Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information*

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Abstract: This paper derives sufficient conditions for a class of games of incomplete information, such as first price auctions, to have pure strategy Nash equilibria (PSNE). The paper treats games between two or more heterogeneous agents, each with private information about his own type (for example, a bidder’s value for an object or a firm’s marginal cost of production), and the types are drawn from an atomless joint probability distribution which potentially allows for correlation between types. Agents’ utility may depend directly on the realizations of other agents’ types, as in Milgrom and Weber’s (1982) formulation of the “mineral rights” auction. The restriction we consider is that each player’s expected payoffs satisfy the following single crossing condition: whenever each opponent uses a nondecreasing strategy (that is, an opponent who has a higher type chooses a higher action), then a player’s best response strategy is also nondecreasing in her type.

The paper has two main results. The first result shows that, when players are restricted to choose among a finite set of actions (for example, bidding or pricing where the smallest unit is a penny), games where players’ objective functions satisfy this single crossing condition will have PSNE. The second result demonstrates that when players’ utility functions are continuous, as well as in mineral rights auction games and other games where “winning” creates a discontinuity in payoffs, the existence result can be extended to the case where players choose from a continuum of actions.

The paper then applies the theory to several classes of games, providing conditions on utility functions and joint distributions over types under which each class of games satisfies the single crossing condition. In particular, the single crossing condition is shown to hold in all first-price, private value auctions with potentially heterogeneous, risk-averse bidders, with either independent or affiliated values, and with reserve prices which may differ across bidders; mineral rights auctions with two heterogeneous bidders and affiliated values; a class of pricing games with incomplete information about costs; a class of all-pay auction games; and a class of noisy signaling games. Finally, the formulation of the problem introduced in this paper suggests a straightforward algorithm for numerically computing equilibrium bidding strategies in games such as first price auctions, and we present numerical analyses of several auctions under alternative assumptions about the joint distribution of types.

Keywords: Games of incomplete information, pure strategy Nash equilibrium, auctions, pricing games, signaling games, supermodularity, log-supermodularity, single crossed, affiliation.

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1. Introduction

This paper derives sufficient conditions for a class of games of incomplete information, such as first price auction games, to have pure strategy Nash equilibria (PSNE). The class of games is described as follows: there are $I$ agents with private information about their types, and the types are drawn from a joint distribution which allows for correlation between types. Types are drawn from a convex subset of $\mathcal{R}^l$, and the joint distribution of types is atomless. We allow for heterogeneity across players in the distribution over types as well as the players’ utility functions, and the utility functions may depend directly on other players’ types. Thus, the formulation includes the “mineral rights” auction (Milgrom and Weber (1982)), where bidders receive a signal about the underlying value of the object, and signals and values may be correlated across players. The existence result does not make any assumptions about quasi-concavity or differentiability of the underlying utility functions of the agents, nor does it require that each agent has a unique optimal action for any type.

The main restriction studied in this paper is what we call the single crossing condition for games of incomplete information: for every player $i$, whenever each of player $i$’s opponents uses a pure strategy such that higher types choose (weakly) higher actions, player $i$’s expected payoffs satisfy Milgrom and Shannon’s (1994) single crossing property. In particular, when choosing between a low action and a high action, if a low type of agent $i$ weakly (strictly) prefers the higher action, then all higher types of agent $i$ will weakly (strictly) prefer the higher action as well. This condition implies that in response to nondecreasing strategies by opponents, each player will have a best response strategy where higher types choose higher actions. The single crossing condition contrasts with that studied by Vives (1990), who shows that a sufficient condition for existence of PSNE is that the game is supermodular in the strategies, in the following sense: if one player’s strategy increases pointwise (almost everywhere), the best response strategies of all opponents must increase pointwise (a.e). However, in Vives’ analysis, the strategies themselves need not be monotone in types. Vives’ condition is applicable games where players have supermodular utility functions, but not in the auctions and log-supermodular pricing games highlighted in this paper.

The paper has four parts. The first part shows that when a game of incomplete information satisfies the single crossing condition described above, but when the players are restricted to choose from a finite action space, a PSNE exists. Next, we show that under the further assumption that players’ utility functions are continuous, or in a class of “winner-take-more” games such as first price auctions, we can find a sequence of equilibria of finite-action games which converges to an equilibrium in a game where actions are chosen from a continuum. The third part of the paper builds on Athey (1995, 1996) to study conditions on utility functions and type distributions under which commonly studied games satisfy the single crossing condition. Games which satisfy the single crossing condition include any first price auction, where values are
private, bidders are (weakly) risk averse, and the types are independent or affiliated.\footnote{Weber (1994) studies mixed strategy equilibria in auction a class of auction games where the affiliation inequality fails.} In the class of “mineral rights” auctions, the conditions are satisfied when there are two heterogeneous bidders whose types are affiliated. Other applications which satisfy the conditions include all-pay auctions and multi-unit auctions with heterogeneous bidders and independent private values, noisy signaling games (such as limit pricing with demand shocks), and a class of supermodular and log-supermodular quantity and pricing games with incomplete information about costs. The final part of the paper focuses on numerical computation of equilibria, showing that equilibria to first price auctions may be easily computed for games with a finite number of potential bids. Several examples of auctions are provided, considering alternative scenarios for heterogeneity, private versus common values, and correlated versus independent values.

The existence result for finite action spaces analyzed in the first part of this paper is straightforward to prove using a reformulation of the problem, which allows us to simplify the game to a finite-dimensional problem and then apply standard fixed point theorems. The existence result proceeds in two steps. First, we observe that if a player uses a nondecreasing strategy which maps types into actions, the strategy will be a nondecreasing step function. Thus, we can restate the player’s problem as determining at which realizations of his type the strategy will “step” to the next highest action, as well as what action is taken at the “step point.” Once the problem has been reformulated in this way, we can view a nondecreasing strategy as a subset of finite-dimensional Euclidean space, and the existence result then relies on Kakutani’s fixed point theorem. The single crossing condition plays two roles in this analysis. First, it simplifies the strategies enough such that they can be represented with vectors of “step points.” Second, we show that it implies that the set of vectors which represent optimal actions is convex. The paper also demonstrates that the logic of the argument can be extended to games with nonmonotonic strategies. For example, a PSNE will exist in games where every player’s best response to a U-shaped strategy by the opponents is U-shaped. However, this result requires an additional assumption to guarantee that the best response correspondence is convex: we assume that players are never indifferent between two actions over an open interval of types.

Thus, the first part of this paper shows that with finite actions, PSNE exist quite generally. The next step is to derive conditions under which these results extend to continuous action spaces. We show that when there exists an equilibrium in nondecreasing strategies for every finite game, and when the players’ objectives are continuous, there exists a limit of a sequence of equilibrium strategies for finite games which is an equilibrium in a game with continuous actions. It is
important to know that the strategies are monotonic (or satisfy related properties such as a U-shape) because when the strategies are of bounded variation, a sequence of equilibrium strategies for finite games has an almost-everywhere convergent subsequence (by Helly’s selection theorem (Billingsley, 1968)). However, in auctions and related games where winning gives a discrete change in payoffs, the players’ objectives are not in general continuous in their own action, and thus establishing existence in such games requires additional work. Using properties of the equilibria to finite-action games, we show that if an auction game satisfies the single crossing property described above, so that best responses to nondecreasing strategies are nondecreasing, plus some additional regularity conditions, then existence result from the finite action auction game will extend to auctions with continuous action spaces.

Analyzing existence in games with a continuum of types and a continuum of actions is difficult because the strategy space is not a finite subset of Euclidean space. Many previous results about existence of pure strategy equilibria are concerned with issues of topology and continuity in the relevant strategy spaces. For example, Milgrom and Weber (1985) show that pure strategy equilibria exist when type spaces are atomless and players choose from a finite set of actions, types are independent conditional on some common state variable (which is finite-valued), and each player’s utility function depends only on his own type, the other players’ actions, and the common state variable (the utility cannot depend on the other players’ types directly). They also study a condition which they call “continuity of information.” Similarly, Radner and Rosenthal (1982) show that players choose from a finite set of actions, types are independent, and each player’s utility function depends only on his own type, but the type distributions are atomless, then a pure strategy equilibrium will exist. The authors then provide several counter-examples of games which fail to have pure strategy equilibria, in particular games where players’ types are correlated. The counter-examples of Radner and Rosenthal fail our sufficient conditions for existence in games with finite actions a different reason: the best response of one player is always a little bit more complicated than the strategy of his opponent.

In contrast, our analysis allows arbitrary correlation between types, to the extent that the joint distributions over types lead to expected payoff functions which satisfy the single crossing property of incremental returns. Thus, any required restrictions on the distribution (such as affiliation) have economic interpretations.

Now consider the special case of first price auctions. The issue of the existence of pure strategy equilibria in first price auctions with heterogeneous agents (and continuous actions spaces)

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2 Radner and Rosenthal (1982) also treat the case where each player can observe a finite-valued “statistic” about a random variable which affects the payoffs of all agents.
has challenged economists for many years. Recently, several authors have made substantial progress, establishing existence and sometimes uniqueness for asymmetric independent private values auctions (Maskin and Riley (1993, 1996); Lebrun (1995, 1996); Bajari (1996a)), as well as affiliated private values or common value auctions with conditionally independent signals (Maskin and Riley, 1996). Lizzeri and Persico (1997) have independently shown that a condition closely related to the single crossing condition is sufficient for existence and uniqueness of equilibrium in two-player mineral rights auction games with heterogeneous bidders, but their approach does not extend to more than two bidders without symmetry assumptions. Many interesting classes of auctions with heterogeneous bidders are not treated by the existing analysis, and even for the auctions where existence is known, computation of equilibrium (which involves numerically computing the solution to a system of nonlinear differential equations with two boundary points) can be difficult due to pathological behavior of the system.\(^3\)

Two main approaches to existence have been used in the case of first price auctions: (i) establishing that a solution exists to a set of differential equations (Lebrun (1995), Bajari (1996a), Lizzeri and Persico (1997)), and (ii) establishing that an equilibrium exists when either types or actions are drawn from finite sets, and then invoking limiting arguments (Lebrun (1996), Maskin and Riley (1992)). Since there exist games which have pure strategy equilibria for every finite action set, but where there is no pure strategy equilibrium in the infinite case (for example, see Fullerton and McAfee (1996)), these limiting arguments generally involve more work and use the special structure of the game at hand.\(^4\)

The strategy taken in this paper is different from that of the existing literature, in that we treat the issue of existence of equilibrium separately from the issue of monotonicity of strategies in different classes of auctions. The second part of this paper establishes existence under the assumption that the single crossing condition holds, while the third part derives conditions under which the single crossing condition holds for different classes of auctions. This distinction is useful because in some classes of auctions (i.e. private value auctions), it is relatively easy to establish that the single crossing condition holds, while it can be challenging in other classes (for example, mineral rights auctions with more than two bidders, where players form expectations about their own value from winning based on their signal and the other players’ actions). Knowing that the single crossing condition implies existence can also be helpful if it is possible to demonstrate that the single crossing condition holds for a range of parameter values, or if it is

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\(^3\) Marshall et al (1994) summarize some of the problems; they provide and implement an algorithm which alleviates these difficulties for a simple class of distributions, \(F(x|a) = x^a\). See Section 5 of this paper for more discussion.

\(^4\) Simon and Zame (1990) provide an elegant treatment of limiting arguments in the context of mixed strategy equilibria.
possible to establish that it holds in the relevant region using numerical analysis.

