.1 shows something slightly stronger: for every ordinal $\alpha$ there is a set $T \subseteq \kappa$ such that $\alpha \in \iota(T)$ and there is a regressive function $f: T \rightarrow \kappa$ such that for every $\rho$, $|f^{-1}(\{\rho\})| \leq |\rho| + \aleph_0$.

The next strengthening of Claim 5.3.1 comes handy in the proof:

Lemma 5.5. Let $\iota: \mathcal{M} \rightarrow \mathbb{N}$ be a finite normal iteration with critical points $\bar{\mu} = \langle \mu_i \mid i < n \rangle$. For every nowhere stationary $T \in \mathcal{P}(i(\kappa))^N$ with $T \cap \bar{\mu} = \emptyset$ there is another nowhere stationary $T \in \mathcal{P}(\kappa)^\mathcal{M}$ such that $T \subseteq \iota(T)$

Proof. Put $\mu_n := \iota(\kappa)$. Since $T \in \mathcal{P}(i(\kappa))^N$, $T$ decomposes as $T = \bigcup_{i < n} T_i$ where $T_i := T \cap [\mu_i, \mu_{i+1})$. Clearly, each $T_i$ is nowhere stationary in $\mu_{i+1}$.

To establish the lemma it will suffice to find, for each $i < n$, a nowhere stationary set $\bar{T}_i \in \mathcal{P}(\kappa)^\mathcal{M}$ with $T_i \subseteq \iota_{i+1}(\bar{T}_i)$. So, fix $i < n$. 
Let \( g: \kappa^{i+1} \to \mathcal{P}(\kappa)^M \) be a map in \( \mathcal{M} \) representing \( T_i \); namely, \( T_i = \iota(g)(\mu_0, \ldots, \mu_i) \). Without loss of generality, \( g(\bar{x}) \) is nowhere stationary for all \( \bar{x} \in \kappa^{i+1} \). Define \( B := \{ \bar{x} \in \kappa^{i+1} \mid \max(\bar{x}) < g(\bar{x}) \} \) and
\[
T_i := \bigcup \{ g(\bar{x}) \backslash \max(\bar{x}) + 1 \mid \bar{x} \in B \}.
\]
Clearly, \((\mu_0, \ldots, \mu_i) \in \iota_{i+1}(B)\) so that, as \( T \cap \bar{\mu} = \emptyset \),
\[
\iota_{i+1}(T_i) \supseteq \iota(g)(\mu_0, \ldots, \mu_i) \backslash \mu_i + 1 = T_i \backslash (\mu_i + 1) = T_i.
\]
Also, \( T_i \) is nowhere stationary by the same argument as in Claim 5.3.1. \( \square \)

The following serves as a warm-up for Lemma 5.8 and provides a complete cartography of the measures in the generic extension \( \mathcal{K}[S] \):

**Lemma 5.6.** Let \( S \subseteq S \) be a \( \mathcal{K} \)-generic filter and \( \mathcal{U} \) a \( \kappa \)-complete ultrafilter on \( \kappa \) in \( \mathcal{K}[S] \). Then, there is a finite iteration \( \langle i_{m,n} \mid m \leq n \leq k + 1 \rangle \) of measures in \( \mathcal{K} \) with critical points \( \langle \mu_0, \ldots, \mu_k \rangle \), \( \kappa = \mu_0 < \mu_1 < \cdots < \mu_k \), a function \( f: \kappa^{k+1} \to \kappa \) in \( \mathcal{K} \) and \( \langle a_0, \ldots, a_k \rangle \in \prod_{i \leq k} H(\mu_i^+) \) such that
\[
\mathcal{U} = \{ \hat{X}^S \subseteq \kappa \mid \exists p \in S (\iota(p) \cup \{ \langle \mu_i, a_i \rangle \mid i \leq k \}|_{\iota(S)} \models \epsilon \in \iota(\hat{X})^S) \},
\]
where \( \epsilon = \iota_k(f)(\mu_0, \ldots, \mu_k) \).

**Proof.** Let \( \mathcal{U} \in \mathcal{K}[S] \) be a \( \kappa \)-complete ultrafilter on \( \kappa \) and \( j: \mathcal{K}[S] \to M \) be the corresponding ultrapower embedding. By our anti-large-cardinal hypothesis, \( j \upharpoonright \mathcal{K} \) is an iteration of normal measures in \( \mathcal{K} \). Let \( \epsilon := [\text{id}]_{\mathcal{U}} \).

**Claim 5.6.1.** \( j \upharpoonright \mathcal{K} \) is a finite iteration.

**Proof of claim.** We commence observing that \( M = \mathcal{K}^M[S'] \) for \( S' \subseteq j(S) \) a \( \mathcal{K}^M \)-generic filter. First, by elementarity, \( M = N[S'] \) for some \( N \)-generic filter \( S' \subseteq j(S) \). Second, by absoluteness of the core model, \( \mathcal{K}^{\mathcal{K}[S]} = \mathcal{K} \), and so \( j(\mathcal{K}) = \mathcal{K}^M \). Finally, by the Laver-Woodin theorem on definability of the ground model [Lav07, Woo11], “\( \mathcal{K} \) is the ground model of \( \mathcal{K}[S] \)” is first-order-expressible in \( \mathcal{K}[S] \), hence \( N = \mathcal{K}^M \) is the ground model of \( M \).

We are now in conditions to argue that \( j \upharpoonright \mathcal{K} \) is a finite iteration. Clearly, \( M \) is closed under \( \kappa \)-sequences in \( \mathcal{K}[S] \), hence also closed under \( \omega \)-sequences. Since \( j(S) \) is \( \omega_1 \)-closed it must be the case that \( \bar{x} \in {}^\omega M \cap \mathcal{K}[S] \subseteq \mathcal{K}^M \). However, if \( j \upharpoonright \mathcal{K} \) is an infinite iteration then the sequence of its first \( \omega \)-many critical points \( \bar{x} = \langle \mu_n \mid n < \omega \rangle \) belongs to \( M^\omega \cap \mathcal{K}[S] \) but not to \( \mathcal{K}^M \). \( \square \)

So, let \( \iota = \langle i_{m,n}: \mathcal{K}_m \to \mathcal{K}_n \mid m \leq n \leq k + 1 \rangle \) be this iteration. For each \( i \leq k \) put \( \mu_i := \text{crit}(i_{i,i+1}) \). Without loss of generality \( \iota \) is normal.

We would like to analyse the \( \mathcal{K}^M \)-generic filter \( S' \), or equivalently, the function \( \bigcup S' \). Let us consider
\[
G_{i_{k+1}} := \bigcup \{ j(\text{dom}(s)) \mid s \in S \}.
\]
Clearly, \( S \) satisfies the assumptions of Lemma 5.3 and thus \( G_{i_{k+1}} = \iota_{k+1}(\kappa) \sans \{ \mu_0, \ldots, \mu_k \} \). Since \( \mathcal{U} \) is the measure induced by \( j \) we have that
\[
\mathcal{U} = \{ \hat{X}^S \subseteq \kappa \mid \exists q \in S' (q|_{\iota(S)} \models \epsilon \in j(\hat{X})^S) \}.
\]
Let $f: \kappa^{k+1} \to \kappa$ be such that $\iota_k(f)(\mu_0, \ldots, \mu_k) = \epsilon$. Using Lemma 5.3 we find $\langle a_0, \ldots, a_k \rangle \in \left( \prod_{i < k} H(\mu^+_i) \right)^M$ such that for every condition $q \in S'$ there is $p \in S$, $q \geq j(p) \cup \{ (\mu_i, a_i) \mid j \leq k \}$. Therefore,

$$U = \{ \hat{X}^S \subseteq \kappa \mid \exists p \in S (j(p) \cup \{ (\mu_i, a_i) \mid i \leq k \}) \models j(S) \epsilon \in j(\hat{X}^S) \},$$

which completes the proof. \hfill \square

The following useful observation arises from the previous proof:

**Lemma 5.7.** Suppose $j: \mathcal{K}[S] \to M$ is an arbitrary elementary embedding such that $\iota = j \restriction \mathcal{K}$ is a normal iteration of measures in $\mathcal{K}$, say $\langle \iota_{\alpha, \beta} \mid \alpha \leq \beta \leq \delta \rangle$. Then, there is an iteration $\langle j_{\alpha, \beta} \mid \alpha \leq \beta \leq \gamma \rangle$ such that each embedding $j_{\alpha, \beta}$ lifts the corresponding embedding $\iota_{\alpha, \beta}$.

**Proof.** Let us prove by induction that each embedding $\iota_{\alpha, \alpha+1}: \mathcal{K}_\alpha \to \mathcal{K}_{\alpha+1}$ extends to an embedding $j_{\alpha, \alpha+1}: \mathcal{K}_\alpha[j(S) \restriction \mu_\alpha] \to \mathcal{K}_{\alpha+1}[j(S) \restriction \mu_{\alpha+1}]$.

Indeed, by the arguments of Lemma 5.6, in order to lift this embedding, we must pick a value for the point $\text{crit}(\iota_{\alpha, \alpha+1})$ in the generic filter for $\iota_{\alpha+1}(S)$. This value is fully determined by the generic $j_{\alpha, \alpha+1}(S)$. \hfill \square

The next lemma characterizes $\kappa$-complete measures in $\mathcal{K}[S]$-generic extensions by non-stationary support iterations of Tree Prikry forcings. This will replace the nice-name-argument used in Theorem 4.2 when claiming that the Laver function captures our $\omega$-sequence of $\mathbb{P}_\kappa$-names for ultrafilters.

Specifically, Lemma 5.8 shows that every measure $U \in \mathcal{K}[S][G]$ admits a code in $H(\kappa^+)$, hence in the range of the generic Laver function. This is necessary, as in the present context $\mathbb{P}_\kappa$ is not $\kappa$-cc nor there are $(\kappa+2)$-strong embeddings capturing the names for the ultrafilters.

**Lemma 5.8 (Coding Lemma).** Let $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, Q_\beta \mid \alpha < \beta < \kappa \rangle$ be a non-stationary-supported iteration of tree Prikry forcings in $\mathcal{K}[S]$. Assume that, for each $\alpha < \kappa$, the iteration has the following properties:

1. $|\mathbb{P}_\alpha| \leq 2^\alpha$ and $1 \models_{\mathbb{P}_\alpha} \langle Q_\alpha, \leq^* \rangle$ is $\alpha$-closed;
2. $1 \models_{\mathbb{P}_\alpha} \forall p,q \in Q_\alpha \text{ compatible } p \land q$ exists.\hfill (1)

For each generic filter $G \subseteq \mathbb{P}_\kappa$ there are $\kappa^+$-many $\kappa$-complete ultrafilters in $\mathcal{K}[S][G]$. Moreover, for each such ultrafilter $U \in \mathcal{K}[S][G]$ there are

- (a) a finite sub-iteration $\iota: \mathcal{K}[S] \to \mathcal{K}^{M}[\iota(S)]$ of $j_U \restriction \mathcal{K}[S]$,
- (b) an ordinal $\bar{\epsilon} < \iota(\kappa)$ with $\bar{\epsilon} \in \text{range}(k)$,
- (c) $r \in \iota(S \ast \mathbb{P}_\kappa)$ with finite support compatible with all conditions in $\iota$" $G$.

Moreover, for every $p \in G$, $\iota(p) \land r$, and for each $p \in G$, $p \models_{\mathbb{P}_\kappa} \langle \hat{X} \in U \rangle$ if and only if

$$\exists q \in \iota(\mathbb{P}_\kappa)(q \leq^* \iota_\alpha(p) \land r \land \text{supp}(q) = \text{supp}(\iota(p) \land r) \land q \models_{\mathbb{P}_\kappa} \bar{\epsilon} \in \iota_\alpha(\hat{X})).$$

---

\textsuperscript{10} Note that $M$ might not be closed under countable sequences.