The theoretical and computational difficulties in analyzing auctions with heterogeneous bidders have confounded attempts to apply and test auction theory in real-world problems,\(^5\) where heterogeneity and correlation between types are the rule rather than the exception, and as well the private values assumption may be tenuous. For example, in construction auctions, the number of contracts which a company currently holds significantly affects its value for new work. Likewise, the companies face common uncertainty about the difficulties which will arise in completing the construction project. In timber auctions, different companies have different transportation costs due to the different locations of their mills. Further, even if bidders are ex ante symmetric, if they collude or engage in joint bidding, asymmetries will arise (Marshall et al., 1994).

Since very little is known about the theoretical properties of general auction games with heterogeneous bidders, numerical computation can also play an important role in suggesting avenues for future theorizing. Further, the growing literature on structural empirical analyses of auctions (i.e. Laffont, Ossard, and Vuong (1995), Bajari (1996b)) requires that equilibria to auctions be computed in each iteration of an econometric procedure. The final section of this paper analyzes the computation of equilibria to auctions. For example, the representation of nondecreasing strategies with finite actions as a vector of “step points” implies that equilibria to auction games can be computed with very simple algorithms. There are well-developed numerical techniques for approximating a fixed point in finite-dimensional problems (Judd, forthcoming). We provide a numerical analysis of several examples of first price auctions with alternative assumptions about heterogeneity and the type distribution.

2. Existence of Equilibrium with Finite Actions

This section derives sufficient conditions for the existence of a PSNE in a game of incomplete information, where the types of the players are drawn from an atomless distribution and the players are restricted to choose from a finite set of actions. Consider a game of incomplete information between \(I\) players, \(i=1, \ldots, I\), where each player first observes their own type \(t_i \in T_i = [t_i, \bar{t}_i] \subseteq \mathbb{R}\) and then takes an action \(a_i\) from an action space \(A_i \subseteq \mathbb{R}\). Each player’s utility, \(u_i(a, t)\), may depend on the actions taken as well as the types directly. The joint density over player types is \(f(t)\), with conditional densities \(f(t_i|t)\). The objective function for player \(i\) is then specified as follows. Given any set of strategies for the opponents, \(\alpha_j; [t_j, \bar{t}_j]) \rightarrow A_j, j \neq i\), we can write player \(i\’s\)

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objective function as follows (using the notational convention \( (a_i, \alpha_{-i}(t_{-i})) = (\ldots, \alpha_{i,1}(t_{i,1}), \ldots) \)):

\[
U_i(a_i, \alpha_{-i}(\cdot), t_i) = \int_{t_{i,1}} u_i((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}^i) dt_{-i}
\]

The following basic assumptions are maintained throughout the paper:

The types have joint density with respect to Lebesgue measure, \( f(t) \), which is bounded and
has no mass points.\(^6\) Further, \( \int_{t_{i,1} \in S} u_i((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}^i) dt_{-i} \) exists and is finite for
all \( S \) and all \( \alpha_j^i; [t_j, \tilde{t}_j] \rightarrow \mathcal{A}^j, j \neq i \).

We proceed by proving two results. The first looks for pure strategy equilibria in
nondecreasing strategies, while the second potentially allows for more complicated strategies.
However, the second result requires that players are never indifferent between two strategies over
an open interval of types (this assumption would not be satisfied for an auction without additional
structure, since two actions might both win with probability zero against more aggressive opponent).

### 2.1. Pure Strategy Equilibrium in Nondecreasing Strategies

This section studies games with finite action spaces which satisfy the single crossing condition.
The single crossing will play two roles in the analysis. First, it will guarantee that we can
represent each agent’s strategy with a vector of finite dimension. Second, it will be used to
 guarantee that each player’s best response correspondence is convex (recall that we have not made
assumptions which could be used to guarantee a unique optimum). These two properties of games
with the single crossing condition allow us to use Kakutani’s fixed point theorem to guarantee
existence of a PSNE.

Before beginning our analysis of equilibria in nondecreasing strategies, we introduce the
definition of Milgrom and Shannon’s (1994) single crossing property of incremental returns (SCP-
IR), as well as the corresponding theorem which states that SCP-IR is sufficient for a monotone
comparative statics conclusion to hold in problems which may be non-differentiable, non-concave,
or have multiple optima. Consider the following definition:

**Definition 2.1** \( h(x, \theta) \) satisfies the (Milgrom-Shannon) single crossing property of
incremental returns (SCP-IR) in \( (x; \theta) \) if, for all \( x_0 > x \), \( g(\theta) = h(x_0, \theta) - h(x, \theta) \) satisfies the

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\(^6\) In games with finite actions, condition (2.1) can be relaxed to allow for mass points at the lower end of the
distribution, so long as for each player, there exists a \( k > t_j \) such that the lowest action chosen by player \( j \) is chosen
throughout the region \([t_j, k]\).
following conditions for all $\theta_h > \theta_i$: (a) $g(\theta_i) \geq 0$ implies $g(\theta_h) \geq 0$, and (b) $g(\theta_i) > 0$ implies $g(\theta_h) > 0$.

The definition requires that if $x_H$ is (strictly) preferred to $x_L$ for $\theta_i$, then $x_H$ must still be (strictly) preferred to $x_L$ if $\theta$ increases from $\theta_i$ to $\theta_H$. In other words, the incremental returns to $x$ cross zero at most once, from below, as a function of $\theta$ (Figure 1). Note the relationship is not symmetric between $x$ and $\theta$. Milgrom and Shannon show that SCP-IR implies that the set of optimizers is nondecreasing in the Strong Set Order (SSO), defined as follows:

**Definition 2.2** A set $A \subseteq \mathbb{R}$ is greater than a set $B \subseteq \mathbb{R}$ in the **strong set order** (SSO), written $A \trianglerighteq B$, if, for any $a \in A$ and any $b \in B$, $\max(a, b) \in A$ and $\min(a, b) \in B$. A set-valued function $A(\tau)$ is nondecreasing in the strong set order (SSO) if for any $\tau_H > \tau_i$, $A(\tau_H) \trianglerighteq A(\tau_i)$.

**Lemma 2.1** (Milgrom and Shannon, 1994) Let $h: \mathbb{R}^2 \to \mathbb{R}$. Then $h$ satisfies SCP-IR if and only if $x'(\theta) = \arg\max_{x \in \mathbb{R}} h(x, \theta)$ is nondecreasing in the strong set order for all $B$.

Under SCP-IR, there might be a $x' \in x^*(\theta_i)$ and a $x'' \in x^*(\theta_H)$ such that $x' > x''$, so that some selection of optimizers is decreasing on a region; however, if this is true, then $x' \in x^*(\theta_H)$ as well. Using Definition 2.1, we can state the sufficient condition for existence of a pure strategy Nash equilibria in nondecreasing strategies.

We say that a game satisfies the **Single Crossing Condition (SCC)** for games of incomplete information if, for $i = 1, \ldots, I$, given any set of $I - 1$ nondecreasing functions $\alpha_j: [L_j, H_j] \to \mathcal{A}^i, j \neq i$, player $i$’s objective function, $U_i(a_i, \alpha_j(\cdot), t_i)$, satisfies single crossing of incremental returns (SCP-IR) in $(a_i; t_i)$.  \hspace{1cm} (2.2)

The first result is then stated as follows:

**Theorem 2.1** Consider the game of incomplete information described above, where (2.1) and the Single Crossing Condition (2.2) hold. If $\mathcal{A}^i$ is finite for all $i$, this game has a PSNE, where each player’s equilibrium strategy, $\beta_i(t_i)$, is a nondecreasing function of $t_i$.

The proof of Theorem 2.1 relies on the fact that, when players choose from a finite set, any nondecreasing strategy $\alpha_i(t_i)$ is simply a nondecreasing step function. Then, the strategy can be described simply by naming the values of the player’s type $t_i$ at which the player “jumps” from one action to the next higher action. To formalize this idea, consider the following representation of nondecreasing strategies. For simplicity, we treat the case where each player has the same feasible
action space, but this assumption is made purely to conserve notation. Let \( \mathcal{A} = \{A_0, A_1, \ldots, A_M\} \) be the set of potential actions, in ascending order, where \( M+1 \) is the number of potential actions. Let \( T_i^M = \bigtimes_{m=1}^{M} [t_i^m, t_i^*] \), and define the set of nondecreasing vectors in \( T_i^M \) as follows: \( \Sigma_i^M = \left\{ x \in T_i^{M+2} \left| x_0 = t_i^1, x_1 \leq x_2 \leq \cdots \leq x_M, x_{M+1} = t_i^* \right. \right\} \). Further, let \( \Sigma = \Sigma_1^M \times \cdots \times \Sigma_i^M \).

Consider a candidate nondecreasing strategy for player \( i \), \( \alpha_i : [t_i^1, t_i^*] \to \mathcal{A} \). For any such function \( \alpha_i \), we can assign a corresponding vector \( x \in \Sigma_i^M \) according to the following algorithm (illustrated in Figure 2).

**Definition 2.3**  
(i) Given a nondecreasing strategy \( \alpha_i(t_i) \), we say that “the vector \( x \in \Sigma_i^M \) represents \( \alpha_i(t_i) \)” if \( x_m = \inf \{ t_i | \alpha_i(t_i) \geq A_m \} \) whenever there is some \( n \geq m \) such that \( \alpha_i(t_i) = A_n \) on an open interval of \( T_i \), and \( x_m = t_i^* \) otherwise.

(ii) Given \( x \in \Sigma_i^M \), let \( \{x\} \) denote the set \( \{t_i^1, x_1, \ldots, x_M, t_i^*\} \), and let \( m^*(t, x) = \max \{m | x_m < t\} \). Then, we say a nondecreasing “strategy \( \alpha_i(t_i) \) is consistent with \( x \)” if \( \alpha_i(t_i) = A_{m^*(t, x)} \) for all \( t_i \in T_i \setminus \{x\} \).

Each component of \( x \) is therefore a point of discontinuity, or a “jump point,” of the step function described by \( \alpha_i \). Simply by naming the \( x \) which corresponds to \( \alpha_i \), we have specified what action the player takes everywhere except at the jump points and endpoints, that is, the actions are specified on \( T_i \setminus \{x\} \).

To interpret part (ii), note that \( \alpha_i(t_i) \) is consistent with \( x \) if action \( A_m \) is used between \( x_m \) and \( x_{m+1} \). Since \( x \) does not specify behavior for \( t_i \in \{x\} \), a given \( x \in \Sigma_i \) might correspond to more than one nondecreasing strategy. However, because there are no atoms in the distributions of types, a player’s behavior on the set \( \{x\} \) (which has measure zero) will not affect the best responses of other players. The proof makes use of this fact, proceeding by finding a fixed point in the space of nondecreasing best responses which can be described by elements of \( \Sigma \), and then filling in optimal behavior for each player at the jump points. Behavior at the jump points can be assigned arbitrarily since it won’t affect the best responses of the other players; thus, it can be assigned to be optimal without loss of generality, making the resulting strategies best responses for all types.

Now consider notation for the objective function faced by a given player. Let \( X \) denote the vector \((x^1, \ldots, x^i)\), which describes a set of step functions for each player. Consider player 1 with
type \( t_i \), and let \( V_i(a_i;X,t_i) \) denote the expected payoffs to player 1 when player 1 chooses \( a_i \in \mathcal{A} \) and players 2,...,\( I \) use strategies which are consistent with \( (x^2,..,x^I) \). Then \( V_i(a_i;X,t_i) \) can be written as follows:

\[
V_i(a_i;X,t_i) = \sum_{m_z=0}^M \sum_{m_y=0}^M \sum_{t_{-i}} \int u_i(a_i,A_{m_z},...,A_{m_y},t) \cdot 1_{(x^2_{m_z}..x^I_{m_y})=x(t_{-i})} (t_{-i}) \cdot f(t_{-i}|t_i) dt_{-i}
\]

(2.3)

where \( 1_{S}(y) \) is an indicator function which takes the value 1 if \( y \in S \) and 0 otherwise.

By (2.1), the behavior of opponents on sets of measure zero do not affect player \( i \), so player \( i \) views each opponent as using a nondecreasing strategy whenever they use actions consistent with \( x^i \) on \( T_i \{x^i \} \). Then, by (2.2), \( V_i(a_i;X,t_i) \) satisfies the SCP-IR in \( (a_i,t_i) \). Let \( a_{BR}^i(t_i|X)= \arg \max \limits_{a_i \in \mathcal{A}} V_i(a_i;X,t_i) \); this is nonempty for all \( t_i \) by finiteness of \( \mathcal{A} \). By Lemma 2.1, there exists a selection, \( \gamma(t_i) \in a_{BR}^i(t_i|X) \), from the set which is nondecreasing in \( t_i \) (in particular, the lowest and highest members of this set are nondecreasing; see Milgrom and Shannon (1994)). As we argued above, there exists a \( y \in \Sigma_i \) which represents \( \gamma(t_i) \), so that \( \gamma(t_i) \) is consistent with \( y \). Thus, there is at least one best response vector \( y \) which represents an optimal strategy. Now define the set of all such vectors as follows:

\[
\Gamma_i(X) = \{ y : \exists \alpha(t_i) \text{ which is consistent with } y \text{ such that } \alpha(t_i) \in a_{BR}^i(t_i|X) \}.
\]

The existence proof proceeds by showing that a fixed point exists for this correspondence. Once the problem is formulated in this way, it is straightforward to verify that the correspondence \( \Gamma(X) \) is nonempty and has a closed graph. However, convexity of the correspondence requires additional work. Observe that even if the player’s payoff function is strictly quasi-concave, and
even if a change in the player’s type changes payoffs everywhere, the player still might be indifferent between two actions over a set of types, as shown in Figure 4. Thus, it is important to address the issue of multiple optimal actions. The proof makes use of an important consequence of the SCP-IR: it implies that the set of best response actions is increasing in the strong set order. In Figure 3, notice that as \( t_i \) increases, higher actions come into the set of optimizers, but once a lower action leaves the set of optimizers, it never reappears. Further, once a given action has entered the set of optimizers, no lower actions enter the set for the first time. When this property is satisfied, it is straightforward to show that the set of vectors which represent optimal behavior will be convex.

In the figure, \( \mathbf{x} \) and \( \mathbf{y} \) are both vectors of jump points representing optimal behavior; the arrows in the figure show convex combinations of \( x_m \) and \( y_m \) for \( m=1, \ldots, 4 \), and any such convex combination also represents optimal behavior.

**Lemma 2.2:** Define \( \Gamma_i \) as above, \( i=1, \ldots, I \). Then there exists a fixed point of the correspondence \( (\Gamma_1(\mathbf{X}), \ldots, \Gamma_I(\mathbf{X})):\Sigma \rightarrow \Sigma \).

**Proof of Lemma:** The proof proceeds by checking that the correspondence \( \Gamma=(\Gamma_1(\mathbf{X}), \ldots, \Gamma_I(\mathbf{X})) \) is nonempty, has a closed graph, and is convex-valued. Then Kakutani’s fixed point theorem will give the result. The details are in the Appendix.

With this result in place, all that remains in proof of Theorem 2.1 is to assign strategies to players which are consistent with the fixed point of \( \Gamma(\mathbf{X})=\mathbf{X} \). By definition, \( \Gamma(\mathbf{X}) \) describes strategies for each player which are optimal almost everywhere in response to behavior by the other players which is consistent with \( \mathbf{X} \), and that each player does not care what the other players do on a set of measure 0. Thus, for each player \( i \) we can assign any behavior we like to the “jump points,” \( \{\mathbf{x}'\} \), without affecting the best responses of the opponents. Then, all that remains is to fill in the behavior of each player at the “jump points.” Consider an \( \mathbf{X} \) such that \( \mathbf{X} \in \Gamma(\mathbf{X}) \). The correspondence \( \Gamma \) was constructed so that each \( \mathbf{x}' \) represents a nondecreasing, optimal strategy for player \( i \) given \( \mathbf{X} \), call it \( \beta_i(t_i) \). Then the vector of nondecreasing strategies \( (\beta_1(t_1), \ldots, \beta_I(t_I)) \) is a pure strategy Nash Equilibrium of the original game, since \( \beta_i(t_i) \in a^{\text{BR}}_i(t_i|\mathbf{X}) \) for \( i=1, \ldots, I \) and \( t_i \in T_i \).

### 2.2. An Existence Theorem for Strategies of Limited Complexity

Now, we turn to a generalization of Theorem 2.1 beyond games with the single crossing condition (2.2). The basic idea is that an equilibrium will exist if we can find bounds on the “complexity” of each player’s strategy such that each player’s best response stays within those bounds when the opponents use strategies which can be represented within those bounds. The formalization we use for representing strategies builds directly on our representation of nondecreasing strategies (there are, of course, alternative representations). We will need a
Definition 2.4 The strategy \( \alpha_i(t_i) \) has at most \( K \) direction changes if there exists a nondecreasing vector \( z \in \Sigma_i^K \), such that for all \( 0 \leq k \leq K \) the following holds:

(i) If \( k \) is even, then for all \( z \leq t'_i \leq t''_i \leq z_{k+1} \), \( \alpha_i(t'_i) \leq \alpha_i(t''_i) \).

(ii) If \( k \) is odd, then for all \( z \leq t'_i \leq t''_i \leq z_{k+1} \), \( \alpha_i(t'_i) \geq \alpha_i(t''_i) \).

We say that such a \( z \) represents the direction changes of \( \alpha_i(t_i) \).

This definition, illustrated in Figure 5, merely formalizes the idea that a strategy changes from nondecreasing to nonincreasing, or vice versa, at most \( K \) times. Thus, a nondecreasing strategy has at most 0 direction changes. An important feature of strategies with at most \( K \) direction changes is that they are functions of bounded variation; this will imply that our existence results extend to games with continuous action spaces in Section 3.

We can then generalize condition (2.2) as follows:

A game of incomplete information satisfies the Limited Complexity Condition if there exists a vector of nonnegative integers \( K=(K_1,\ldots,K_i) \), such that, for all \( i \), if each of player \( i \)'s opponents \( j \neq i \) use strategies which have at most \( K_j \) direction changes, and \( \mathcal{A} \) is finite, then there exists a best response for player \( i \) which has at most \( K_i \) direction changes. Further, when opponents use such strategies, then for all \( a \neq a'_i \), there is no open interval \( T'_i \subseteq T_i \) such that \( U_i(a_i,\alpha_i(\cdot),t_i)=U_i(a'_i,\alpha_i(\cdot),t_i) \) for all \( t_i \in T'_i \). (2.4)

Unlike Theorem 2.1, where we could prove convexity of the best response correspondence from the SCP-IR, here we require an additional assumption which implies that the best response action is unique for almost all types. This in turn implies that there will be a unique vector of jump points representing the best response strategy. To see why this assumption is required, consider Figure 6, where a player is indifferent between two actions over two regions of types. There are two vectors, \( x \) and \( y \), both of which represent optimal
behavior. However, the convex combination of \( x \) and \( y \) would assign the player to use action \( A_i \) in the region \( (\lambda x_1+(1-\lambda)y_1,\lambda x_2+(1-\lambda)y_2) \), which is not optimal. Condition (2.4) rules this out.

While the Limited Complexity Condition may arise naturally in some problems (the case of U-shaped strategies seems especially promising), another possible motivation for the condition could be bounded rationality on the part of the players.

Under condition (2.4), we can extend the logic of Theorem 2.1 in a straight-forward way. The representation is a natural extension of the one developed for nondecreasing strategies; the vector which represents our strategy will have \( K \) subvectors, each of which describes behavior on a monotonic portion of the strategy. This is formalized in the appendix, as part of the proof of the following theorem.

**Theorem 2.2** Consider a game of incomplete information, where (2.1) and the Limited Complexity Condition (2.4) hold for some \( K \). Then this game has a PSNE where each player’s strategy, \( \beta(\cdot) \), has at most \( K \) direction changes.

It is interesting to discuss the relationship between this result and the results from the existing literature, especially Radner and Rosenthal (1982). On the one hand, condition (2.4) seems quite general: we do not need it to hold for all \( K \), but rather just for some \( K \). What it rules out is games where a given player’s response to a “simple” strategy becomes ever more complicated. Radner and Rosenthal maintain assumption (2.1) and finite actions, and further they require independence of types. Under those conditions, they find existence of a pure strategy equilibrium. They then present counterexamples, such as an incomplete-information variant of matching pennies, where correlated information leads to nonexistence of a pure strategy equilibrium.

Since our results do not place ex ante restrictions on the joint distribution over types, it is interesting to revisit their example. The setup is as follows: the game is zero-sum, and each player can choose actions \( A_0 \) or \( A_1 \). When the players match their actions, player 2 pays $1 to player 1, while if they do not match, the players each receive zero. The types do not directly affect payoffs, and are uniformly distributed on the triangle \( 0 \leq t_1 \leq t_2 \leq 1 \). We now argue that this game fails condition (2.4). Since player \( i \) is only indifferent between the actions if \( \Pr(a_j = A_i | t_j) = .5 \), and since \( \Pr(a_j = A_0 | t_j) \) cannot be constant in \( t_j \) when player \( j \) uses a finite number of direction changes, the there will be no open interval on which player \( i \) is indifferent between the two actions. The issue is that whenever player 1 uses a strategy with \( K \) direction changes (i.e. alternates between \( A_0 \) and \( A_1 \) \( K \) times, starting with \( A_0 \)), player 2 potentially has the incentive to switch between \( A_0 \) and \( A_1 \) \( K \) times as well, but starting with \( A_1 \) due to the incentive of player 1 to avoid a match. This can be represented as \( K + 1 \) direction changes using Definition 2.4, with the first change degenerate. However, in response to an arbitrary strategy by player 2 with \( K + 1 \) direction changes, player 1 wishes to use a strategy with \( K + 1 \) direction changes. In turn, such a strategy by player 1 will
induce a response by player 2 represented by $K + 2$ direction changes. Notice that this logic does not change if we reorder the action space for player 2; the fact that player 2 tries to “run away” from player 1’s strategy, but then player 1 tries to “catch” player 2, leads to a situation where strategies can potentially get progressively more complex, and thus our condition (2.4) fails.

3. Existence of Equilibrium in Games with a Continuum of Actions

This section shows that the results about existence in games with a finite number of actions can be used to construct equilibria of games with a continuum of actions. As discussed in the introduction, the assumption about finite actions plays a dual role in our analysis. First, it guarantees existence of an optimal action for every type. The second role is to simplify the description of strategies so that they can be represented with finite-dimensional vectors. In games where payoff functions are continuous in actions, we no longer rely on the assumption of finite actions for the first purpose. The arguments in this section then show how the existence of equilibrium in a sequence of finite games will imply existence in a game with a continuum of actions. We also extend the results to classes of games, such as first price auctions, which potentially have discontinuities in payoffs when an increase in a player’s action causes a discrete change in the probability of “winning.”

The properties of the equilibrium strategies implied by the Single Crossing Condition or the Limited Complexity Condition play a special role in this section. While arbitrary sequences of functions need not have convergent subsequences, sequences of nondecreasing functions (or more generally, functions of bounded variation, which can be expressed as the difference between two nondecreasing functions (Billingsley, 1986, p. 435)) do have almost-everywhere convergent subsequences by Helly’s Theorem. Thus, our restrictions play two roles in the existence results: first, they guarantees that the strategies in finite games can be represented with a finite vector so that Kakutani’s fixed point theorem can be applied, and second, they guarantee that sequences of equilibria to finite games have almost-everywhere convergent subsequences. All that remains to show is that the limits of these sequences are in fact equilibria to the continuous action game.