\textsuperscript{11} Here $p \land q$ stands for the $\leq^*$-greatest condition $r \leq p, q$, if such a condition exists.
During the proof of the lemma we will be able to describe $r$ explicitly, based on the finite iteration $\iota$ (see Definition 5.14). Nevertheless, we do not know whether in general $r$ can be computed from the iteration $\iota$ itself.

We postpone the proof of this lemma to the end of §5.3 (see p. 32).

5.2. Interlude on non-stationary-supported iterations. In the road to prove Lemma 5.8 we will need some abstract results about non-stationary-supported iterations. Essentially, we want a lemma saying the following: Given a finite normal iteration $\iota: \mathcal{M} \to \mathcal{N}$, a non-stationary supported iteration $P \in \mathcal{M}$, and a dense open set $D \in \mathcal{N}$ for $\iota(P)$, then it suffices to perform a finite modification of the conditions in $\iota"G$ to enter $D$.

To make this precise we introduce the following handy notation:

**Definition 5.9.** Let $P_\kappa = \langle P_\alpha, Q_\beta \mid \beta < \alpha < \kappa \rangle$ be an iteration of forcings. For $p, q \in P_\kappa$ and $\Gamma \subseteq \kappa$, write $q \leq \Gamma p$ if $q \leq p$ and
\[
\{ \alpha < \kappa \mid q \upharpoonright \alpha \upharpoonright P_\alpha p(\alpha) = q(\alpha) \} \subseteq \Gamma.
\]

**Remark 5.10.** Notice that for $p, q \in P_\kappa$, if $q \leq \Gamma p$ then $\text{supp } q \subseteq \text{supp } p \cup \Gamma$.

We will eventually apply the forthcoming lemma to a Prikry-type iteration of Tree Prikry forcings endowed with its direct extension order $\leq^*$. For an exposition on non-stationary-supported iterations of Prikry-type forcings we refer the reader to Ben-Neria and Unger’s paper [BNU17].

**Lemma 5.11.** Let $P_\kappa = \langle P_\alpha, Q_\beta \mid \beta < \alpha < \kappa \rangle$ be a non-stationary-supported iteration satisfying, for each $\alpha < \kappa$, the following properties:

1. $|P_\alpha| \leq 2^\alpha$ and $Q_\alpha$ is trivial unless $\alpha$ is inaccessible;
2. $1 \models_{P_\alpha} \langle Q_\alpha, \leq \rangle$ is $\alpha$-closed;
3. $1 \models_{P_\alpha} \forall p, q \in Q_\alpha$ compatible $p \land q$ exists.

Let $\iota: \mathcal{M} \to \mathcal{N}$ be a finite iteration of normal measures and consider $\langle \iota_{k, \ell} \mid k \leq \ell \leq n \rangle$ the collection of all sub-iterations. Put
\[
\mu_i := \text{crit}(\iota_{i,i+1}) \text{ and } \Gamma := \{ \iota_{k, \ell}(\mu_k) \mid k \leq \ell \leq n \}.
\]

Let $D \in \mathcal{N}$ be a dense open subset of $\iota(P_\kappa)$ and $r \in \iota(P_\kappa)$ a condition with $\text{supp } r \subseteq \Gamma$. Let us assume, in addition, that $p \in P_\kappa$ is a condition such that, for each $q \leq p$, $r$ is compatible with $\iota(q)$.

Then, there is a dense open set $E \subseteq P_\kappa$ below $p$ such that for every $q \in E$,

1. $\iota(q) \land r$ exists, hence $\iota(q) \land r \leq \Gamma \iota(q)$;
2. there is $q' \leq \Gamma \iota(q) \land r$ such that $q' \in D$.

**Proof.** Before addressing the proof of the lemma we need some preliminary considerations. Given $u, v \in P_\kappa$ with disjoint supports we write
\[
(u \cup v)(\alpha) := \begin{cases} u(\alpha), & \alpha \in \text{supp } u; \\ v(\alpha), & \alpha \in \text{supp } v. \end{cases}
\]

Clearly, $u \cup v \in P_\kappa$. Similarly, if $u \in P_\kappa$ and $X \cup Y \supseteq \text{supp } u$ with $X, Y$ disjoint, we can define $u \upharpoonright X, u \upharpoonright Y$ in the natural way so that
\[
u = (u \upharpoonright X) \cup (u \upharpoonright Y).
\]
Claim 5.11.2. Let \( p \leq D \) and \( \alpha \) be our non-stationary-supported iteration, \( \beta \) be a finite set of inaccessible cardinals. Put \( \Gamma := \{ \gamma_0, \ldots, \gamma_{m-1} \} \) and put \( \gamma_{-1} := 0 \) and \( \gamma_m := \iota(\kappa) \).

By standard facts about the term-space forcing there is a canonical projection from \( A(\mathbb{I}, \mathbb{R}_0) \times (\prod_{\alpha=0}^{\alpha} A(\mathbb{R}_n, \iota(\mathbb{P}_\gamma)_{\alpha})) \times A(\mathbb{R}_m, \{ \mathbb{I} \}) \) to \( \iota(\mathbb{P}_\kappa) \), where \( \mathbb{R}_n \) is the iteration \( \iota(\mathbb{P}_\kappa) \). In particular, there is \( D_1 \times (\prod_{\alpha=0}^{\alpha} D_n) \times D_m \) which is dense and open in this product and that it is contained in \( D \).

Let \( q' \leq_{(\iota(\kappa))} q \). By induction on \( n \in [-1, m] \) define \( r \leq_{(\iota(\kappa))} q \) such that \( q'' := r \upharpoonright \gamma_0 \in (s(\gamma_0)) \upharpoonright (\gamma_0, \gamma_1) \in (s(\gamma_1)) \upharpoonright (\gamma_m-1, \gamma_m) \) is a condition where \( r \upharpoonright \gamma_0 \in D_1, (s(\gamma_1)) \upharpoonright (\gamma_m-1, \gamma_m) \in D_m \). This is possible by density and openness of the \( D_n \)'s. Clearly, \( q'' \leq_{\Gamma} r \) and \( q'' \in D \). Therefore there is a \( \leq_{(\iota(\kappa))} q' \) -dense open set \( D' \) as in the claim.

By the claim, it is enough to show that for every \( \leq_{(\iota(\kappa))} q' \) -dense open set \( D \in \mathcal{N} \) the set \( E \subseteq \mathbb{P} \) of all conditions \( q \) such that \( \iota(q) \land r \in D \) is dense: Indeed, fix \( D \in \mathcal{N} \) and \( r \in \iota(\mathbb{P}_\kappa) \) be as in the lemma. Invoking Claim 5.11.1 with respect to \( D \) we get in return a \( \leq_{(\iota(\kappa))} q' \) -dense open set \( D' \) with the above-mentioned property. Using our assumption we have that the set \( E \subseteq \mathbb{P} \) of all \( q \)'s such that \( \iota(q) \land r \in D' \) is dense. By the properties of \( D' \) there is a condition \( q' \leq_{\Gamma} \iota(q) \land r \in D \), so we are done.

We will eventually prove the lemma by induction on the length of the iteration. Let us first deal with the case of a single ultrapower. In order to make the inductive hypothesis run smoothly we are going to tweak it slightly. Specifically, the set \( \Gamma \) of the forthcoming proof will contain the critical points of the embedding, their images and other possible cardinals.

So, let \( \iota : \mathcal{M} \rightarrow \mathcal{N} \) be a normal ultrapower embedding with critical point \( \mu \) and let \( \Gamma' \) be a finite set of inaccessible cardinals. Put \( \Gamma := \iota(\Gamma') \cup \{ \mu, \iota(\mu) \} \). Let \( \mathbb{P}_\kappa \) be our non-stationary-supported iteration, \( r \in \iota(\mathbb{P}_\kappa) \) with \( \text{supp} r = \Gamma \) and \( D \subseteq \iota(\mathbb{P}_\kappa) \) a dense open set with respect to \( \leq_{(\iota(\kappa))} \Gamma \).

Claim 5.11.2. Let \( E \) be the collection of all conditions \( q \) such that

1. either there is \( \alpha \in \Gamma \) such that \( \iota(q) \land r \leq_{\Gamma} \iota(q)(\alpha) \) is incompatible with \( r(\alpha) \)
2. or \( \iota(q) \land r \in D \).

Then \( E \) is \( \leq_{(\iota(\kappa))} \Gamma \) -dense open.

Proving the claim suffices to infer the first step of the induction: In effect, suppose that \( p \in \mathbb{P}_\kappa \) is such that \( \iota(q) \) is compatible with \( r \) for all conditions \( q \leq p \). In that case, \( E \cap (\mathbb{P}_\kappa/p) \) is a dense open set \( \leq \)-below \( p \) consisting of conditions \( q \) for which clause (2) above holds.

\text{References:}
\text{[Cum10, Proposition 2.25].}
\text{[Formally speaking, D contains the image of D_1 \times (\prod_{\alpha=0}^{\alpha} D_n) \times D_m under the canonical projection.]}
Proof of Claim 5.11.2. \( E \) is open in \( \leq (\kappa \setminus \Gamma \cup \{\mu\}) \) so we shall show it is dense.

Fix \( d \) and \( \bar{r} \) be functions with domain \( \mu \) representing \( D \) and \( r \), respectively; i.e., \( \iota(d)(\mu) = D \) and \( \iota(\bar{r})(\mu) = r \). Without loss of generality we may assume that, for each \( \alpha < \mu \), \( \text{supp}\, \bar{r}(\alpha) = \Gamma' \cup \{\alpha, \mu\} \) and that \( d(\alpha) \leq \kappa \setminus \Gamma \cup \{\alpha, \mu\} \)-dense open. In what follows \( q \) will be a fixed condition in \( \mathbb{P}_\kappa \).

We inductively define a \( \leq \kappa \setminus \Gamma \cup \{\mu\} \)-decreasing sequence \( \langle q_\alpha \mid \alpha < \mu \rangle \subseteq \mathbb{P}_\kappa \) below \( q \) together with a continuously-decreasing sequence of clubs \( \langle C_\alpha \mid \alpha < \mu \rangle \) on \( \mu \) such that \( (\text{supp}\, q_\alpha) \cap C_\alpha = \emptyset \). For each \( \alpha < \mu \) we stipulate that

\[
\gamma_\alpha := \min(C_\alpha \setminus (\cup_{\beta < \alpha}^\text{sup} \gamma_\beta) + 1).
\]

Our mission is to find a dense subset of conditions \( e(\gamma_\alpha) \subseteq \mathbb{P}_{\gamma_\alpha + 1} \) with the following property: for all \( s \in e(\gamma_\alpha) \) such that \( s \leq q_\alpha \upharpoonright \gamma_\alpha + 1 \), put \( q'_s := s \cup q_\alpha \upharpoonright [\gamma_\alpha + 1, \kappa) \). Then,

- \( (\alpha) \) either \( q'_s \) is incompatible with \( \bar{r}(\gamma_\alpha) \),
- \( (\beta) \) or \( q'_s \wedge \bar{r}(\gamma_\alpha) \) exists, \( q'_s \wedge \bar{r}(\gamma_\alpha) \leq \Gamma' \cup \{\alpha, \mu\} \) \( q'_s \) and \( q'_s \wedge \bar{r}(\gamma_\alpha) \in d(\gamma_\alpha) \).

We will inductively maintain that \( q_\alpha \upharpoonright \gamma_\alpha + 1 = q_\beta \upharpoonright \gamma_\alpha + 1 \) for all \( \alpha < \beta \). In particular, \( \gamma_\alpha \) will never be in the support of \( q_\beta \), as \( (\text{supp}\, q_\alpha) \cap C_\alpha = \emptyset \). Moreover, we will assume that \( q_\alpha \) agrees with \( q \) on \( \Gamma' \cup \{\mu\} \). We proceed with the construction of \( q_\alpha \) in a fusion-like fashion.