3.1. Games with Continuous Payoff Functions

This section extends the results of Section 2 to games with a continuum of actions. The following theorem shows that in a game of incomplete information, if payoffs are continuous and all finite-action games have equilibria in functions of bounded variation, then the continuum-action game will have an equilibrium as well.
Theorem 3.1 Consider a game of incomplete information which satisfies (2.1). Suppose that (i) $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_I)$, where $\mathcal{A}_i = [a_i, \bar{a}_i]$, (ii) for all $i$, $u_i(a, t)$ is continuous in $a$ on $[a_i, \bar{a}_i]$, and (iii) for any finite $\mathcal{A} \subset \mathcal{A}$, a PSNE exists in the game where the players choose actions from $\mathcal{A}$, where the equilibrium strategies $\beta_{n}(t)$ are functions of bounded variation.

Then a PSNE exists in the game where players choose actions from $\mathcal{A}$.

Corollary 3.1.1 Consider a game of incomplete information which satisfies (2.1). Suppose that (i) $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_I)$, where $\mathcal{A}_i = [a_i, \bar{a}_i]$, and (ii) for all $i$, $u_i(a, t)$ is continuous in $a$ on $[a_i, \bar{a}_i]$. Then:

(i) If the Single Crossing Condition (2.2) holds, then there exists a PSNE, $\beta^*(t)$, where each player’s strategy is nondecreasing in her type.

(ii) If the Limited Complexity Condition (2.4) holds for some $K$, then there exists a PSNE, $\beta^*(t)$, where each player i’s strategy has at most $K_i$ direction changes.

While Corollary 3.1.1 does require the regularity assumption that $u_i$ is continuous and integrable, it does not require differentiability or quasi-concavity, two assumptions which often arise in alternative approaches. Furthermore, it does not place any additional restrictions on the correlation structure between types above what is required for the (economically interpretable) condition that best responses are nondecreasing.

Despite the generality of Corollary 3.1.1, the restriction that payoffs be continuous still rules out many interesting games. The next section extends this result to games with discontinuous payoffs such as auctions.

3.2. Games with Discontinuities: Auctions and Pricing Games

This section studies games whose payoffs exhibit a particular type of discontinuity. A player sees a discrete change in her payoffs depending on whether she is a “winner” (i.e., the high bidder in a single-unit auction or the low price in a pricing game), or a “loser.” Examples include auctions (first price or all-pay), pricing games, and more general mechanism design problems where the goal is to allocate resources between multiple players. Winners receive payoffs $v_1(a, t)$, while losers receive payoffs $v_2(a, t)$. The allocation rule $q_i(a)$ specifies the probability that the player receives the winner’s payoffs as a function of the actions taken by all players. We will restrict attention to games with only two outcomes. Under our assumptions, then, a player’s payoffs given a realization of types and actions is given as follows:

$$u_i(a, t) = q_i(a) \cdot v_1(a, t) + (1 - q_i(a)) \cdot v_2(a, t) \quad (3.1)$$

This formulation highlights the second assumption implicit in this formulation: payoffs depend on the other players actions only through the allocation, not through payoffs directly. This assumption can be relaxed, but it sufficiently complicates the analysis that we will not consider it.
Each player chooses from a set of actions, \( \mathcal{A} = Q \cup \{a_i, \bar{a}_i\} \). The action \( Q < \min_i \{a_i\} \) guarantees each player zero expected payoffs. We will maintain the following assumption about payoffs:

\[
\bar{v}_i(Q, t) = v_i(Q, t) = 0 \quad \text{for all } t. \tag{3.2}
\]

For all \( a_i \in \{a_i, \bar{a}_i\} \) and all \( t \),

(i) \( \bar{v}_i(a_i, t) > 0 \) implies \( \bar{v}_i(a_i, t) > v_i(a_i, t) \), and (ii) \( \bar{v}_i(a_i, t) \leq 0 \) implies \( v_i(a_i, t) \leq 0. \) \( \tag{3.3} \)

There are several classes of examples which have this structure. In a first-price auction, the winner receives the object and pays her bid, while losers get payoffs of \( v_i(a_i, t) = 0 \). In Milgrom and Weber’s (1982) formulation of the mineral rights auction, \( \bar{v}_i(a, t) \) represents the expected payoffs to the bidder conditional on the vector of type realizations, and the vector \( t \) is interpreted as a vector of signals about each player’s true value for the object (where signals and values may be correlated). In an all-pay auction, the player pays her bid no matter what, but the winner receives the object. In some pricing games, the lowest price (the highest action) implies that the firm captures a segment of price-sensitive consumers, while having a higher price implies that the firm only serves a set of local customers.

We will restrict attention to allocation rules which take the following form:

\[
q_i(a) = \sum_{\{s_i, s_j \mid i \neq j \neq l \neq \cdots \mid l \}} \left[ \mathbf{1}_{|s_i| > \cdots > |s_l|} + \mathbf{1}_{|s_i| > |s_j| > |s_l|} \rho(\sigma_T) \right] \prod_{j \neq i} \mathbf{1}_{m_j(a_i) < m_{a_i}(a_i)} \prod_{j \neq i} \mathbf{1}_{m_j(a_i) = m_{a_i}(a_i)} \tag{3.4}
\]

In this expression, \( m_i(\cdot) \) is a strictly increasing function. Player \( i \) receives the object with probability zero if \( k \) or more opponents choose actions such that \( m_i(a_i) > m_i(a_j) \), and with probability 1 if \( I-k \) or more opponents choose actions such that \( m_i(a_i) < m_i(a_j) \). The remaining events are “ties,” in which case \( \rho(2^{(I-k)}) \rightarrow [0,1] \) is the probability that player \( i \) wins. We further assume that:

If \( |\sigma_i| > 0 \), then \( \rho(\sigma_i) < 1. \) \( \tag{3.5} \)

This assumption requires that if player \( i \) ties for winner with a non-empty subset of players, no player wins with probability 1. This last assumption simplifies the proof, but can be relaxed.

To interpret expression (3.4), consider the example of a first price auction for a single object. In this case, \( k = 1 \) and \( m_i(a_i) = a_i \). That is, the player wins with probability zero if 1 or more opponents place a higher bid. If ties are broken randomly, the probability of winning given that no opponents place a higher bid and further the player ties with the subset of opponents \( \sigma_i \) is given by \( \rho(\sigma_i) = \frac{1}{|\sigma_i|}. \) More general mechanism design problems also fall into this framework. When a player’s payoffs satisfy the single crossing property, only direct mechanisms in which the allocation rule is monotonic will be incentive compatible; thus, any incentive-compatible rule for
allocating \( k \) winners from \( I \) players will take the form of (3.4). Further, consider an optimal mechanism to allocate a single object between risk neutral agents whose (independently distributed) types represent their valuations for the object. In this case, the actions are reports of types, and the allocation rule determines the winner according to the agent who has the highest “virtual type” (Myerson, 1983).

Thus, the players’ expected payoffs can be written as follows:

\[
U_i(a_i, \alpha_{i-1}(\cdot), t_i) = \int u_i(a_i, \alpha_{i-1}(t_{i-1}), t) f(t_{i-1} \mid t_i) dt_{i-1} = \int [q_i(a_i, \alpha_{i-1}(t_{i-1})) \cdot \nu_i(a_i, t) + (1 - q_i(a_i, \alpha_{i-1}(t_{i-1}))) \cdot \nu_i(a_i, t)] \cdot f(t_{i-1} \mid t_i) dt_{i-1} = \int \nu_i(a_i, t) \cdot f(t_{i-1} \mid t_i) dt_{i-1} + \int [\nu_i(a_i, t) - \nu_i(a_i, t)] \cdot q_i(a_i, \alpha_{i-1}(t_{i-1})) \cdot f(t_{i-1} \mid t_i) dt_{i-1}
\]

We maintain the following additional assumptions:

For all \( i \) and \( a_i \in [a_i, \bar{a}_i] \), \( \bar{v}_i(a_i, t) \) and \( v_i(a_i, t) \) are bounded and continuous in \( (a_i, t) \), and \( \Delta v_i(a_i, t) = \bar{v}_i(a_i, t) - v_i(a_i, t) \) is nondecreasing in \( t \) and strictly increasing in \( (a_i, \bar{t}_i) \). (3.6)

For all \( a_i \in [a_i, \bar{a}_i] \), all \( k_1, k_2 \in \text{supp}(f(t_{i-1})) \), \( E[\Delta v_i(a_i, t) \mid t_i, k_1 \leq t_{i-1} \leq k_2] \) is strictly increasing in \( t_i \) and nondecreasing in \( k_i, k_2 \). (3.7)

Since (3.7) is the most restrictive of these assumptions, it is worth pausing to note that it has in fact been characterized by Milgrom and Weber (1982) for the case where \( \Delta v_i \) is nondecreasing in \( t \), as assumed in (3.6) (log-supermodularity of densities is discussed in more detail in Section 4 and in the Appendix).

**Lemma 3.2.1** Consider a conditional density \( f(t_{i-1} \mid t_i) \) which is log-supermodular a.e. Then \( E[g_i(t) \mid t_i, k_1 \leq t_{i-1} \leq k_2] \) is nondecreasing in \( (t_i, k_i, k_2) \) for all \( g_i \) nondecreasing if and only if \( f(t_{i-1} \mid t_i) \) is log-supermodular a.e. (equivalently, \( t \) is affiliated).

With these assumptions in place, we can state our existence result, proved in the Appendix:

**Theorem 3.2** Consider a game which satisfies (2.1). Let the action space for each player be \( A = Q \cup [a_i, \bar{a}_i] \), assume that the support of the distribution \( F(t) \) is a product set, and assume (3.1)-(3.7). Suppose further that the Single Crossing Condition, (2.2), holds for \( a_i \in [a_i, \bar{a}_i] \). Then, there exists a PSNE, \( \beta^*(t) \), where each player’s strategy is nondecreasing in her type.

Once existence is established for the continuous-action game, standard arguments can be used to verify the usual regularity properties. For example, strategies are strictly increasing on the interior of the set of actions played with positive probability, and no player sees a gap in the set of actions played with positive probability by opponents. Further, with appropriate differentiability assumptions, we can use a differential equations approach to characterize the equilibrium.
The intuition behind the proof of Theorem 3.2 can be summarized as follows. The only reason that Theorem 3.1 cannot be applied directly is that the game has a potential discontinuity in a player’s own action. If the opponents use a particular action \( a \) with positive discontinuity, then increasing one’s own action to be just higher than \( a \) will lead to a discontinuous increase in the probability of winning. However, observe that if each \( \beta_i^*(t_i) \) is strictly increasing on \( T_i \), expected payoffs will be continuous in \( a_i \), since the Lebesgue measure of the sets \( \{ t_i | \beta_i^*(t_i) \leq k \} \) and \( \{ t_i | \beta_i^*(t_i) < k \} \) changes continuously in \( k \), and since we have assumed that the type distributions are atomless. If, in contrast, \( \beta_i^*(t_i) \) is constant at \( b \) on a subinterval \( S \) of \( T_i \), then each player \( j \neq i \) sees a discontinuity in their expected payoffs at \( a_j = b \).