- Set \( q_0 := q \) and let \( C_0 \subseteq \mu \) be any club disjoint from \( \text{supp}\, q_0 \).

- Assume that \( \langle q_\alpha \mid \alpha < \beta \rangle \) and \( \langle C_\alpha \mid \alpha < \beta \rangle \) were constructed for some \( \beta > 0 \). If \( \beta \) is limit then, by our inductive assumption, there is a condition \( q_\beta^* \) that serves as a lower bound for \( \langle q_\alpha \mid \alpha < \beta \rangle \) and whose support is disjoint from \( \bigcap_{\alpha < \beta} C_\alpha \). Note that this latter is a club for all \( \beta < \mu \). Likewise, for \( \beta = \mu \), the assumption on freezing initial segments ensures that this lower bound exists and that its support is disjoint from \( \Delta_{\beta < \mu} C_\alpha \).

Let \( \beta < \mu \) be arbitrary. Let us show how to define \( q_\beta \) together with the dense set \( e(\gamma_\beta) \). Let us consider \( \langle s_\zeta \mid \zeta < \zeta_* \rangle \), an enumeration of all the conditions in \( \mathbb{P}_{\gamma_\beta + 1} \) which are below \( q_\beta^* \upharpoonright \gamma_\beta + 1 \). By our departing assumption upon \( \mathbb{P}_\kappa \), \( \zeta_* \leq \max(\beta_\eta, \kappa) \), which is below the degree of closure of \( \mathbb{P}_\kappa / \mathbb{P}_{\gamma_\beta + 1} \). This enable us to define a decreasing sequence \( \langle q_\zeta^\beta \mid \zeta < \zeta_* \rangle \) as follows:

- If \( s_0 \cup q_\beta^* \upharpoonright [\gamma_\beta + 1, \kappa) \) is incompatible with \( \bar{r}(\gamma_\beta) \) then stipulate \( q_0^\beta := q_\beta^* \).
- Otherwise, \( (s_0 \cup q_\beta^* \upharpoonright [\gamma_\beta + 1, \kappa)) \wedge \bar{r}(\gamma_\beta) \) exists and we can \( \leq \kappa \setminus \Gamma' \cup \{\alpha, \mu\} \)-extend it to a condition \( q'' = d(\gamma_\beta) \). In this case define

\[
q_0^\beta := (q_\beta^* \upharpoonright \gamma_\beta + 1) \cup (q'' \upharpoonright [\gamma_\beta + 1, \kappa)).
\]

- In general, let \( r_\eta \) be a lower bound for \( \langle q_\zeta^\beta \mid \zeta < \eta \rangle \) and proceed exactly as before with respect to the condition \( s_\eta \cup r_\eta \upharpoonright [\gamma_\beta + 1, \kappa) \).

Finally, let \( q_\beta \) be the condition

\[
q_\beta \upharpoonright (\gamma_\beta + 1 \cup \Gamma' \cup \{\mu\}) \cup q_\beta^* \upharpoonright (\left[\gamma_\beta + 1, \kappa\right) \setminus \Gamma'),
\]
and $e(\gamma_\beta)$ be the dense open set defined as

$$\{s \mid s \subseteq (q_\beta^\beta \cup \gamma_\beta + 1)\} \cup$$

$$\{s \mid \exists \xi (s = s_\xi \land s \cup q_\beta^\beta \cup [\gamma_\beta + 1, \kappa] \text{ witnesses } (\alpha) \text{ or } (\beta))\}.$$  

Applying $\iota$, for densely many $s \in \iota(e)(\mu)$, which are below $q_\mu \mid [\mu + 1 = \iota(q_\mu) \mid [\mu + 1$, the condition $s \cup \iota(q_\mu) \mid [\mu + 1, \iota(\kappa)]$ satisfies the requirements of the claim (here we use the fact that $\iota(q_\mu)$ is stronger than $\iota(q_\mu)$). But $s \leq q_\mu \mid [\mu + 1$ and therefore $s \cup q_\mu \mid [\mu, \kappa] \leq q$ belongs to $E$.  

As mentioned earlier we will prove the lemma by induction on the length of the iteration $\iota: \mathcal{M} \to \mathcal{N}$. Recall that it is enough to show it for $\leq_{(\kappa)}^\Gamma$-dense open sets. So assume by induction that the previous claim holds for iterations of length $n - 1$. Let $\iota: \mathcal{M} \to \mathcal{N}$ be an iteration of length $n$, $D \in \mathcal{N}$ a $\leq_{(\kappa)}^\Gamma$-dense open set and $r \in \iota(\mathbb{P}_n)$ with $\text{supp } r \subseteq \Gamma$. Let $\mu$ be the last critical point of $\iota$ and denote by $\iota_{n-1}: \mathcal{M} \to \mathcal{N}_{n-1}$ and $\iota_{n-1,n}: \mathcal{N}_{n-1} \to \mathcal{N}_n$ the first $n - 1$ ultrapowers and the last ultrapower of $\iota$, respectively.

Let $\Gamma_{n-1}$ be the collection of critical points of $\iota_{n-1}$ together with their images. Then $\Gamma = \iota_{n-1,n}(\Gamma_{n-1}) \cup \{\mu, \iota_{n-1,n}(\mu)\}$. Applying Claim 5.11.2 to $\iota_{n-1,n}$ we get a $\leq_{(\kappa)}^\Gamma \Gamma_{n-1} \cup \{\mu_{n-1}\}$-dense open set $E'$ with the above-stated properties. Next we invoke our induction hypothesis to $\iota_{n-1}$, $E'$ and $r \restriction [\mu_{n-1}]$ and find a dense open $E$ satisfying the conclusion of Claim 5.11.2.

Let $q \in E$ be compatible with $r \restriction [\mu_{n-1}]$. Then, $\iota_{n-1}(q) \land r \restriction [\mu_{n-1}]$ is $\leq_{\Gamma_{n-1}} E$ below $\iota_{n-1}(q)$ and there is $q'' \leq_{\Gamma_{n-1}} \iota_{n-1}(q) \land r \restriction [\mu_{n-1}]$ in $E'$.

Note that

$$\iota_{n-1,n}(q') \leq_{\iota_{n-1}(\Gamma_{n-1})} \iota_{n-1,n}(\iota_{n-1}(q) \land r \restriction [\mu_{n-1}]) = \iota(q) \land r \restriction [\mu_{n-1}],$$

hence $\iota_{n-1,n}(q')$ is compatible with $r$. Thus, by the property of $E'$, there is $q'' \leq_{\Gamma} \iota_{n-1}(q') \land r$ in $D$. Since $\iota_{n-1}(q') \land r \leq_{\Gamma} \iota_{n}(q) \land r$ this yields a condition $q'' \leq_{\Gamma} \iota(q) \land r$, which accomplishes the proof.  

5.3. Proof of the coding lemma. After a short digression with non-stationary-supported iterations we focus on proving Lemma 5.8.

**Setup.** For the rest of this section our ground model is $\mathcal{K}[S]$ and $\mathbb{P}_\kappa$ is a non-stationary-supported iteration (in $\mathcal{K}[S]$) as described in Lemma 5.8.

We will fix $G \subseteq \mathbb{P}_\kappa$ a $V$-generic filter and consider $j: V[G] \to M[H]$, a $V[G]$-elementary embedding with $\text{crit}(j) = \kappa$. Notice that, by virtue of our anti-large-cardinal assumption, $j \restriction \mathcal{K}$ is a normal iteration and thus so is $j \restriction V$ (by Lemma 5.7), but those iterations can be infinite$^{15}$. Invoking Lemma 5.2, for each $x \in M$, there is a finite iteration $\iota: V \to N$ such that $k \circ \iota = j \restriction V$ and $x \in \text{range}(k)$, where $k: N \to M$ is the factor map between $\iota$ and $j$ (we will say that $\iota$ factors $j \restriction V$).

---

$^{14}$Recall that we were assuming the existence of a condition $p \in \mathbb{P}_\kappa$ such that $\iota(q)$ is compatible with $r$ for all $q \leq p$. Since $E$ is dense, any condition in $E \cap (\mathbb{P}_\kappa/q)$ works.

$^{15}$In fact, they must be quite long, as the forcing $j(\mathbb{P}_\kappa)$ adds many $\omega$-sequences.
Lemma 5.12. Suppose that $i: V \to N$ is a finite normal iteration with critical point at least $\kappa$. Then, $N[G]$ is closed under $<\kappa$-sequences in $V[G]$.

Moreover, for every such sequence $\alpha$ there is $\zeta < \kappa$ such that $\vec{\alpha} \in N[G|\zeta]$.

Proof. As usual, let $\vec{\mu}$ be the sequence of critical points. Since $\min \vec{\mu} \geq \kappa$, $V_{\kappa+1} \subseteq N$ and thus $\mathbb{P}_\kappa$ is computed correctly in $N$. In particular, the model $N[G]$ is well-defined and it is the generic extension of $N$ by $G$.

Let $\langle \alpha_i \mid i < i_* \rangle$ be a sequence of ordinals in $V[G]$ with $i_* < \kappa$. For each $i < i_*$ fix a function $V \ni f_i: \kappa^+ \to \text{Ord}$ such that $\alpha_i = \iota(f_i)(\vec{\mu})$. Everything boils down to show that $\langle f_i \mid i < i_* \rangle$ belongs to $V[G | \gamma]$ for some $\gamma < \kappa$.

Invoking the Axiom of Choice in $V$ we let $h: \text{dom}(h) \to V$, $h \in V$, with $\text{dom}(h) \subseteq \text{Ord}$ and $f_i \in \text{range}(h)$ for all $i < i_*$. This choice is possible in that each $f_i$ belongs to the ground model. Note that a priori, we do not put any bound on $\text{dom}(h)$. Put $a_* := \{ h^{-1}(f_i) \mid i < i_* \}$ and note that this is a set of ordinals in $V[G]$ whose order-type is $< \kappa$.

Claim 5.12.1. Let $a$ be a set of ordinals in $V[G]$ with $\text{otp} a < \kappa$. Then, there is $\zeta < \kappa$ and a function $g: \zeta \to \text{Ord}$ in $V$ such that $a \subseteq \text{range} g$.

Proof. First, we may find a function $\tilde{g}: \mathbb{P}_\kappa \times \text{otp} a \to \text{sup} a$ in $V$ such that $a \subseteq \text{range} \tilde{g}$. In effect, let for example $\tilde{g}$ be send each pair $(p, \xi) \in \mathbb{P}_\kappa \times \text{otp} a$ to the $\xi$-th element of $a$ decided by $p$, if this exists; otherwise, set $\tilde{g}(p, \xi) := 0$. Since $|\mathbb{P}_\kappa| = 2^\kappa$, by well ordering $\mathbb{P}_\kappa$, we may assume that

$$\text{dom}(\tilde{g}) = (2^\kappa)^V = (\kappa^+)^V.$$

Since $(\kappa^+)^V = (\kappa^+)^{V[G]}$ is regular in $V[G]$, $\text{sup} a < \kappa^+$. In particular, there must be some ordinal $\zeta' < \kappa^+$ such that $\text{range}(\tilde{g} \upharpoonright \zeta')$ already covers $a$.

Note that this $\zeta'$ might be determined only in the generic extension $V[G]$.

Next, let us pick a surjection $\varphi: \kappa \to \zeta'$ in $V$. Then, $\tilde{g} \circ \varphi: \kappa \to \text{sup} a$ covers $a$. But $\kappa$ is regular in $V[G]$ as well, so we apply the same argument and obtain some $\zeta < \kappa$ such that $g := \tilde{g} \circ \varphi \upharpoonright \zeta'$ already covers $a$. \qed

Using the above claim we have that $a_* \subseteq \text{range} g$ for some $g: \zeta \to V$ in $V$ with $\zeta < \kappa$. Thus, in order to decide the value of $a_*$, it is enough to decide the value of $g^{-1}(a_*)$ which is a subset of $\zeta$.

Using the closure of the $\leq^*$-order, we may decide the values of $a_*$ up to the forcing $\mathbb{P}_\kappa \upharpoonright \zeta + 1$. This is achieved by repeatedly strengthening the components of the condition above $\zeta$. For details, see Claim 5.11.2.

Since $a_* \in V[G \upharpoonright \zeta + 1]$ and $h \in V$ it follows that $h^* a_* = \{ f_i \mid i < i_* \}$ belongs to $V[G \upharpoonright \zeta + 1]$, hence it also does $\langle f_i \mid i < i_* \rangle$. Now, the embedding $\iota$ lifts uniquely (as the forcing $\mathbb{P}_\kappa \upharpoonright (\zeta + 1)$ is small) to an embedding $\iota_*$. Thus, $\iota_*(\langle f_i \mid i < i_* \rangle) = \langle \iota(f_i) \mid i < i_* \rangle \in N[G \upharpoonright \zeta + 1 \subseteq N[G]$. By plugging the generators $\vec{\mu}$ to this sequence we infer that $\langle \alpha_i \mid i < i_* \rangle \in N[G]$. \qed

The next will help us isolate the condition $r$ from Lemma 5.8(γ):
Lemma 5.13. Let $\iota: V \rightarrow N$ be a finite iteration and $k: N \rightarrow M$ be such that $k \circ \iota = j \upharpoonright V$. Then, the set $A_\iota$ defined as
\[ \{ \alpha \in [\kappa, \iota(\kappa)) \mid \exists r \in k^{-1}(H) \forall p \in G (k(r(\alpha)))_{H \upharpoonright k(\alpha)} < j(p)(k(\alpha))_{H \upharpoonright k(\alpha)} \}^{16} \]
is finite. Moreover, for each $\alpha \in A_\iota$
\[ \bar{H}_\alpha = \{ \text{stem}(k(r(\alpha)))_{H \upharpoonright k(\alpha)} \mid r \text{ witnesses } \alpha \in A_\iota \} \]
is finite and has $\bigcup \bar{H}_\alpha$ as a maximal element.

Proof. Let us show that $\bar{H} := \bigcup_{\alpha \in A} \bar{H}_\alpha$ is finite. Assume towards a contradiction that $\bar{H}$ is infinite and let $\langle k(\hat{\eta}_n)_{H \upharpoonright k(\alpha_n)} \mid n < \omega \rangle \in V[G]$ be a 1-1 sequence witnessing this. Put $\bar{\alpha} := \langle \alpha_n \mid n < \omega \rangle \in V[G]$. Note that we allow $\bar{\alpha}$ to exhibit repetitions, despite $\bar{H}$ is assumed to be infinite.

For each $n < \omega$, let $r_n \in k^{-1}(H)$ be such that
\[ \text{stem}(k(r(\alpha_n)))_{H \upharpoonright k(\alpha_n)} = k(\eta_n)_{H \upharpoonright \alpha_n} \]
and
\[ k(r_n(\alpha_n))_{H \upharpoonright \alpha_n} < j(p)(k(\alpha_n))_{H \upharpoonright \alpha_n} \].

In particular, as $k(r_n) \in H$, $k(\hat{\eta}_n)_{H \upharpoonright k(\alpha_n)}$ is an initial segment of the $k(\alpha_n)$-th Prikry sequence introduced by $H$.

By the proof of Lemma 5.12, there is $\zeta < \kappa$ such that $\langle \hat{\alpha}_n \mid n < \omega \rangle, \langle \hat{\eta}_n \mid n < \omega \rangle \in N[G \upharpoonright \zeta]$. Clearly, $k$ lifts to $k: N[G \upharpoonright \zeta] \rightarrow M[G \upharpoonright \zeta]$ so that $\langle k(\alpha_n) \mid n < \omega \rangle, \langle k(\hat{\eta}_n) \mid n < \omega \rangle \in M[G \upharpoonright \zeta] \subseteq M[H]$. Also, $\langle j(p_n) \mid n < \omega \rangle \in M[H]$.

Combining all of these inputs we find $q \in H$ such that, for each $n < \omega$,
\[ q \upharpoonright k(\alpha_n) \Vdash_{j(p_n)_{k(\alpha_n)}} \text{"}k(\hat{\eta}_n)\text{ is an initial segment of stem}(q(k(\alpha_n)))\text{"}. \]
and
\[ q \upharpoonright k(\alpha_n) \Vdash_{j(p)_{k(\alpha_n)}} \text{"}\forall p \in G (k(\eta_n) \not\subseteq \text{stem}(j(p)(k(\alpha_n)))\text{"}. \]

We now produce a contradiction by distinguishing two cases: either $\bar{\alpha}$ is finite or it is not. In the first case there is $\alpha_* \in \bar{\alpha}$ and $I \in [\omega]^\omega$ such that $\alpha_n = \alpha_*$ for all $n \in I$. In particular $\langle k(\hat{\eta}_n)_{H \upharpoonright k(\alpha_*)} \mid n \in I \rangle$ is bounded by $\text{stem}(q(k(\alpha_*)))_{H \upharpoonright k(\alpha_*)}$ and thus there are $n, m \in I$ such that $k(\hat{\eta}_n)_{H \upharpoonright k(\alpha_*)} = k(\hat{\eta}_m)_{H \upharpoonright k(\alpha_*)}$. This contradicts injectivity of $\langle k(\hat{\eta}_n)_{H \upharpoonright k(\alpha_*)} \mid n < \omega \rangle$.

Thus, assume that $\bar{\alpha}$ is infinite.

Claim 5.13.1. There is $p_* \in G$ such that $\bar{\alpha} \setminus \bar{\mu} \subseteq \text{dom}(\iota(p_*))$.

Proof of claim. Since $\bar{\alpha} \setminus \bar{\mu} \in V[G]$ there is $\zeta < \kappa$ with $\bar{\alpha} \setminus \bar{\mu} \in N[G \upharpoonright \zeta]$. A chain condition argument with $\mathbb{P}_\zeta$ allows us to find a set $B \subseteq [\kappa, \iota(\kappa))$, $|B| < \kappa$ such that $1 \Vdash_{\mathbb{P}_\zeta} \bar{\alpha} \setminus \bar{\mu} \subseteq B$ and $B \cap \bar{\mu} = \emptyset$. By Lemma 5.5 there must be a condition $p_* \in G$ with $B \subseteq \text{dom}(\iota(p_*))$. Thus, $p_*$ is as wished. \(\square\)

---

16Here $p < q$ stands for $p \leq q$ and $|\text{stem}(p)| > |\text{stem}(q)|$.
Let $s \leq q, j(p_\ast)$ be in $H$. Then for some $\alpha_{n_\ast} < \omega$
\[
s \upharpoonright k(\alpha_{n_\ast}) \models \downarrow j(P)[k(\alpha_{n_\ast})] s(\alpha_{n_\ast}) \leq^* j(P)[k(\alpha_{n_\ast})]
\]
and
\[
s \upharpoonright k(\alpha_{n_\ast}) \models \downarrow j(P)[k(\alpha_{n_\ast})] s(\alpha_{n_\ast}) \leq^* q(k(\alpha_{n_\ast})).
\]
Combining this with (1) and (2) above we have
\[
s \upharpoonright k(\alpha_{n_\ast}) \models \downarrow j(P)[k(\alpha_{n_\ast})] s(\alpha_{n_\ast}) \leq^* j(P)[k(\alpha_{n_\ast})]
\]
and
\[
s \upharpoonright k(\alpha_{n_\ast}) \models \downarrow j(P)[k(\alpha_{n_\ast})] s(\alpha_{n_\ast}) \leq^* q(k(\alpha_{n_\ast})).
\]
Finally, we argue that
\[
\text{Proof of claim.}
\]

In fact, there is
\[
\text{Claim 5.13.2. Every two members of } H_\alpha \text{ are } \sqsubseteq\text{-compatible.}
\]
\[
\text{Proof of claim. Suppose } k(\dot{\eta})_{H|\alpha}, k(\dot{\sigma})_{H|\alpha} \in H_\alpha \text{ and let } r, r' \in k^{-1}(H) \text{ witnessing this. Then there is } H \ni s \leq k(r), k(r') \text{ such that }
\]
\[
s \upharpoonright k(\alpha) \models \downarrow j(P)[k(\alpha)] "k(r(\alpha)) and } k(r'(\alpha)) are compatible",
\]
which yields $k(\dot{\eta})_{H|k(\alpha)} \sqsubseteq k(\dot{\sigma})_{H|k(\alpha)}$ or the other way around. 

This is the end of the proof.

By the previous lemma we can associate to each finite iteration $\iota: V \rightarrow N$ (factoring $j \upharpoonright V$) a finite list of names for stems $\vec{\eta} := \langle \dot{\eta}_\alpha \mid \alpha \in A_i \rangle$ where $k(\dot{\eta}_\alpha)_{H|k(\alpha)}$ is the maximal element of $H_\alpha$. We may assume, in addition, that $\eta_\alpha$ is forced by the weakest condition of $\iota(P_\alpha)$ to be a potential stem at coordinate $\alpha$. This makes the following to be a sound definition:

**Definition 5.14.** Let $\iota: V \rightarrow N$ be a finite iteration factoring $j \upharpoonright V$.

We will denote by $r_\iota$ the condition with $\text{supp } r_\iota = A_i$ such that $r(\alpha)$ is a $\iota(P_\alpha)_{\alpha\ast}$-name for a maximal tree with stem $\dot{\eta}_\alpha$ and $A_i$ being as in Lemma 5.13.

The following gives better control on the support of $r_\iota$:

**Lemma 5.15.** Let $\iota: V \rightarrow N$ be a finite iteration factoring $j \upharpoonright V$.

Using the notations of Lemmas 5.11 and 5.13, $A_i \subseteq \Gamma_\iota$.

**Proof.** Assume otherwise and let $\alpha < \iota(\kappa)$ be such that $\alpha \in A_i$ but $\alpha \notin \Gamma_\iota$.

Let $\tau$ be a $\iota(P_\alpha) \upharpoonright \alpha$-name for the ordinal max $\dot{\eta}_\alpha$. We may assume that this name is forced by the trivial condition to be strictly below $\alpha$. Let $D_0$ be the set of all conditions $q \in \iota(P_\alpha)$ with $\alpha \in \text{supp } q$ such that
\[
1 \models \iota(P_\alpha)_{\alpha} \min(\text{Lev}_1(T^q(\alpha))) > \tau.
\]

An easy application of the mixing lemma shows that $D_0$ is $\leq^*$-dense open in $\iota(P_\alpha)$. The goal is to find a condition $p \in G$ with $\iota(p) \in D_0$.

Recall that for $q, q' \in \iota(P_\alpha)$ we write $q' \leq^* q$ if $q' \leq^* q$ and
\[
\{ \beta < \iota(\kappa) \mid q' \upharpoonright \beta \not\vDash \iota(P_\alpha)_{\beta} q'(\beta) = q(\beta) \} \subseteq \Gamma_\iota.
\]
Define $D := \{ q \in \iota(\mathcal{P}) \mid \forall q' \leq^*_{\Gamma_i} q \ (q' \in D_0) \}$. It follows that $D$ is $\leq^*$-dense open in $\iota(\mathcal{P})$. Also, if $q \in \iota(\mathcal{P})$, $\alpha \in \text{supp } q$ and $D \ni q' \leq^*_{\Gamma_i} q$ then $q \in D_0$.

Using Lemma 5.11, for each each $r \in \iota(\mathcal{P})$ with $\text{supp } r \subseteq \Gamma_i$, there is a $\leq^*$-dense open set $E \subseteq \mathcal{P}$ with the following property: for every $p \in E$ there is $D \ni q' \leq^*_{\Gamma_i} \iota(p) \wedge r$. Thus, if $\alpha \in \text{supp } \iota(p)$ then $\iota(p) \in D_0$.