Let \( \beta^*(t) \) denote the equilibrium strategies. Our argument establishes that for each player \( i \), almost every type \( t_i \) sees zero probability of a tie at her optimal action \( \beta_i^*(t_i) \). Ruling out the possibility of such mass points in the limit involves showing that mass cannot be “adding up” close to a particular action as the action space gets finer. Recall that a sequence of nondecreasing functions has an almost-everywhere convergent subsequence, and further this subsequence converges uniformly except on a set of arbitrarily small measure. Thus, if a player’s limiting strategy involves a mass point at some action \( b \), then given a \( d > 0 \), a positive mass of players must be using a strategy on \( [b-d, b+d] \) as the action grid gets finer. But then, other players have an incentive to “jump over” that player’s action \( b \) rather than using an action less than \( a' - d \): with an action only slightly higher than \( b+d \), the other players can beat all of the types playing on \( [b-d, b+d] \). But this will in turn undermine the incentives of the first player to choose an action on \( [b-d, b+d] \).

This result then generalizes the best available existence results about first price auctions. Previous studies (Maskin and Riley (1993, 1996); Lebrun (1995, 1996), Bajari (1996a)) have analyzed independent private values auctions, as well as affiliated private values auctions and common value auctions with conditionally independent signals about the object’s value (Maskin and Riley (1993, 1996). The work closest to ours is Lizzeri and Persico (1997), who have independently established existence and uniqueness of equilibria in a class of games similar to the one studied above, but with the restriction to two bidders (further, their approach is based on differential equations, while differentiability of utility functions is not assumed here). The approach taken in this paper is different from those of the existing literature, in that it separates out the issue of monotonicity of strategies and existence, showing that monotonicity implies existence. Thus, the only role played by assumptions about the joint distribution over types is to guarantee that the single crossing property holds.

The next section analyzes applications, including first price auctions. To preview, however,
the weakest general conditions for private value auctions requires (i) log-supermodular utility functions, which amounts to log-concave utility functions if utility takes the form \( V(t_i - a_i) \), and (ii) affiliated types. In the more general “mineral rights” auctions, which allow for utility functions of the form \( v_i(a_i, t_i) \), for two bidders we require that \( v_i(a_i, t_i) \) be supermodular in \((a_i, t_1)\) and \((a_i, t_2)\), strictly increasing in \((-a_i, t_i)\), and that values are affiliated. We are not aware of general conditions for monotonicity in the mineral rights auction with more than two bidders; however, our numerical results indicate that monotonicity holds in the relevant range (i.e., near equilibrium) for several examples with log-normally distributed values and signals.

4. Characterizing the Single Crossing Condition in Applications

This section characterizes the single crossing condition in several classes of games of incomplete information. It applies results from Athey (1995, 1996) to describe conditions on the primitives of a game, that is, the utility functions of the players and the joint distribution of types, which are sufficient for the expected value of the utility function to satisfy SCP-IR when all other players use nondecreasing strategies. Thus, applying Theorems 2.1, 3.1, and 3.2, we are able to characterize classes of games which have PSNE.

The results in this section are grouped according to the structure of the problem: additively separable problems (such as investment games), multiplicatively separable problems (such as private value auctions), and non-separable problems (such as mineral rights auctions). Table 7.1 in the Appendix summarize our analysis from this section, stating the conditions to check for games of incomplete information which take a variety of structures. The results are applications of theorems about comparative statics in stochastic problems from Athey (1995, 1996). All of these results can be applied to derive sufficient, and sometimes necessary, conditions on payoff functions and type distributions so that the game satisfies the Single Crossing Condition.

The results make use of the properties supermodularity and log-supermodularity. Since we will be interested in product sets, we will state the definitions for that case. Let \( X = \times_{n=1}^{N} X_n \) and consider an order for each \( X_n \), which will be denoted \( \succeq \). Suppose that each set \( X_n \) is a totally ordered set. An example of such a product set is \( \mathbb{R}^N \) with the usual order, where \( x \succeq y \) if \( x_n \geq y_n \) for \( n = 1, \ldots, N \). We will use the operations “meet” (\( \lor \)) and “join” (\( \land \)), defined for product sets as follows: \( x \lor y = (\max(x_1, y_1), \ldots, \max(x_N, y_N)) \) and \( x \land y = (\min(x_1, y_1), \ldots, \min(x_N, y_N)) \).

**Definition 4.1** A function \( h: X \rightarrow \mathbb{R} \) is **supermodular** if, for all \( x, y \in X \),

\[
h(x \lor y) + h(x \land y) \geq h(x) + h(y)
\]

A non-negative function \( h: X \rightarrow \mathbb{R} \) is **log-supermodular**\(^7\) if, for all \( x, y \in X \),

\[
h(x \lor y) \cdot h(x \land y) \geq h(x) \cdot h(y)
\]

---

\(^7\) Karlin and Rinott (1980) called log-supermodularity multivariate total positivity of order 2.
Clearly, a non-negative function \( h(x) \) is log-supermodular if \( \ln(h(x)) \) is supermodular. When \( h: \mathbb{R}^n \to \mathbb{R} \), and we order vectors in the usual way, Topkis (1978) proves that if \( h \) is twice differentiable, \( h \) is supermodular if and only if \( \frac{\partial^2}{\partial x_i \partial x_j} h(x) \geq 0 \) for all \( i \neq j \). For the moment, four additional facts about these properties are important: (1) if \( h(x,t) \) is supermodular or log-supermodular, then \( h(x,t) \) satisfies SCP-IR; (2) sums of supermodular functions are supermodular, while products of log-supermodular functions are log-supermodular; (3) if \( h(x) \) is supermodular (resp. log-supermodular), then so is \( h(\alpha_1(x_1), \ldots, \alpha_n(x_n)) \), where \( \alpha_i(\cdot) \) is nondecreasing; (4) a density is log-supermodular if and only if the random variables are affiliated (as defined in Milgrom and Weber, 1982).\(^8\)

4.1. General Characterizations

4.1.1. Additively separable expected payoffs

First consider games where payoffs are given by \( u(a,t) = g_i(a,t) + h_i(a,t) \), and \( U_i(a,\alpha_\cdot(\cdot),t) = g_i(a,t) + H_i(a,t) \). A game which fits into this framework is an all-pay auction, where \( g_i(a,t) \) is the expected utility from losing the auction and having to pay \( a_i \), while \( H_i(a_i,t) \) gives the expected returns to winning as opposed to losing the auction, weighted by the probability of winning.

Since the sum of supermodular functions is supermodular, and since supermodularity implies the SCP-IR, supermodularity of \( g_i \) and \( H_i \) will imply the SCP-IR. The following result, due to Milgrom and Shannon (1994), shows function that supermodularity is the “right” property for additive problems.

**Lemma 4.1:** Consider \( g_i, H_i: \mathbb{R}^2 \to \mathbb{R} \). \( g_i(a_i,t_i) + H_i(a_i,t_i) \) satisfies single crossing of incremental returns (SCP-IR) in \( (a_i,t_i) \) for all \( H_i \) which are supermodular in \( a_i \), if and only if \( g_i \) is supermodular.

For example, in an all-pay auction, we might wish to characterize conditions on \( g_i \) which are sufficient for the SCP-IR to hold without specifying any additional structure on opponent strategies besides monotonicity, and allowing for general type distributions. Supermodularity is the weakest condition on \( g_i \) which guarantees SCP-IR of payoffs across an unrestricted class of functions \( H_i \).

The next step is to characterize when expected values of payoff functions are supermodular. The following Lemma applies results from Athey (1995, 1996) to this problem:

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\(^8\) See Whitt (1982) and Karlin and Rinott (1980) for related discussions of this property in statistics. Log-supermodularity of a density is also equivalent to the as monotone likelihood ratio property (MLRP) (Milgrom, 1981). A probability density \( f(s;\theta) \) satisfies the MLRP if the likelihood ratio \( f(t;\theta_2)/f(t;\theta_1) \) is nondecreasing in \( t \) for all \( \theta_2 \geq \theta_1 \).
Lemma 4.2  (i) $\int_{t_j} k_j(a_i, t) f_j(t | t) dt_j$ is supermodular in $(a_i, t)$ for all $k_j$ which are supermodular in $(a_i, t_j)$, $j=1,..,J$, if and only if $\int_{t_i} S_j(t | t) dt_i$ is nondecreasing in $t_i$ for all $S$ such that $1_S(t_i)$ is nondecreasing.

(ii) If $f(t)$ is log-supermodular, then $\int_{t_i} S_j(t | t) dt_i$ is nondecreasing in $t_i$ for all $S$ such that $1_S(t_i)$ is nondecreasing $t_i$.

Athey (1995) provides a more thorough characterization of supermodularity in stochastic problems based on alternative assumptions about the payoff functions; Lemma 4.2 is most applicable for games of incomplete information, and it does not rely on higher order derivatives of the payoffs.

Together, the above results can be used to characterize the single crossing condition in a class of games with additively separable payoffs.

Theorem 4.3 Consider a game of incomplete information which satisfies (2.1). Suppose that payoffs are given by $u(a, t) = g_i(a_i, t) + h_i(a_i, t)$. Then if (i) for all $j$, $g_i$ and $h_i$ are supermodular in $(a_i, t_j)$, (ii) $h_i$ is supermodular in $(a_i, t_i)$ and $(a_i, t_j)$, $i \neq j$, and (iii) the joint distribution over types $f(t)$ is log-supermodular, then the game satisfies the Single Crossing Condition (2.2).

Proof: Apply Lemmas 4.1 and 4.2, letting $k(a_i, t) = h_i(a_i, \alpha_i(t), t)$, recalling Fact (3) from above, which guarantees that for nondecreasing $\alpha_i(t)$, supermodularity of $h_i$ implies supermodularity of $k_i$.

While Theorem 4.3 can be used to establish existence of equilibrium, the games studied in Theorem 4.3 also satisfy Vives’ (1990) sufficient condition for existence of PSNE. When payoffs are supermodular, a pointwise increase in $\alpha_i(t_j)$ increases the returns to $a_i$ for all $t_j$, and thus the game is supermodular in strategies under the pointwise order. Under those conditions, Vives applies theorems about existence of equilibria in supermodular games without relying on the Single Crossing Condition, and thus without reference to assumptions about the joint distribution over types. However, Vives’ result relies crucially on the assumption that payoffs are supermodular, and it will not be applicable to the other classes of games, including games with log-supermodular or single-crossing payoff functions, as well as the additively separable all-pay auction with independent private values (which relies on Lemma 4.1 directly). Further, because monotonicity of strategies is of independent interest in many games, we will at times be interested in applying Lemma 4.2 and Theorem 4.3 even in games with supermodular payoffs.

In terms of applications, many of the supermodular games with complete information which have been studied by economists (see Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) for examples) can also be studied as games of incomplete information. For example, pricing or quantity games have variations where firms have incomplete information about their
rivals production costs or information about demand. Games between two players whose choices are strategic substitutes can also be considered, such as a Cournot quantity game between two firms whose quantities decrease the marginal revenue of the opponents, but where the firms have incomplete information about rivals costs.

**4.1.2. Multiplicatively separable expected payoffs**

The next class of games we consider is a class where player \( i \)’s payoffs can be written as the product of two nonnegative terms, so that \( u_i(a_1, \ldots, a_I) = g_i(a_i, t_i) \cdot h_i(a_1, \ldots, a_I) \) and \( U_i(a, \alpha, \beta, t) = g_i(a_i, t_i) \cdot H_i(a_i, t_i) \). A game which fits into this framework is a private-value first price auction, where \( H_i(a_i, t_i) = \Pr\{\text{bid } a_i \text{ wins the auction given } t_i\} \); other examples include pricing games where firms’ products are imperfect substitutes.

The theory of comparative statics for additively separable problems can be applied to multiplicatively separable problems merely by taking logarithms, i.e., \( \ln(U_i(a, \alpha, \beta, t)) = \ln(g_i(a_i, t_i)) + \ln(H_i(a_i, t_i)) \), and applying Lemma 4.1. Thus, for multiplicatively separable problems, log-supremodularity is the right property to require in order to guarantee the SCP-IR.