We note that $\alpha \in \text{supp } p$ for $\leq^*$-densely many $p \in E$: by Claim 5.3.1, as $\alpha \not\in \Gamma_i$ (and in particular $\alpha$ is not one of the critical points of the iteration) there is a nowhere dense set $T$ such that $\alpha \in \iota(T)$. Thus, if $\text{supp } p \supseteq T$, $\alpha \in \text{dom } \iota(p)$. Clearly, the collection of conditions $p$ with support $\text{supp } p$ containing a given nowhere stationary set $T$ is dense open.

By density, there is $p \in G \cap E$ with $\iota(p) \in D_0$. Now use that $\alpha \in A_i$ to find $r \in k^{-1}(H)$ such that $k(\iota(\alpha))_{H|k(\alpha)} \leq j(p)(k(\alpha))_{H|k(\alpha)}$. On one hand,

\[ r \upharpoonright \alpha \mathbin{|}_{\iota(p)} \mathbin{\alpha} \max(\text{stem}(r(\alpha))) \leq \tau. \]

On the other hand, since $\iota(p) \in D_0$,

\[ r \upharpoonright \alpha \mathbin{|}_{\iota(p)} \mathbin{\alpha} \max(\text{stem}(r(\alpha))) > \tau. \]

This is a contradiction.

The previous lemma acknowledges that the condition $r_\iota$ of Definition 5.14 complies with the requirements of Lemma 5.11.

We need a further technical lemma. The following lemma gives us almost a complete description of the conditions in a finite iteration that are going to be sent to $H$.

**Lemma 5.16.** Let $\iota: V \to N$ be a finite iteration factoring $j \upharpoonright V$.

For every $q \in H$ with $q \in \text{range}(k)$, there is $p \in G$ and $q' \in k^{-1}(H)$ with $k(q') \leq q$ such that $q' \leq^*_{\Gamma_i} \iota(p) \wedge r_\iota$.

**Proof.** Set $\bar{q} := k^{-1}(q)$. Since $\text{supp } \bar{q}$ is a nowhere stationary set, there is $p \in G$ such that $\iota(\text{supp } p)$ covers $\text{supp } \bar{q} \setminus \Gamma_i$. In addition, we may assume that $p \leq q \upharpoonright \kappa$ and that $p \mathbin{\models}_\mathcal{P} \forall \bar{q}' \in G \ (\iota(\bar{q}')$ is compatible with $r_\iota$).

We would like to use Lemma 5.11: First, the set $D \subseteq \iota(\mathcal{P})$ of all conditions that are either $\leq^*$-incompatible with $\bar{q}$ or $\leq^*$-stronger than $\bar{q}$ is $\leq^*$-dense open. Second, for every $q \leq p$, $\iota(q)$ is compatible with $r_\iota$ because, by our choice of $p$, $\bar{q} \mathbin{\models}_\mathcal{P} \forall \bar{q}' \in G \ (\iota(\bar{q}')$ is compatible with $r_\iota$). Thus, we may appeal to Lemma 5.11 with respect to $\langle D, r_\iota, p \rangle$ and get a $\leq^*$-dense open set $E \subseteq \mathcal{P}$ below $p$ such that, for each $\bar{q}' \in E$, there is a $\leq^*_{\Gamma_i}$-extension of $\iota(\bar{q}') \wedge r_\iota$ in $D$. Extending $p$ we assume that there is a $\leq^*_{\Gamma_i}$ extension of $\iota(p) \wedge r_\iota$ in $D$.

Let us define an auxiliary condition $q''$ as follows: $q''(\alpha) := \iota(p)(\alpha) \wedge \bar{q}(\alpha)$ for $\alpha \not\in \Gamma_i$ and otherwise $q''(\alpha) := r_\iota(\alpha) \wedge \bar{q}(\alpha)$.

Let $q' := \iota(p) \upharpoonright (\iota(\kappa) \setminus \Gamma_i) \cup q'' \upharpoonright \Gamma_i$.

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17 Because we are assuming that $\alpha \not\in \Gamma_i$.

18 More precisely, we define $q''(\alpha)$ this way assuming that the outcome is a condition. Otherwise, we stipulate $q''(\alpha)$ to be the trivial condition.
By elementarity, for each $\alpha$, $k(q''(\alpha))$ is forced to be the maximal condition in $j(\mathbb{Q}_1)$, stronger than $j(p)(\alpha) \land k(r_i)(\alpha)$ and $q(\alpha)$, if such a condition exists. Let $s$ be a condition in $H$ stronger than $j(p), k(r_i)$ and $q$. Then, clearly, $s$ forces that $k(q'') \leq j(p) \land k(r_i), q$ and that $k(q'') \in H$. We would like to show that $s \Vdash_{j(\mathcal{P}_1)} k(q') = k(q'')$. To this end, fix $u \leq^*_\mathcal{P} \alpha(p) \land r_i$.

**Claim 5.16.1.** $u \leq^* \tilde{q}$.

**Proof.** Otherwise, $u$ is $\leq^*$-incompatible with $\tilde{q}$ and thus so are $k(u)$ and $q$. Without loss of generality, this is forced by $s$. Let us now look at $k(q'')$. The only coordinates in which we don’t have $k(q'')$, it will be able to infer our key coding lemma (Lemma 5.8). By the Prikry lemma this is a forcing that, for some $\dot{\epsilon}$ with $\dot{\epsilon} \leq \epsilon$, the condition $\dot{\epsilon}$ is independent of their choice. By Lemma 5.14, we may take $q$ so that, for some $p \in G$, $q \leq^*_\mathcal{P} \alpha(p) \land r_i$ and supp $q = \text{supp}(\alpha(p) \land r_i)$.

**Proof.** Suppose that the lemma is false. Fix a finite iteration $\iota: V \rightarrow N$ with $\epsilon \in \text{Im}(k)$, and set $\bar{\epsilon} = k^{-1}(\epsilon)$. For each $\mathbb{P}_\kappa$-name $\dot{X}$ for a subset of $\kappa$,

$$D_{\dot{X}, \epsilon} := \{ s \in \mathbb{P}_\kappa \mid s \Vdash_{\mathbb{P}_\kappa} \bar{\epsilon} \in \dot{X} \lor s \Vdash_{\mathbb{P}_\kappa} \bar{\epsilon} \notin \dot{X} \}.$$ 

By the Prikry lemma this is a $\leq^*$-dense open set and $D_{\dot{X}, \epsilon} \in N$.

Denote by $\Gamma_i$ the set from Lemma 5.11. Likewise, $r_i$ denotes the condition arising from Definition 5.14; namely, $r_i$ is the weakest condition with support $A_i$ and stem $\bar{\eta}_\alpha$ at each $\alpha \in A_i$. By Lemma 5.15, $A_i \subseteq \Gamma_i$, hence supp $r_i \subseteq \Gamma_i$.

Let $p_0$ be any condition in $G$ forcing that $\bar{\iota}(q)$ is compatible with $r_i$, for every $q \in G$. Invoking Lemma 5.11 for the direct extension order $\leq^*$ of $\mathbb{P}_\kappa$, the $\leq^*$-dense open set $D_{\dot{X}, \epsilon}$ and the conditions $r_i$ and $p_0$ we obtain a condition $p \in G$, $p \leq p_0$ such that $\bar{\iota}(p) \land r_i$ admits a $\leq^*_\Gamma_i$-extension $s \in D_{\dot{X}, \epsilon}$ with supp $s = \text{supp}(\bar{\iota}(p) \land r_i)$. Note that any two conditions $s$’s as above are compatible, hence the decision on “$\bar{\epsilon} \in \bar{\iota}(\dot{X})$” is independent of their choice.

If for every $\dot{X}_G \in \mathcal{U}$ there was a condition $s \in D_{\dot{X}, \epsilon}$ with $s \leq^*_\Gamma_i \bar{\iota}(p) \land r_i$ (for some $p \in G$) and $k(s) \in H$ then the conclusion of Lemma 5.17 would hold, thus contradicting our departing assumption. Hence, there must be some $\dot{X}_G \in \mathcal{U}$ such that if $s \in D_{\dot{X}, \epsilon}$ and $s \leq^*_\Gamma_i \bar{\iota}(p) \land r_i$ then $k(s) \notin H$. 


Claim 5.17.1. The following are true:

(1) \( \{ \beta \, |\, k'(s')^H(\beta) \subseteq k(s) \} \subseteq k(\Gamma') \cap k'(\Gamma''). \)

(2) For each \( \beta \) in the above set, 
\[
\text{stem}(k'(r')^H(\beta)) \neq \text{stem}(k(r)^H(\beta)).
\]

Proof. Let us stipulate that \( r := r_\iota, r' = r_{\iota'} \), \( \Gamma := \Gamma_\iota \) and \( \Gamma' = \Gamma'' \).

(1) Let \( \beta \) be such that \( k'(s')^H(\beta) \subseteq k(s)^H(\beta) \). Clearly, \( \beta \geq \kappa \).

Since any two conditions in \( j(Q)^H(\beta) \) with the same stem are compatible it must be the case that \( k(s)^H(\beta) \) and \( k(s')^H(\beta) \) have different stems. To simplify notations, let us put \( q := j(p)^H(\beta) \) and \( q' := j'(p')^H(\beta) \).

First, \( \beta \in k(\Gamma) \): Otherwise, \( \beta \notin k(\Gamma) \) and so \( q = j(p)^H(\beta) \), which belongs to the generic filter \( H_\beta \subseteq j(Q)^H \). Since \( q' \in H_\beta \) (because \( k'(s') \in H \)) we infer that \( q' \) and \( q \) are compatible, hence obtaining a contradiction.

Second, \( \beta \in k'(\Gamma') \): Otherwise, \( q' = j(p')^H(\beta) \). Since the condition \( j(p') \) is compatible with \( r, (j(p') \land k(r))^H(\beta) \) is well-defined and extends \( j'(p')^H(\beta) \). Now, \( q \) is an \( \leq^* \)-extension of \( j(p) \land k(r)^H(\beta) \) so in \( M[H \upharpoonright \beta] \), 
\[
\text{stem } q = \text{stem } (j(p) \land k(r)(\beta)) = \text{stem } (j(p') \land k(r)(\beta)),
\]
where the last equality follows from the maximality of the stem of \( r \). Thus, \( q \) is compatible with \( (j(p') \land k(r)(\beta)) \), which in turn extends \( q' \). Thereby, \( q \) and \( q' \) are compatible, hence yielding the sought contradiction.

(2) By the previous arguments, the stems of \( k(s)^H(\beta) \) and \( k'(s')^H(\beta) \) must be different. Also, since \( \beta \in k(\Gamma) \cap k'(\Gamma') \), these stems respectively coincide with those of \( k(r)^H(\beta) \) and \( k'(r')^H(\beta) \), which shows that the stems of \( k(r)^H(\beta) \) and \( k'(r')^H(\beta) \) must be different. \( \square \)

For \( X \in U \) and \( : V \rightarrow N \) a finite iteration with \( \epsilon \in \text{range}(k) \) we say that \( X \) is good for \( \iota \) if there is \( s \in D_{X,\iota} \) with \( s \leq^* r \), \( \iota(p) \land r \) (some \( p \in G \)) such that \( k(s) \in H. \)

Alternatively, \( X \) is bad for \( \iota \) if it is not good.

The above argument shows that if \( X \) is bad for \( \iota \) then there is another finite iteration \( \iota' \) for which \( X \) is good and this discrepancy entails \( k(\Gamma_\iota) \cap k'(\Gamma_\iota') \neq \emptyset \). Moreover, at least one of the \( \beta \)'s in this intersection witnesses a disagreement between the length of the stems of \( k(r_\iota) \) and \( k'(r_{\iota'}) \).

\(^{19}\)Note that if \( s \in D_{X,\iota} \) and \( k(s) \in H \) then \( k(s) \models j(p) \epsilon \in j(X) \).
Working in $V[G]$ let us define $\langle \langle \iota_\alpha, X_\alpha \rangle \mid \alpha < (2^{\aleph_0})^+ \rangle$ as follows:

1. $\iota_\alpha : V \to N_\alpha$ is a finite iteration with $\epsilon \in \text{range}(k_\alpha)$;
2. $\langle X_\alpha \mid \alpha < (2^{\aleph_0})^+ \rangle \subseteq U$ is a $\subseteq$-decreasing sequence;
3. for all $\alpha < (2^{\aleph_0})^+$,
   (a) $\bigcap_{\beta < \alpha} X_\beta$ is good for $\iota_\alpha$;
   (b) $X_\alpha$ is bad for $\iota_\alpha$, for all $\beta \leq \alpha$.

In particular, by (3)(a), for each $\beta < \alpha < (2^{\aleph_0})^+$, $k_\alpha(\Gamma_{\iota_\alpha}) \cap k_\beta(\Gamma_{\iota_\beta}) \neq \emptyset$.

Let $\iota_0$ be any finite iteration witnessing (1). Since we are assuming that the lemma is false there is $X_0 \in U$ which is bad for $\iota_0$. So, suppose that $\langle \langle \iota_\beta, X_\beta \rangle \mid \beta < \alpha \rangle$ was defined. By $\kappa$-completeness, $X^0_\alpha := \bigcap_{\beta < \alpha} X_\beta \in U$. Arguing as before, there is a finite iteration $\iota_\alpha$ for which $X^0_\alpha$ is good. Also, since we are assuming that the conclusion of the lemma is false, there is $X_\alpha \subseteq X^0_\alpha$ that is bad for $\iota_\alpha$. Finally, $X_\alpha$ is bad for all $\beta \leq \alpha$: Otherwise, there is $s \in D_{X_{\alpha,\iota_\beta}}$ such that $k_\beta(s) \in H$. However, $D_{X_{\alpha,\iota_\beta}} \subseteq D_{X_{\beta,\iota_\beta}}$, which would contradicts that $X_{\beta}$ is bad for $\iota_\beta$.

Let us now look at the sets $\Gamma_\alpha := \Gamma_{\iota_\alpha}$. By the argument above, for every $\alpha < \beta$, every disagreement point between (the evaluations of) $k_\alpha(r_\alpha)$ and $k_\beta(r_\beta)$ is a member of $k_\alpha(\Gamma_\alpha) \cap k_\beta(\Gamma_\beta)$. Let us apply the $\Delta$-system lemma to $\{k_\alpha(\Gamma_\alpha) \mid \alpha < (2^{\aleph_0})^+ \}$. This way we obtain a set $I$ of cardinality $(2^{\aleph_0})^+$ and a root $\Delta$ such that for all $\alpha, \beta \in I$, $k_\alpha(\Gamma_\alpha) \cap k_\beta(\Gamma_\beta) = \Delta$. Fix a coloring $c : ( (2^{\aleph_0})^+ )^2 \to \Delta$ sending $\langle \alpha, \beta \rangle$ to the least coordinate $\zeta$ in the root $\Delta$ which exhibits an incompatibility between the stems of $k_\alpha(r_\alpha)^H|\zeta$ and $k_\beta(r_\beta)^H|\zeta$. By the Erdős-Rado theorem this coloring admits a homogeneous set $J \subseteq I$ of order type $\omega_1 + 1$. So, let $\gamma \in \Delta$ be such that $c(J = \{\gamma\})$; namely, $\gamma$ is the common least coordinate witnessing a disagreement between the stems of $k_\alpha(r_\alpha)^H|\gamma$ and $k_\beta(r_\beta)^H|\gamma$, for all $\langle \alpha, \beta \rangle \in |J|^2$. Since for each such $\alpha, \beta$ the lengths of $\text{stem}(k_\alpha(r_\alpha)^H|\gamma)$ and $\text{stem}(k_\beta(r_\beta)^H|\gamma)$ are finite and different we derive a contradiction with our departing assumption. \( \square \)

Given Lemma 5.17 the proof of our coding lemma is easy:

**Proof of Lemma 5.8.** Let $U \in V[G]$ be a $\kappa$-complete ultrafilter on $\kappa$ and let $j : V[G] \to M[H]$ be the induced ultrapower. By our setup considerations in page 25 there is a finite iteration $\iota : V \to N$ such that $k \circ \iota = j \upharpoonright V$ and $\epsilon := [\text{id}]_U$ belongs to $\text{range}(k)$. Invoke Lemma 5.17 with respect to these inputs and define $V$ to be the set of all $X \in \mathcal{P}(\kappa)^{V[G]}$ such that there are $q \in \kappa^{-1}(H)$ and $p \in G$ such that

\[
q \leq_{\text{F}_{\Gamma_\iota}} (\iota(p) \wedge r_\iota) \quad \text{and} \quad q \Vdash_{\iota(\mathcal{P}_\kappa)} \epsilon \in \iota(\check{X}).
\]

Note that any two conditions $q, q'$ witnessing the above admit an explicit $\leq_{\text{F}_{\Gamma_\iota}}$-extension. In particular, $\mathcal{V}$ is a filter. Also, by the moreover part of Lemma 5.17, $\mathcal{V}$ contains the $\kappa$-complete ultrafilter $U$. Thus, $\mathcal{V} = U$. \( \square \)
Before proving the consistency of the \(\omega\)-gluing property (Theorem 5.19 below) let us pose an attractive conjecture, which is a strengthening of Lemma 5.16:

**Conjecture 5.18.** Let \( j: V[G] \to M[H] \) be an ultrapower embedding with critical point \( \kappa \), such that \( j \restriction V \) is a normal iteration using normal measures.

Then, there is a finite sub-iteration \( \iota: V \to N \) factoring \( j \restriction V \), with critical points \( \check{\mu} \), and a condition \( r \in \check{\iota}(\check{\mathcal{P}}_{\kappa}) \) such that \( \text{supp } r = \check{\mu} \) and for all \( s \leq^* r \) in \( N \) with \( \text{supp } s = \check{\mu} \) then \( \kappa(s) \in H \).

The meaning of this conjecture is the following: There is a finite sub-iteration \( \iota \) of \( j \) for which there is a \( \check{\iota}(\check{\mathcal{P}}_{\kappa}) \)-generic filter that can be defined from \( G \) together with finitely-many additional information, and all its conditions are sent to \( H \) by \( k \). Moreover, this filter is \( \leq^* \)-generic.

If our conjecture holds then, in particular, the above-mentioned filter will decide all statements of the form “\( \epsilon \in \check{\iota}(\check{X}) \)" for a \( \check{\mathcal{P}}_{\kappa} \)-name \( \check{X} \) for a subset of \( \kappa \). Thus the filter will determine the measure \( \check{\mathcal{U}} \) defining \( j \).

Unfortunately we do not know how to prove the conjecture, but we got pretty close to it, by showing that its corollary holds; to wit, there is a finite iteration that determines \( \check{\mathcal{U}} \) (Lemma 5.17 above).

### 5.4. A model for the \(\omega\)-gluing property from \( o(\kappa) = \omega_1 \)

We are finally in conditions to derive the main result of Section 5.

**Theorem 5.19 (V = \( K \)).** Let \( \kappa \) be a measurable cardinal with \( o(\kappa) = \omega_1 \) and assume that there are no other measurables of Mitchell order \( \geq \omega_1 \).

Then, there is a forcing extension in which the \(\omega\)-gluing property holds.

**Proof.** Let \( \check{\mathcal{U}} = \langle \mathcal{U}(\mu, \zeta) \mid \mu \leq \kappa, \zeta < \check{\mathcal{J}}(\mu) \rangle \) be the coherent sequence of normal measures in \( K \). Force with \( S \) and, working in \( K[S] \), let \( F = \bigcup S \) be the obtained generic function. Let us define a non-stationary support iteration of Prikry type forcings. The forthcoming definition will ensure that \( Q_{\alpha} \) exists in the generic extension by \( S \upharpoonright \alpha \times \mathcal{P} \upharpoonright \alpha \).

We say that a tuple \( \langle \check{\rho}, \check{\zeta}, f, g \rangle \) codes a measure on \( \alpha \) if:

1. \( \check{\rho} = \langle \rho_0, \ldots, \rho_{n-1} \rangle \) where \( \rho_i: \alpha^i \to (\alpha + 1) \) and \( \rho_0(()) = \alpha \),
2. \( \check{\zeta} = \langle \zeta_0, \ldots, \zeta_{n-1} \rangle \) is a sequence of countable ordinals,
3. \( f: \alpha^n \to S \upharpoonright \alpha \ast \check{\mathcal{P}}_{\alpha} \),
4. \( g: \alpha^n \to \alpha \),
5. For each \( i < n \), \( \rho_i(\langle \mu_{j} \mid j < i \rangle) = \mu_{i} \) and \( \zeta_i < o^{N_{i}}(\mu_{i}) \), where \( \iota_{i+1}: N_{i} \to N_{i+1} \cong \text{Ult}(N_{i}, \iota_{i}(\check{\mathcal{U}})(\mu_{i}, \zeta_{i})) \), being \( \iota_{i+1} = \iota_{i+1} \circ \iota_{i} \),

and moreover setting \( r = \iota_{n}(f)(\langle \mu_{i} \mid i < n \rangle), \epsilon = \iota_{n}(g)(\langle \mu_{i} \mid i < n \rangle) \), and

\[
U = \{ \check{X}^{S \upharpoonright \alpha \ast \mathcal{G} \upharpoonright \alpha} \subseteq \alpha \mid \exists p \in S \upharpoonright \alpha \ast \mathcal{G} \upharpoonright \alpha, \quad s \leq_{\mathcal{P}_{\alpha}} \iota_{n}(p) \land \iota_{n}(s) \in N_{n}, \quad N_{n} \models s \models \iota_{n}(S \upharpoonright \alpha \ast \mathcal{P}_{\alpha}) \epsilon \in \iota_{n}(\check{X}) \}\]
it is the case that $U$ is an $\alpha$-complete ultrafilter on $\alpha$. In this case, we will say that the measure $U$ is coded by $\langle \vec{\rho}, \vec{\zeta}, f, g \rangle$. Note that a code $\langle \vec{\rho}, \vec{\zeta}, f, g \rangle$ for a measure on $\alpha$ belongs to $H(\alpha^+)$. In particular, $\langle \vec{\rho}, \vec{\zeta}, f, g \rangle$ is a legitimate value for $s(\alpha)$ provided $s \in S$ (see our definition in page 16).

Let us define $Q_\alpha$.20 If $F(\alpha)$ is not a countable sequence of codes for measures on $\alpha$, we let $Q_\alpha$ to be trivial. Otherwise, let $Q_\alpha$ be the tree Prikry forcing of Section 4 defined with respect to the measures coded by $F(\alpha)$.

Let us show that in $V[\hat{S} \times G]$ the $\omega$-gluing property holds.

Indeed, by Lemma 5.8, in the generic extension, every $\kappa$-complete ultrafilter on $\kappa$ has a code. Fix a countable sequence of measures in the generic extension, $\langle U_n \mid n < \omega \rangle$, and let $c_n = \langle \vec{\rho}_n, \vec{\zeta}_n, f_n, g_n \rangle \mid n < \omega \rangle$ be a sequence of codes for them. Let $\zeta_\ast$ be a countable ordinal larger than all the countable ordinals which are mentioned in all $\vec{\zeta}_n$'s.

Let $i : V \rightarrow \text{Ult}(V, U(\kappa, \zeta_\ast)) = M$. By Lemma 5.3 regarded with respect to $S$, one can lift $i$ to $\hat{i} : V[\hat{S}] \rightarrow M[\hat{S}]$, in a way that

$$(\bigcup \hat{S})(\kappa) = \langle c_n \mid n < \omega \rangle.$$ 

Let us verify that, in $M$, each $c_n = \langle \vec{\rho}_n, \vec{\zeta}_n, f_n, g_n \rangle$ is still a code for $U_n$. Fix a code $c_\ast = c_n$. By our choice of $\zeta_\ast$ (i.e., above $\sup_{n<\omega} \vec{\zeta}_n$), each measure which is mentioned in the process of decoding $c_\ast$ exists in $M$ and in its iterations. Since $M$ is closed under $\kappa$ sequences, the computations of the models $N_i$ and the embeddings $i_i$ are the same. Given the embedding and the models, the definition of $U_n$ is clearly absolute.