The methods for analyzing log-supremodularity in stochastic problems are different from those for supermodular functions; however, the sufficient conditions are in the end quite similar. The following characterization theorem follows from Athey (1996):

**Lemma 4.4:** Let \( k_i: \mathcal{R}^I \rightarrow \mathcal{R}_+ \), where \( f(t) \) is a probability density. For \( i = 1, \ldots, I \), let \( k_i: \mathcal{R}^{I-1} \rightarrow \mathcal{R} \) and let \( f_i(t_{-i} | t_i) \) be the conditional density of \( t_{-i} \) given \( t_i \). Then the following conditions hold:

1. \( \int k_i(a_i, t) f_i(t_{-i} | t_i) dt_{-i} \) is log-supremodular in \( (a_i, t_i) \) for all \( i = 1, \ldots, I \) and all \( k_i \), log-supremodular, if and only if \( f(t) \) is log-supremodular.
2. \( \int k_i(a_i, t) f_i(t_{-i} | t_i) dt_{-i} \) is log-supremodular in \( (a_i, t_i) \) for all \( f(t) \) log-supremodular, if and only if \( k(a_i, t) \) is log-supremodular.

Log-supremodularity is an especially convenient property for working with expectations because it is preserved by multiplication; thus, multiplying the integrand by an indicator function \( 1_S(t_{-i}) \) preserves log-supremodularity so long as the set \( S \) is a sublattice (see Topkis, 1978); a common example of a sublattice is a cube in \( \mathcal{R}^n \). For more discussion see Athey (1996).

We can pull these results together into the following set of sufficient conditions for the Single Crossing Condition to hold, in a manner similar to Theorem 4.3.

**Theorem 4.5** Consider a game of incomplete information which satisfies (2.1). Suppose that payoffs are given by \( u_i(a_1, \ldots, a_I) = g_i(a_i, t_i) \cdot h_i(a_1, \ldots, a_I) \). Then if (i) for all \( i \), \( g_i \) and \( h_i \) are nonnegative and log-supremodular, and (ii) the joint distribution over types \( f(t) \) is log-supremodular, then the game satisfies the Single Crossing Condition (2.2).

We will apply this result to private values auctions and pricing games in Section 4.2.
4.1.3. Nonseparable Expected Payoffs

This section analyzes games which do not fit into the classes of additively or multiplicatively separable games analyzed above. Of course, we can always apply the above results about payoffs of the form \( u(a, t) = g(a, t) + h(a, t) \) or \( u(a, t) = g(a, t) \cdot h(a, t) \), letting \( g(a, t) = 1 \). However, the requirements on \( h \) which must be satisfied to apply Theorems 4.3 and 4.5 are stronger than necessary if \( g(a, t) \) is constant. Further, some games (such as the mineral rights auction) do not satisfy the conditions of Theorems 4.3 and 4.5. This section shows that weaker conditions on the utility function suffice, if either (i) there are only two players, as in the noisy signaling game studied in Section 4.2.3, or if (ii) \( u(a, t) \) takes a very special form. In particular, in many-player games, \( u_i \) must depend on the opponents’ types and actions through a single index, denoted \( s_i \). That is, \( u_i(a_i, \alpha_{-i}(t_i), t) = k_i(a_i, s_i, t_i; \alpha_{-i}) \). For example, in a first price mineral rights auction with identical bidders using symmetric strategies, \( \alpha_{-i} = \alpha_{-i} \) for all \( j, l \neq i \). Then, \( k_i(a_i, s_i, t_i; \alpha_{-i}) = E_{t_i \rightarrow [s]} [v(a_i, t_i | \max (l_{ij} | j \neq i) = s_i] \cdot 1_{[a_i, \alpha_{-i}]}(s_i) \). That is, \( s_i \) is the value of the highest opponent type, and payoffs depend on opponent types only through the realization of this type and the associated action. In a multi-unit auction, \( s_i \) might be a different order statistic of the distribution. In other applications, \( s_i \) might be a sufficient statistic for \( t_i \).

Our theorem makes use of a weak version of the SCP-IR. Formally:

**Definition 4.2** \( h(x, \theta) \) satisfies (Milgrom and Shannon’s) weak single crossing property of incremental returns (WSCP-IR) in \((x, \theta)\) if, for all \( x_i > x_j \), \( g(\theta) = h(x_i, \theta) - h(x_j, \theta) \) satisfies the following condition for all \( \theta_j > \theta_i \): \( g(\theta_i) > 0 \) implies \( g(\theta_j) \approx 0 \).

The following Lemma is proved in Athey (1996).

**Lemma 4.6:** Consider a function \( k_i : \mathbb{R}^3 \rightarrow \mathbb{R} \) and a conditional density \( f_{s_i}(s_i | t_i) \) whose support does not change with \( t_i \), and suppose \( f_{s_i}(s_i | t_i) \) is log-supermodular. Suppose further that \( k_i(a_i, s_i, t_i) \) is supermodular in \((a_i, t_i)\) and satisfies WSCP-IR in \((a_i, s_i)\). Then

\[
\int k_i(a_i, s_i, t_i) \cdot f_{s_i}(s_i | t_i) \, dt_i \text{ satisfies SCP-IR in } (a_i, t_i). \]

Lemma 4.6 can be applied to games of incomplete information, and the following theorem will be used in our study of mineral rights auctions as well as noisy signaling games.

**Theorem 4.7** Consider a game of incomplete information which satisfies (2.1). Suppose that for all \( i=1, \ldots, l \), there exists a random variable \( s_i \) and a family of functions \( k_i(\cdot; \alpha_{-i}) : \mathbb{R}^3 \rightarrow \mathbb{R} \) indexed by opponent strategies, such that (i) \( U(a_i, \alpha_{-i}(\cdot); t_i) = E_{s_i} [k(a_i, s_i, t_i; \alpha_{-i}) | t_i] \); (ii) when \( \alpha_{-i} \) is nondecreasing, \( k(a_i, s_i, t_i; \alpha_{-i}) \) is supermodular in \((a_i, t_i)\) and satisfies WSCP-IR in \((a_i, s_i)\); and (iii) the conditional density \( f_{s_i}(s_i | t_i) \) is log-supermodular and the support does not change with \( t_i \). Then the game satisfies the Single Crossing Condition (2.2).

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9 Athey (1996) further shows that none of the hypotheses can be weakened without strengthening the others; see Table 7.1 in the Appendix.
4.2. Applications

4.2.1. Auctions

4.2.1.1. Private Values Auctions

Consider a first-price, private value auction, where each player \( i \) \((i=1,\ldots,I)\) observes his own value \( t_i \), and values are drawn from a distribution satisfying (2.1). Each player's utility function is given by \( V_i(a_i,t_i) \). Suppose players \( j \neq i \) use nondecreasing strategies. Writing the objective function when ties are resolved randomly is cumbersome, so for the moment consider the game where the auctioneer keeps the object in the case of ties. Then, let \( H_i(a_i|t_i) = \Pr(a_i \geq a_j(t_i) \text{ for } j \neq i|t_i) \).

To see when the conditions of Theorem 4.5 are satisfied, consider first the utility function. If utility takes the form \( V_i(a_i,t_i)=V_i(t_i-a_i) \), where \( V_i(0) \geq 0 \) and \( \ln(V_i) \) is concave and strictly increasing, it is straightforward to verify that \( V_i \) will be log-supermodular in \((t_i,a_i)\). Next, consider the issue of when \( H(a_i|t_i) \) is log-supermodular. Since the strategies are nondecreasing (and recalling Fact 3, that log-supermodularity is preserved by monotone transformations of the variables), \( H(a_i|t_i) \) will be log-supermodular if the joint distribution of types, \( F(t) \), is log-supermodular. A sufficient (but not necessary) condition for this is that \( f(t) \) is log-supermodular, i.e., types are affiliated (this can be shown using Lemma 4.4, letting \( g_i(a,t)=1_{|t-a|}(t) \)).

Now, consider the case where ties are resolved uniformly. Then define \( h_i(a,t) \) as follows (and likewise for players \( 2,\ldots,I \)):

\[
h_i(a) = \sum_{j_i=0}^1 \cdots \sum_{j_i=0}^1 \frac{1}{1 + \sum_{k=2}^I j_k} \prod_{k=2}^I ((1-j_k) \cdot 1_{a_k < a_i} + j_k \cdot 1_{a_k = a_i})
\]

Thus, the probability that player \( i \) wins the auction when she has type \( t_i \) and chooses an action \( a_i \) is given as \( \Pr\{a_i \text{ wins}|t_i\} = H(a_i|t_i) = E_{t_i \sim \alpha_i}[h(a_i, \alpha_{-i}(t_{-i}))] \). If the types are affiliated, then Lemma 4.4 implies that this probability will be log-supermodular in \((a_i,t_i)\) if the integrand is log-supermodular in \((a_i,t)\). Log-supermodularity can be verified pairwise, so that (using the symmetry) we need only to check log-supermodularity of \( h(a_i, \alpha_{-i}(t_{-i})) \) in \((a_i,t_i)\) and \((t_j,t_i)\) for \( 1 \neq j \neq i \). This can be verified directly.

Thus, we have the following proposition (the assumption of the proposition that \( V_i(a_i,t_i) \) is nonnegative is innocuous since the agent knows \( V_i(a_i,t_i) \) after observing her type and thus can always choose an action \( a_i \) which yields nonnegative payoffs).

**Proposition 4.8** Consider a private values, first price auction, where (2.1) holds. Suppose further that (i) for all \( i \), the utility functions \( V(a_i,t_i) \) are non-negative and log-supermodular, (ii) types are affiliated (\( f(t) \) is log-supermodular), and (iii) ties are broken uniformly. Then:

1. The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-
action games.

(2) If, in addition, for all i (iv) \( V_i \) is strictly increasing in \((-a_i, t_i)\), (v) \( V_i \) is bounded and continuous, and (vi) the support of \( f(t) \) is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

Now consider a generalization of Proposition 4.3 to multi-unit auctions, where each agent demands a single unit. For example, in a 2-unit auction, the players with the highest two bids win an object. Unfortunately, this complicates the analysis of log-supermodularity of the function \( \Pr(a_i \text{ wins} | t_i) \). However, if the types are drawn independently, then \( \Pr(a_i \text{ wins} | t_i) \) does not depend on \( t_i \), and the expected payoff function reduces to \( V_i(a_i, t_i) \cdot \Pr(a_i \text{ wins}) \). This is always log-supermodular if \( V_i(a_i, t_i) \) is log-supermodular; thus, for the case of independent types, the extension to multi-unit auctions is immediate.

4.2.1.2. All-Pay Auction

In this section, we consider an alternative auction format, the all-pay auction. In this auction, the highest bidder receives the object, but all bidders pay their bids. This game has been used to model activities such as lobbying. To keep the analysis simple, we will use an independent private values formulation. Let \( V_i(a_i, t_i) \) take the form \( V_i(t_i \& a_i) \). Then a player’s expected payoffs from action \( a_i \) can be written as follows:

\[
V_i(-a_i) + V_i(t_i-a_i) \cdot \Pr(a_i \text{ wins})
\]

This game has an additively separable form, and thus by Lemma 4.1, we look for the components of the objective function to be supermodular. Since \( \Pr(a_i \text{ wins}) \) is nonnegative and nondecreasing in \( a_i \), it is straightforward to verify that \( V_i(t_i-a_i) \cdot \Pr(a_i \text{ wins}) \) is supermodular if \( V_i(t_i-a_i) \) is increasing in \( t_i \) and supermodular in \( (a_i, t_i) \). In turn, \( V_i(t_i-a_i) \) is supermodular if and only if it is concave, that is, the bidder is risk averse.