Finally, we argue as in Section 4. Let $\bar{V}$ be the measure on $\kappa^\omega$ defined by $\bar{X} \in V$ if and only if there is a direct extension of $i(p)$ forcing the canonical name of the generic of $\bar{i}(Q)\kappa$ to be in $\bar{i}(\bar{X})$. \hfill $\Box$

6. A LOWER BOUND FOR THE $\omega$-GLUING PROPERTY

**Theorem 6.1.** Assume that $\kappa$ is a measurable cardinal having the $\omega$-gluing property. If there is no inner model of “$\exists \alpha (o(\alpha) = \alpha)$” then $\mathcal{O}(\kappa) \geq \omega_1$.

**Proof.** Assume that there is no inner model of “$\exists \alpha (o(\alpha) = \alpha)$”. Through the proof $\langle U_{\kappa, \xi} \mid \xi < \mathcal{O}(\kappa) \rangle$ will denote an enumeration of the normal measures in $\mathcal{K}$ indexed by their corresponding Mitchell orders. We aim to produce —by induction on $\beta < \omega_1$— a sequence $\langle V_\beta \mid \beta < \omega_1 \rangle$ of $\kappa$-complete $\mathcal{K}$-normal measures on $\kappa$ such that $\langle V_\beta \cap \mathcal{K} \mid \beta < \omega_1 \rangle$ is Mitchell-order increasing. By maximality of the core model, if $V_\beta$ is $\mathcal{K}$-normal and $\kappa$-complete then $V_\beta \cap \mathcal{K} \in \mathcal{K}$. In particular, every $V_\beta$ will admit an index $\xi_\beta < \mathcal{O}(\kappa)$ such that $V_\beta \cap \mathcal{K} = U_{\kappa, \xi_\beta}$. The sequence $\langle V_\beta \cap \mathcal{K} \mid \beta < \omega_1 \rangle$ will thus produce an evidence for $\mathcal{O}(\kappa) \geq \omega_1$.

Let $V_0$ be a normal $\kappa$-complete measure in $V$. Clearly, $V_0 \cap \mathcal{K}$ is $\mathcal{K}$-normal so that $V_0 \cap \mathcal{K} \in \mathcal{K}$. Next, let us assume that we have built $\langle V_\beta \mid \beta < \omega_1 \rangle$.

---

20Recall that this definition is in the generic extension by $S \cup (\alpha + 1) \ast \mathcal{P} \cup \alpha$. 
If \( \alpha \) takes the form \( \alpha = \gamma + 1 \) then we apply the \( \omega \)-gluing property to the constant sequence \( \vec{V} = \langle \mathcal{V}_\gamma \mid n < \omega \rangle \). Otherwise, we fix \( \langle \beta_n \mid n < \omega \rangle \) a cofinal sequence in \( \alpha \) and apply the \( \omega \)-gluing property to the sequence \( \vec{V} = \langle \mathcal{V}_{\beta_r(n)} \mid m < \omega \rangle \), where \( r: \omega \to \omega \) is a function such that \( |r^{-1}(\{n\})| = \aleph_0 \) for all \( n < \omega \).

Let \( \mathcal{V}_\alpha \) be the obtained gluing measure and \( j = j_\alpha: V \to M \) be its ultraforce map. By virtue of our anti-large cardinal hypothesis, a classical theorem of Mitchell [Mit84] implies that \( j \upharpoonright \mathcal{K} : \mathcal{K} \to \mathcal{K}_M \) is a normal iteration of measures in \( \mathcal{K} \). Denote this latter by \( \langle \iota_{\gamma, \delta} \mid \gamma \leq \delta < \delta_* \rangle \), being \( \iota_{\gamma, \gamma + 1}: \mathcal{K}_\gamma \to \mathcal{K}_{\gamma + 1} \). In addition, we put \( \mu_\gamma := \text{crit}(\iota_{\gamma, \gamma + 1}) \).

Since \( \mathcal{V}_\alpha \) glues the sequence \( \vec{V} \) there is, by Definition 1.1, an increasing sequence of ordinals \( \langle \eta_n \mid n < \omega \rangle \) such that

\[
\vec{V}(n) = \{ X \subseteq \kappa \mid \eta_n \in j(X) \}.
\]

Let \( \eta_\omega := \sup_{n < \omega} \eta_n \) and define

\[
\mathcal{V}_\alpha := \{ X \subseteq \kappa \mid \eta_\omega \in j(X) \}.
\]

By our induction hypothesis \( \vec{V}(n) \) is a normal measure, hence \( \vec{V}(n) \cap \mathcal{K} \) contains the club filter \( \text{Club}_\mathcal{K}^\kappa \), and thus \( \eta_n \in \bigsqcup j^* \text{Club}_\mathcal{K}^\kappa \) for all \( n < \omega \).

For \( m < \omega \) and a function \( f: \kappa^m \to \kappa \) in \( \mathcal{K} \), the set of its closure points \( \{ \alpha < \kappa \mid f^\alpha \kappa^m \subseteq \alpha \} \) belongs to \( \text{Club}_\mathcal{K}^\kappa \). In particular, all the \( \eta_n \)'s and \( \eta_\omega \) are closure points of \( j(f) \), for every such function \( f \in \mathcal{K} \).

**Claim 6.1.1.** For \( n < \omega \), there is \( \gamma_n < \delta_* \) with \( \iota_{\gamma_n}(\kappa) = \text{crit}(\iota_{\gamma_n, \gamma_n + 1}) = \eta_n \).

**Proof.** We may assume without loss of generality that \( \mu_\gamma \leq i_\gamma(\kappa) \) for \( \gamma \leq \delta_* \): If for some index \( \gamma \), \( \mu_\gamma > i_\gamma(\kappa) \) then the latter must be \( j(\kappa) \), hence above \( \eta_n \) for all \( n < \omega \). So, if that happens, we truncate the iteration at that stage.

Let \( \gamma_n := \gamma \leq \delta_* \) be the maximal ordinal such that \( \mu_\gamma < \eta_n \) for \( \gamma < \gamma_n \).

Recall that every member of \( \mathcal{K}_\gamma \) is represented as \( \iota_{\gamma}(f)(\mu_{\gamma_0}, \ldots, \mu_{\gamma_{m-1}}) \) where \( m < \omega \), \( f: \kappa^m \to \mathcal{K} \) is in \( \mathcal{K} \) and \( \gamma_0, \ldots, \gamma_{m-1} < \gamma \). In particular, any ordinal below \( \iota_\gamma(\kappa) \) is represented by a function \( f: \kappa^m \to \kappa \) in \( \mathcal{K} \).

Let us show that \( \eta_n = \iota_\gamma(\kappa) = \mu_{\gamma + 1} \): For each \( \alpha < \iota_\gamma(\kappa) \) there is a function \( f: \kappa^m \to \kappa \) with \( \alpha = \iota_\gamma(f)(\mu_{\gamma_0}, \ldots, \mu_{\gamma_{m-1}}) \). Clearly, \( \iota_\gamma(f)(\mu_{\gamma_0}, \ldots, \mu_{\gamma_{m-1}}) \leq j(f)(\mu_{\gamma_0}, \ldots, \mu_{\gamma_{m-1}}) \). Also, in that \( \eta_n \) is an accumulation point of \( j(f) \) and \( \mu_{\gamma_{m-1}} < \eta_n \), we have \( \alpha \leq j(f)(\mu_{\gamma_0}, \ldots, \mu_{\gamma_{m-1}}) < \eta_n \). All in all, \( \iota_\gamma(\kappa) \leq \eta_n \).

By maximality of \( \gamma \), \( \eta_n \leq \mu_\gamma \). From altogether, \( \iota_{\gamma_n}(\kappa) = \mu_{\gamma_n} = \eta_n \).

Let \( \gamma_\omega := \sup_{n < \omega} \gamma_n \).

**Claim 6.1.2.** \( \eta_\omega = \iota_{\gamma_\omega}(\kappa) = \text{crit}(\iota_{\gamma_\omega, \gamma_\omega + 1}) \).

**Proof.** On one hand, since \( \iota_{\gamma_n}(\kappa) = \eta_n \) and this latter is the critical point of \( \iota_{\gamma_n, \gamma_n + 1}, \iota_{\gamma_n}(\kappa) = \eta_n \). On the other hand, \( \gamma_\omega < \delta_* \) and \( \text{crit}(\iota_{\gamma_\omega, \gamma_\omega + 1}) = \eta_\omega \).

In effect, the latter assertion will follow from normality of the iteration once we show that \( \gamma_\omega < \delta_* \). Towards a contradiction, suppose that \( \delta_* = \gamma_\omega \).
Then, \( j(\kappa) = \nu_\gamma(\kappa) = \eta_\omega \). However, \( M \) is closed under \( \omega \)-sequences and so \( \text{cof}^V(j(\kappa)) > \omega \). Thus, in particular, \( j(\kappa) \) cannot be \( \eta_\omega \). □

Let us analyse the measures from the iteration of \( K \) which are used during the steps \( \gamma_n \) and \( \gamma_\omega \). By our initial hypothesis, \( \sigma^K(\kappa) < \kappa \) and in particular, \( \sigma^K(\kappa) < (\kappa^+)^K \). Therefore, the normal measures on \( \kappa \) in \( K \) are discrete –there is a sequence of pairwise disjoint sets \( \langle A_\xi \mid \xi < \sigma^K(\kappa) \rangle \), such that

\[
A_\xi \in U_{\kappa,\xi} \setminus \bigcup_{\zeta \neq \xi} U_{\kappa,\zeta}.
\]

Let us look at the measure which is applied at step \( \gamma_n \) of the iteration. As \( \text{crit}(\nu_{\gamma_n,\gamma_{n+1}}) = \nu_\gamma(\kappa) \) this is one of the measures on \( \nu_\gamma(\kappa) \) lying in \( K_{\gamma_n} \). Recall that \( \langle U_{\kappa,\xi} \mid \xi < \sigma^K(\kappa) \rangle \) denotes the sequence of normal measures on \( \kappa \) in \( K \) and that they are indexed according to their Mitchell order.

Since \( \text{crit}(\nu_\gamma) = \kappa > \sigma^K(\kappa) \) we conclude that

\[
\nu_\gamma(\langle U_{\kappa,\xi} \mid \xi < \sigma^K(\kappa) \rangle) = \langle \nu_\gamma(U_{\kappa,\xi}) \mid \xi < \sigma^K(\kappa) \rangle
\]

are the only normal measures on \( \nu_\gamma(\kappa) \) lying in \( K_\gamma \). In particular, the measure that is iterated at stage \( \gamma_n \) must have the form \( \nu_\gamma(U_{\kappa,\zeta_n}) \) for \( \zeta_n < \sigma^K(\kappa) \).