Observe that we have not discussed the interactions between the players’ strategies in determining the function \( \Pr(a_i \text{ wins}) \); since types are independent and bidder valuations are private, this is not important for establishing the single crossing conclusion, in contrast to Theorem 4.3. However, it is not true that a pointwise increase in player \( j \)’s strategy leads to a pointwise increase in player \( i \)’s best response, and thus the game is not supermodular in strategies.

Summarizing, we have the following proposition:

**Proposition 4.9** Consider a private values, all-pay auction, where (2.1) holds. Suppose further that (i) for all i, the utility functions \( V_i(t_i-a_i) \) are concave, and \( V_i(0) \geq 0 \), (ii) types are independent, and (iii) ties are broken uniformly. Then:

(1) The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-action games.

(2) If, in addition, for all i (iv) \( V_i \) is strictly increasing, (v) \( V_i \) is bounded and continuous, and
The support of \( f(t) \) is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

4.2.1.3. First Price Mineral Rights Auctions

We now generalize our analysis of auctions to consider Milgrom and Weber’s (1982) model of a mineral rights auction (their general existence and characterization results treat the 1-bidder, symmetric case and several other special cases), allowing for risk averse, asymmetric bidders whose utility functions are not necessarily differentiable.

We begin with the two bidder case. Suppose each agent’s utility (written \( V(a_i,t_1,t_2) \)) is supermodular in \( (a_i,t_1) \) and \( (a_i,t_2) \); this can be interpreted as the player’s expected payoffs conditional on both players’ signals. When player two uses the bidding function \( \alpha_2(t_2) \), player 1’s expected payoffs given her signal can be written as follows (assuming ties are broken randomly):

\[
U_1(a_1,\alpha_2(),t_1) = \int_{t_2} V_1(a_1,t_1,t_2) \cdot I_{a_1 > \alpha_2(t_2)}(t_2) \cdot f_2(t_2|t_1) dt_2 \\
+ \frac{1}{2} \int_{t_2} V_1(a_1,t_1,t_2) \cdot I_{a_1 = \alpha_2(t_2)}(t_2) \cdot f_2(t_2|t_1) dt_2
\]

To keep things simple, consider the finite-action game. Figure 7 illustrates how player 1’s incremental returns to increasing her bid from \( A_{m_L} \) to \( A_{m_H} \) (where \( m_L < m_H \)) change with the signal of the opponent, given that the opponent’s strategy is consistent with \( x^2 \). There are several regions to consider. When the opponent bids less than \( A_{m_L} \), the outcome of the auction is unchanged by the increase in bid, but player 1 simply pays more. For a risk neutral bidder, this would be a constant.

![Figure 7: Player 1’s payoff function satisfies WSCP-IR in \((a_1,t_2)\).](image)

When the opponent bids more than \( A_{m_H} \), the outcome of the auction is also not changed by the bid increase: player 1 loses in both cases and pays nothing. In the intermediate cases, there are the regions which involve ties at either the low or the high bid, and the region where

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10 To see an example where these assumptions on the payoff function are satisfied, let \( V(a,t) = \int \hat{V}(z-a)g(z,t_1,t_2)dz \), where \( z \) is affiliated with \( t_1 \) and \( t_2 \), and \( \hat{V}_i \) is nondecreasing and concave. To see why, note that \( t_1 \) and \( t_2 \) each induce a first order stochastic dominance shift on \( G \), and \( \hat{V}_i \) is supermodular in \((a,z)\). By Athey (1995), supermodularity of the expectation in \((a_i,t_1)\) and \((a_i,t_2)\) follows.
the bid increase causes player 1 to change from losing for certain, to winning for certain. Within each of these regions, expected payoffs are nondecreasing in the opponent’s type since this increases the expected value of the object to bidder 1.

When bidder two plays a nondecreasing strategy, bidder one’s payoff function given a realization of $t_2$ satisfies WSCP-IR in $(a_1,t_2)$, and it is supermodular in $(a_1,t_1)$ can further apply Lemma 3.2.1 to show that the conditional expected payoffs for a bidder are always strictly increasing in his type so long as $v_i(a,t)$ is nondecreasing in $t$, thus satisfying (3.7). This gives the following result:

**Proposition 4.10** Consider a 2-bidder first price “mineral rights” auction, where (2.1) holds. Suppose further that (i) for all $i$, the utility functions $V_i(a,t)$ are supermodular in $(a_i,t_j)$, $j=1,2$, and nondecreasing in $t$, (ii) types are affiliated ($f(t)$ is log-supermodular), and (iii) ties are broken uniformly. Then:

1. The game satisfies the Single Crossing Condition (2.2), and a PSNE exists in all finite-action games.
2. If, in addition, for all $i$ (iv) $V_i$ is strictly increasing in $(-a_i,t_i)$, (v) $V_i$ is bounded and continuous, and (vi) the support of $f(t)$ is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

What happens when we try to extend this model to more than two bidders? If the bidders face a symmetric distribution, and all opponents use the same bidding function, then only the maximum signal of all of the opponents will be relevant to bidder one. Define $s_1 = \max(t_2,\ldots,t_n)$. Milgrom and Weber (1982) show that $(t_1,s_1)$ are affiliated when the distribution is symmetric. Further, if the opponents are using the same strategies, whichever opponent has the highest signal will necessarily have the highest bid. It is straightforward to extend Milgrom and Weber’s arguments to show that $\hat{V}_i(a_1,t_1,s_1) = E\left[V_i(a_1,t_1,\ldots,t_n)|s_1\right]$ is increasing in $(t_1,s_1)$ and supermodular in $(a_1,t_1)$ whenever $V_i$ is nondecreasing in $t$ and is supermodular in $(a_i,t_i)$, $i=1,\ldots,n$. Then we can apply Proposition 4.9 to this problem exactly as if there were only two bidders.

Unfortunately, when players use asymmetric strategies, affiliation of the signals is not sufficient to guarantee that $(t_1,s_1)$ are affiliated, nor is their joint distribution function log-supermodular. Thus, in some regions, a small increase in the signal received by a given player might increase the likelihood that a given bid is higher than all opponents. However, this potential competing effect may not be the dominant one in particular examples; thus, one can proceed by positing that the single crossing condition holds, characterize the equilibrium under that hypothesis, and then verify that the single crossing condition is in fact satisfied for the functional forms and ranges of parameters of interest. Thus, separating the single crossing condition from the existence theorem allows us to proceed even in the absence of general sufficient conditions for the single crossing condition.
In the case where players are risk neutral, we rewrite the $n$-bidder problem another way in order to highlight the difficulty. Let $V_i(a_i, t) = E\left[z_i | t\right] - a_i$, where $z_i$ is player 1’s true value for the object. This is nondecreasing in $t$ by Lemma 3.2.1. Then, we can rewrite player 1’s expected payoffs as follows (ignoring ties):

$$E \left[ E \left[ z_i | t \right] | t \right] , \alpha_j(t_j) < a_i \cdot Pr(\alpha_j(t_j) < a_i | t) \right]. \tag{4.2}$$

Applying Lemma 4.4 to this problem in a manner analogous to the private value auction, we know that $Pr(\alpha_j(t_j) < a_i | t)$ is log-supermodular when the density is affiliated and the strategies are nondecreasing. To isolate the issue with the first term of (4.2), we draw a distinction between the two ways that $a_i$ affects this term. We can restrict attention to actions by player 1 less than or equal to the player’s conditional expected payoff. Note that $E \left[ E \left[ z_i | t \right] | t \right] , \alpha_j(t_j) < a_i \right] - c$ is log-supermodular in $(t, c)$, since by Lemma 3.2.1, the first term is nondecreasing in $t$. However, our assumptions do not imply that $E \left[ E \left[ z_i | t \right] | t \right] , \alpha_j(t_j) < a_i \right] - c$ is log-supermodular in $(a_i, t_i)$. Thus, the single crossing property will require that the interactions between $a_i$ and $t_i$ which we know work in the right direction are strong enough to outweigh any competing effects.

4.2.2. Pricing Games

This section studies pricing games with incomplete information, where constant marginal costs are the private information of the players of the game. Spulber (1995) recently analyzed how incomplete information about a firms’ cost parameters alters the results of a Bertrand pricing model, showing that firms price above marginal cost and have positive expected profits. Spulber’s model assumes that costs are independently and identically distributed, and that values are private; now we show that this model can be easily generalized to asymmetric, affiliated signals.

Let $t_i$ represent the marginal cost of firm $i$. Consider a general demand function for firm $i$, $D^i(p_1, ..., p_I)$, where the firms produce goods which are only imperfect substitutes. When the opponents use price functions $\rho_j(t_j)$, firm 1’s problem can now be written as follows:

$$\max_{p_i \in P} \left[ p_i - t_i \right] \cdot \int \cdot \int D^i(p_1, \rho_2(t_2), ..., \rho_I(t_I)) f(t_2, ..., t_I | t_1) dt_{-1}$$

By Lemma 4.4, the expected demand function is log-supermodular if the cost parameters are affiliated and $D^i(p_1, ..., p_I)$ is log-supermodular. The interpretation of the latter condition is that the elasticity of demand is a non-increasing function of the other firms’ prices. As discussed in Milgrom and Roberts (1990b), demand functions which satisfy this criteria include logit, CES, transcendental logarithmic, and a set of linear demand functions (see Topkis (1979)). Another special case is the case of perfect substitutes, where demand to the lowest-price firm is given by
D(P), where P is the lowest price offered in the market, and all other firms get zero demand.

Then, since [p_i-t_i] is log-supermodular, we have the following result (noting that the case with perfect substitutes is formally like a first-price auction):

**Proposition 4.11** Consider a pricing game as described above, where (2.1) holds. Suppose further that (i) for all i, the demand functions D(p_1,..,p_i) are non-negative and log-supermodular, (ii) types are affiliated (f(t) is log-supermodular), (iii) in the case of perfect substitutes, ties are broken uniformly. Then:

1. The game satisfies the Single Crossing Condition (2.2), and a PSNE equilibrium exists when either D_i is continuous or when the action set is finite.

2. For the case where goods are perfect substitutes, if in addition, for all i, (iv) D(P) is strictly decreasing in P, (v) D(P) is bounded and continuous, and (vi) the support of f(t) is a product set, then (3.1)-(3.7) are satisfied and there exists a PSNE in continuous-action games.

This example can also be extended to problems with incomplete information about demand elasticities.

### 4.2.3. Noisy Signaling Games

Consider a signaling game between two players, the sender (player 1) and the receiver (player 2), where player 2 observes the signal of player 1 only with noise. Examples of noisy signaling games include games of limit pricing (Matthews and Mirman (1983)), where an entrant does not know the cost of the incumbent, but can draw inferences about the incumbent’s cost by observing a noisy signal of the incumbent’s product market decision (the noise might be due to demand shocks). In another example, Maggi (1996) studies the value of commitment in a game where one player gets to move first, committing herself to an action, but the move is only imperfectly observed by the opponent. This model is used to show that incomplete information about the first mover’s type (i.e. production cost) can restore some value to commitment, in contrast to Bagwell (1995)’s result that commitment has no value in a game of complete information but imperfectly observed actions.