**Claim 6.1.3.** \( \nu_\alpha \) is \( \kappa \)-normal.

**Proof.** Let \( f : \kappa \to \kappa \) be a function in \( K \) with \( \{ \alpha < \kappa \mid f(\alpha) < \alpha \} \in \nu_\alpha \cap K \). By definition of \( \nu_\alpha \) and since \( f \in K \), \( \nu_\delta_\alpha(f(\eta_\omega)) < \eta_\omega \). Also, Claim 6.1.2 yields \( \nu_{\gamma_\omega+1}(f(\eta_\omega)) < \eta_\omega \), so that \( \{ \alpha < \eta_\omega \mid f(\alpha) < \alpha \} \) belongs to \( \nu_{\gamma_\omega}(U_{\kappa,\zeta_\omega}) \), the normal measure iterated at stage \( \gamma_\omega \). Thus, there is \( X \in \nu_{\gamma_\omega}(U_{\kappa,\zeta_\omega}) \) and \( \theta < \eta_\omega = \nu_\gamma(\kappa) \) such that \( \nu_{\gamma_\omega}(f)^\omega X = \{ \theta \} \). By elementarity, there is \( X \in U_{\kappa,\zeta_\omega} \) and \( \theta < \kappa \) such that \( f^\omega X = \{ \theta \} \). Moreover, \( X \in \nu_\alpha \cap K \):

\[
X \in U_{\kappa,\zeta_\omega} \iff \nu_{\gamma_\omega}(X) \in \nu_{\gamma_\omega}(U_{\kappa,\zeta_\omega}) \iff \eta_\omega \in \nu_{\gamma_\omega,\gamma_{\omega+1}}(X) \iff \eta_\omega \in \nu_\delta_\alpha(X). \quad \square
\]

**Claim 6.1.4.** Let \( \zeta_n \) be as above. Then,

\[
\zeta_n = \begin{cases} 
\xi_\gamma, & \text{if } \alpha = \gamma + 1; \\
\xi_{\beta_\zeta(o)}, & \text{if } \alpha \in \text{acc}(\omega_1).
\end{cases}
\]

**Proof.** By discreteness of the measures \( U_{\kappa,\xi} \)'s, \( \zeta_n \) is the unique ordinal \( \zeta \) such that \( \nu_\gamma(A_\zeta) \in \nu_\gamma(U_{\kappa,\zeta_n}) \). By Claim 6.1.1 and normality of the iteration this is equivalent to say that

\[
\eta_n \in \nu_{\gamma_n,\gamma_{n+1}}(\nu_\gamma(A_\zeta)) \iff \eta_n \in j(A_\zeta).
\]

Finally, this is equivalent to \( A_\zeta \in \nu(n) \cap K \). But \( \nu(n) \cap K \) is exactly \( U_{\kappa,\xi_\gamma} \) in the successor case and \( U_{\kappa,\xi_\delta_\zeta(o)} \) in the limit case, so the claim follows. □

Let us look at \( \nu_\alpha \cap K \). Since \( \nu_\alpha \) is \( \kappa \)-complete and \( K \)-normal there is \( \zeta < \sigma^K(\kappa) \) such that \( \nu_\alpha \cap K = U_{\kappa,\zeta} \). We next show that \( \zeta > \zeta_n \) for all \( n < \omega \). From this we shall be able to infer that \( \nu_\beta \cap K \preceq \nu_\alpha \cap K \).

**Claim 6.1.5.** \( \sigma^K_M(\eta_\omega) = \zeta \).
Proof of claim. Arguing as before, the measure used at stage $\gamma_\omega$ of the iteration is of the form $\nu_\gamma(U_{\kappa,\gamma_\omega})$ for some $\zeta_\omega < o^K(\kappa)$. In particular, $o^K(\eta_\omega) = \zeta_\omega$ and so, since $\eta_\omega < \text{crit}(\nu_{\gamma_\omega+1,\delta_\omega})$, $o^K(\eta_\omega) = \zeta_\omega$. Indeed,

$$\sigma^K(\eta_\omega) = \sigma^K(\tau_{\gamma_\omega,\delta_\omega}(\eta_\omega)) = \tau_{\gamma_\omega,\delta_\omega}(\sigma^K(\eta_\omega))$$

, since $\nu_{\gamma_\omega,\delta_\omega}(\eta_\omega) = \eta_\omega$ and $\nu_{\gamma_\omega,\delta_\omega}(\zeta_\omega) = \zeta_\omega$.

Let us now show that $\zeta_\omega = \zeta$. Once again, by discreteness, $\zeta_\omega$ is the unique ordinal such that $A_{\zeta_\omega} \subseteq U_{\kappa,\zeta_\omega}$. By elementarity this is equivalent to $\nu_\omega(A_{\zeta_\omega}) \subseteq \nu_\omega(U_{\kappa,\zeta_\omega})$, which is equivalent to $\eta_\omega \in j(A_{\zeta_\omega})$. All in all, we have that $A_{\zeta_\omega} \subseteq V_\alpha \cap K = U_{\kappa,\zeta}$. Thus, by our choice on $A_{\zeta_\omega}$, $\zeta = \zeta_\omega$. \square

Claim 6.1.6. For each $n < \omega$ there is a set $B_n \in [\omega]^{\omega}$ such that $\zeta_m = \zeta_n$ for all $m \in B_n$.

Proof of claim. Fix $n < \omega$. If $\alpha$ was a successor ordinal then we can let $B_n = \omega$ for in that case $\zeta_m = \xi_m$ for every $m < \omega$. Otherwise, if $\alpha$ is limit, put $B_n := r^{-1}\{r(n)\}$. By our choice upon $r$, $B_n \in [\omega]^{\omega}$. Also, by Claim 6.1.4 $\zeta_m = \xi_{\beta_m(m)} = \xi_{\beta_r(n)} = \zeta_n$. \square

Claim 6.1.7. For each $n < \omega$, $o^K(\eta_\omega) > \zeta_\omega$.

Proof of claim. Fix $n < \omega$ and let $B_n \in [\omega]^{\omega}$ be as in the previous claim. Since $M$ is closed under $\omega$-sequences we have that $\langle \eta_m \mid m \in B_n \rangle \in B$. Thus, working in $M$ we can define the following filter:

$$\mathcal{F}_n := \{X \subseteq \eta_\omega \mid \exists k \forall m \in (B_n \setminus k) \eta_m \in X\}.$$ 

We claim that $\mathcal{F}_n \cap \mathcal{K}^M = \nu_\omega(U_{\kappa,\zeta_\omega})$. Note that if this is the case $\mathcal{F}_n \cap \mathcal{K}^M$ is a measure on $\eta_\omega$ in $\mathcal{K}^M$ of Mitchell order $\zeta_\omega$ and thus $o^K(\eta_\omega) \geq \zeta_\omega + 1$.

Let $X \in \nu_\omega(U_{\kappa,\zeta_\omega})$. Note that $X$ takes the form $\nu_{\gamma_k,\gamma_\omega}(\bar{X}_k)$ for a tail of $k < \omega$ and $\bar{X}_k \in \bar{K}_k$. Let $k$ be the least of such indices and $m \in B_n \setminus k$. Since $X = \nu_{\gamma_m,\gamma_\omega}(\bar{X}_m)$ then, by elementarity, $\bar{X}_m \in \nu_{\gamma_m}(U_{\kappa,\zeta_m}) = \nu_{\gamma_m}(U_{\kappa,\zeta_m})$. Since this latter is the measure in $\mathcal{K}^\gamma$ used at stage $\gamma_m$ we have that $\eta_m \in \nu_{\gamma_m,\gamma_{m+1}}(\bar{X}_m)$, which is equivalent to $\eta_m \in \nu_{\gamma_m}(\bar{X}_m) = X$.

Conversely, let $X \in \mathcal{K}^M$, $X \subseteq \eta_\omega$ be such that $X \notin \nu_\omega(U_{\kappa,\zeta_\omega})$. Since $\text{crit}(\nu_{\gamma_\omega,\gamma_\omega+1}) = \eta_\omega$ we have that $X \notin \mathcal{K}^\gamma$. Fix $k < \omega$. Arguing as before find $m \in B_n \setminus k$ such that $\nu_{\gamma_m,\gamma_\omega}(\bar{X}_m) = X$. By elementarity, $\bar{X}_m \notin \nu_{\gamma_m}(U_{\kappa,\zeta_m})$, hence $\eta_m \notin \nu_{\gamma_m,\gamma_{m+1}}(\bar{X}_m)$, and thus $\eta_m \notin X$. Thereby, $X \notin \mathcal{F}_n$. \square

Thus, the above arguments show that $o^K(\eta_\omega) = \zeta > \zeta_\omega$ for all $n < \omega$.

Claim 6.1.8. $V_\beta \cap K \subseteq V_\alpha \cap K$ for all $\beta < \alpha$.

Proof of claim. If $\alpha = \gamma + 1$ then $V_\gamma \cap K = U_{\kappa,\xi_\gamma} = U_{\kappa,\zeta} \subseteq U_{\kappa,\zeta} = V_\alpha \cap K$.

Suppose that $\alpha$ is a limit ordinal and let $\beta < \alpha$. Choose $\beta_\alpha \in (\beta, \alpha)$ and observe that $\beta_\alpha = \beta_r(m)$ for some (infinitely-many) $m < \omega$. In particular, $V_\beta \cap K \subseteq V_{\beta_\alpha} \cap K = V_{\beta_r(m)} \cap K = U_{\kappa,\zeta_m} \subseteq U_{\kappa,\zeta} = V_\alpha \cap K$. \square

\footnote{Note that this latter equality follows by our choice of $B_n$.}
This completes the argument for the inductive step of the construction and as result the proof of the theorem.

7. Open problems

We would like to conclude the paper mentioning a few open problems.

In §3.2 we gave a forcing argument showing that consistently a cardinal \( \kappa \) which is \( \kappa^+ - \Pi^1_1 \)-subcompact can fail to have the \( (2^\kappa)^+ \)-gluing property. Intuitively, we expect a better result:

**Question 1.** Can the least cardinal \( \kappa \) which is \( \kappa^+ - \Pi^1_1 \)-subcompact have the \( 2^{2^\kappa} \)-gluing property?

While we suspect that the consistency strength of the gluing property is low, it is rather unclear which large cardinals weaker than strongly compact outright imply gluing. For example, a standard core model argument shows that if there is no inner model with a Woodin cardinal then in the core model the least strong cardinal does not have the \( \omega \)-gluing property.

**Question 2.** What large cardinals do have the \( \omega \)-gluing property? More generally, which of them do have the \( \lambda \)-gluing property for every cardinal \( \lambda \)?

In light of the striking structural consequences of Goldberg’s *Ultrapower Axiom (UA)* [Gol22] the following speculation seems natural:

**Question 3.** Is there a characterization, under UA, of those measurable cardinals having the \( \lambda \)-gluing property?

In §5 and §6 we show that the consistency strength of the \( \omega \)-gluing property is exactly “\( \exists \kappa (\omega(\kappa) = \omega_1) \)”. This raises an obvious question:

**Question 4.** What is the consistency strength of “There is a measurable cardinal with the \( \lambda \)-gluing property” for \( \lambda \geq \omega_1 \)?

Our conjecture is that starting with a strong cardinal one should be able to produce a model with a measurable cardinal having the \( \lambda \)-gluing property for all cardinals \( \lambda \). This combined with Gitik’s theorem from §1 would yield the exact consistency strength of this compactness principle.

Another interesting question from the technical perspective is:

**Question 5.** Suppose that \( \kappa \) has the \( \lambda \)-gluing property. Is there a forcing poset destroying this property? In general, using forcing, can we tune the exact amount of gluing that \( \kappa \) carries?

Lastly, in §5, we gave a characterization of the \( \kappa \)-complete ultrafilters on \( \kappa \) in a certain forcing extension of \( K \). By recent intriguing results of Gitik and Kaplan [GK23], we suspect that the assumption \( V = K \) can be dropped (or replaced by a suitable GCH hypothesis).

\[22\] Indeed, if \( j: K \rightarrow K^M \) is an ultrapower embedding and \( M \) is closed under \( \kappa \)-sequences, then it must be a finite iteration of extenders from \( K \). In particular, there are at most finitely many \( \alpha < j(\kappa) \) which belong to \( \bigcap \{ j(C) \mid C \subseteq \kappa, \text{ club} \} \).
Question 6. Let $\mathbb{P}$ be a non-stationary support iteration of tree Prikry forcings over an arbitrary ground model $V$. Is there a full characterization of the $\kappa$-complete ultrafilters on $\kappa$ in the generic extension, similar to Lemma 5.8?

References


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