Consider a game where Player 1 observes his own type and chooses an action. Player 2 observes a noisy signal of player 1’s action and then chooses an action in response. Player 1 receives payoffs \( u_1(a_1,a_2,t_1) \), while player 2 receives payoffs \( u_2(a_1,a_2,t_1) \) (in the limit pricing game, the player’s action is not important, but the type \( t_i \) represents the unknown production cost; in the commitment game, player 1’s type is not important to player 2, but the action (i.e. output) is). Player 2’s “type,” \( t_2 \), is simply a signal about \( a_1 \), which does not directly affect payoffs. Let \( f_1(t_2|a_1) \) be the density of the signal \( t_2 \) conditional on the action \( a_1 \). Thus the expected payoff to player 1 from choosing action \( a_1 \) when player 2 uses strategy \( \alpha_2(t_2) \) is as follows:
Suppose further that (i) \( u \) is supermodular in (\( a_1, \alpha_2(t_2), t_1 \)) and the belief that player 1 uses strategy \( \alpha_1(\cdot) \), calculated using Bayes’ rule. Then, the expected payoffs to player 2 from choosing action \( a_2 \) when player 1 uses strategy \( \alpha_1(t_1) \) is as follows:

\[
\int u_2(\alpha_1(t_1), a_2, t_1) f_2(t_1|t_2, \alpha_1(\cdot)) dt_1.
\]

Let \( f_2(t_1|t_2, \alpha_1(\cdot)) \) be the conditional density of \( t_1 \) given the signal \( t_2 \) and the belief that player 1 uses strategy \( \alpha_1(\cdot) \), calculated using Bayes’ rule. Then, the expected payoffs to player 2 from choosing action \( a_2 \) when player 1 uses strategy \( \alpha_1(t_1) \) is as follows:

\[
\int u_2(\alpha_1(t_1), a_2, t_1) f_2(t_1|t_2, \alpha_1(\cdot)) dt_1.
\]

Suppose that \( \alpha_1(t_1) \) is nondecreasing. Further, suppose that \( f_1(t_1|a_1) \) is log-supermodular (equivalent to the MLRP). Then, the induced density \( f_2(t_1|t_2, \alpha_1(\cdot)) \) will also be log-supermodular. Sufficient conditions for the single crossing property in this scenario are then given as follows:

**Proposition 4.12** Consider a noisy signaling game as described above, where (2.1) holds. Suppose further that (i) \( u_2(a_1, a_2, t_1) \) satisfies WSCP-IR in \((a_2,a_1)\) and \((a_2,t_1)\), (ii) \( f_1(t_1|a_1) \) is log-supermodular and the support does not move with \( a_1 \), and either (iii)(a) \( u_1(a_1, a_2, t_1) \) is supermodular in \((a_2,t_1)\) and in \((a_1,t_1)\), or else (iii)(b) \( u_1(a_1, a_2, t_1) \) is nonnegative and log-supermodular. Then the game satisfies the Single Crossing Condition (2.2).

**Proof:** Lemma 4.6 implies that under our assumption that \( u_2 \) satisfies WSCP-IR in \((a_2,a_1)\) and \((a_2,t_1)\), \( \int u_2(\alpha_1(t_1), t_1, a_2) f_2(t_1|t_2, \alpha_1(\cdot)) dt_1 \) satisfies SCP-IR in \((a_2,t_2)\) when \( \alpha_1(t_1) \) is nondecreasing.

It remains to establish that player 1’s expected payoffs satisfy (2.2). Under assumption (iii)(a), we can check that \( \int u_1(x, \alpha_2(t_2), t_1) f_1(t_1|y) dt_1 \) is supermodular in \((x,t_1)\) and \((y,t_1)\).

Since supermodularity is preserved by arbitrary sums, \( u_1 \) supermodular in \((a_1,t_1)\) implies supermodularity of \( \int u_1(x, \alpha_2(t_2), t_1) f_1(t_1|y) dt_1 \) in \((x,t_1)\). If \( \alpha_1(t_1) \) is nondecreasing, the assumption that \( u_1 \) is supermodular in \((a_2,t_1)\) implies that \( u_1(a_1, \alpha_2(t_2), t_1) \) is supermodular in \((t_2,t_1)\). But Lemma 4.2 implies that \( \int u_1(x, \alpha_2(t_2), t_1) f_1(t_1|y) dt_1 \) is supermodular in \((t_1,y)\).

Under assumption (iii)(b), we can apply Lemma 4.4 to establish that

\[
\int u_1(a_1, \alpha_2(t_2), t_1) f_1(t_1|a_1) dt_1
\]

will be log-supermodular when \( u_1 \) and \( f_1 \) are log-supermodular and \( \alpha_2(\cdot) \) is nondecreasing.

**5. Numerical Computation of Equilibria: First Price Auctions**

In this section, we consider the issue of computation of equilibria of first price auctions. Since the existence result considered in this paper applies to a wider class of economic environments (arbitrary asymmetries, correlated values) than had been previously analyzed theoretically or numerically, we can take a few first steps towards numerically characterizing the equilibria to such auction games. This numerical exploration can potentially suggest avenues for future theorizing about characterizations of equilibria, and we give some examples of suggestive numerical results about affiliated private values and common value auctions with asymmetries in the covariance structure of types.
Prior to Marshall et al (1994), there were no general numerical algorithms available for computing equilibria to asymmetric first price auctions with continuous bidding units, and in fact existence was not known for some of the kinds of asymmetries considered in their paper. Marshall et al (1994) argue that numerical computation of equilibria in asymmetric first price auctions in the independent private values case is difficult, but can be done. They summarize some of the difficulties as follows:

Although these solutions belong to a class of ‘two-point boundary problems’ for which their exist efficient numerical solution techniques, they all suffer from major pathologies at the origin. First, forward extrapolation produces ‘nuisance’ solutions (linear in our case) that do not satisfy the terminal conditions and act as ‘attractors’ on the algorithm. Second, and not unrelated, backward solutions are well-behaved except in neighborhoods of the origin where they become highly unstable with the consequence that standard (backwards) “shooting” by interpolation does not work.

Marshall et al (1994) then devise a technique which makes use of “backward series expansions” and a transformation of the problem which is more numerically stable than the original problem. Their algorithm requires analytical input to transform the problem appropriately. Generalizing their algorithm to the case of more than two type distributions, or to correlated values, would require non-trivial extensions of their numerical and analytic algorithms.

In contrast to the problem of computing the solution to a set of differential equations, the algorithm for computing the equilibria to the auction game with finite actions constructed in this paper involves few conceptual subtleties. We simply want to find the matrix $X$ so that $X=BR(X)$, where the calculation of $BR(X)$ is a simple exercise (both in terms of programming time and computation time); it can be broken down into a sequence of single variable optimization problems, one for each jump point of each player. That is, for each player $i$, let $x_i^1$ be the smallest value of $t_i$ at which the player prefers action $A_i^1$ to $A_i^0$ (computing expected payoffs to each action according to $V_i(a_i;X,t_i)$), and then proceed to find $x_i^2$, searching over $t_i \geq x_i^1$. In mineral rights auctions, where players must compute the expected value of the object conditional on winning against the opponents’ current strategies, the conditional expectations must be numerically computed; we looked at signals and values which had a log-normal distribution.

The more difficult part of the problem is solving the nonlinear set of equations $X=BR(X)$. The theory of numerical analysis suggests a number of standard ways to solving this problem. Thus, here we will only sketch a few of the numerical issues which arise. First, since we have no global

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11 The algorithm also checks for the cases where a particular action is never used for any type.
12 To approximate the relevant conditional expectations and probabilities, we made use of the library of routines made available by Vassilis Hajivassiliou as a companion to Hajivassiliou, McFadden, and Ruud (1996).
13 See Judd (forthcoming) for an excellent treatment of numerical analysis in economics.
“contraction mapping” theorem, the algorithm $X^{k+1} = BR(X^k)$ is not guaranteed to converge for any starting value, and indeed it does not appear to in numerical trials. However, variations on the algorithm $X^{k+1} = \lambda \cdot BR(X^k) + (1-\lambda) \cdot X^k$ were effective at generating starting values for other algorithms. Methods based on quasi-Newton approaches can also be used. There are potentially large computational benefits to using an analytic Jacobian since the Jacobian is sparse. In particular, the point at which player $i$ jumps to action $A_m$, denoted $x'_m$, affects only the following elements of the best response of opponent $j \neq i$: $x'_{m-1}$, $x'_m$, and $x'_{m+1}$. Thus, to determine the effect of changing each component of $x'_i$ on the equation $X = BR(X)$, it is necessary to call the function $BR(\cdot)$ only three times. This allows us to compute a Jacobian of dimension $M \cdot I \times M \cdot I$ with only $I \cdot 3$ function calls. The number of bidding increments can thus be increased without affecting the number of function calls required by the nonlinear equation solver, while the computation time for the function $BR(X)$ will increase linearly in $M$. Even with this modification, computation could be slow for the mineral rights examples considered below. A final alternative is a simplex method, which was reliable in trials.

The first set of approximations we report were motivated by Marshall et al (1994). We computed the same set of auctions studied in their paper, and compared the calculations for equilibrium bid functions and expected revenue. Marshall et al (1994) chose a set of distributional assumptions motivated by the case where five bidders draw values from uniform $[0,1]$, and a subset of the bidders collude, bidding as a group with the group’s value being the maximum of the values of the participants. A comparison of the expected revenue $F(x)$ for coalition $n$ bidders and individual $n$ bidders is given in Table 1.

<table>
<thead>
<tr>
<th>Coalition Bidders</th>
<th>Our results, $M=100$:</th>
<th>Marshall et al Results:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expected Revenue</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Auctioneer</td>
<td>Coalition</td>
</tr>
<tr>
<td></td>
<td>Individual bidders</td>
<td></td>
</tr>
<tr>
<td>$a=1$ $n=4$</td>
<td>.6664</td>
<td>.0334</td>
</tr>
<tr>
<td></td>
<td>.0334</td>
<td>.6668</td>
</tr>
<tr>
<td></td>
<td>.0335</td>
<td>.0333</td>
</tr>
<tr>
<td>$a=2$ $n=3$</td>
<td>.6508</td>
<td>.0344</td>
</tr>
<tr>
<td></td>
<td>.0374</td>
<td>.6510</td>
</tr>
<tr>
<td></td>
<td>.0352</td>
<td>.0371</td>
</tr>
<tr>
<td>$a=3$ $n=2$</td>
<td>.6085</td>
<td>.0405</td>
</tr>
<tr>
<td></td>
<td>.0487</td>
<td>.6089</td>
</tr>
<tr>
<td></td>
<td>.0406</td>
<td>.0488</td>
</tr>
<tr>
<td>$a=4$ $n=1$</td>
<td>.5055</td>
<td>.0574</td>
</tr>
<tr>
<td></td>
<td>.0840</td>
<td>.5057</td>
</tr>
<tr>
<td></td>
<td>.0567</td>
<td>.0860</td>
</tr>
</tbody>
</table>

Figure 8: First Price Auction games between two coalitions.
revenue calculations for several auctions is presented in Figure 8. The table indicates that the expected revenue from the auction with discrete and continuous bidding units is within .001 in all cases. Thus, in these auctions, the difference between the continuous and discrete games is small. Figure 6 shows how the equilibria to the discrete game change as more bidding units are allowed.

Now consider an auction with affiliated private values, where the types are distributed $ln(t) \sim N(\mu, \Sigma)$. There are several potentially interesting asymmetries: those arising from differences in means, and those arising from differences in variances (which also affect the mean of $t$), and those arising from differences in covariances. The numerical results for differences in means and variances are not surprising: in the trials we conducted, types with higher means always bid less aggressively and get higher expected revenue. Changes in variances affect the shape of the distribution, so the higher variance types may be less aggressive in some regions and more in others, but they tend to do better overall. Differences in covariances are perhaps slightly more subtle, but no less intuitive. We will illustrate this case with an example. Suppose that two bidders have types which are highly correlated, while a third bidder has a type which is less correlated with the other two. Then the two bidders with highly correlated types bid more aggressively and win more often than the third bidder, who also gets higher expected revenue. The intuition is simple: the two bidders whose types are more highly correlated are always concerned with competing with one another, even when their values are high.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Variance Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ln(t_1)$</td>
<td>0</td>
<td>1 .5 .25</td>
</tr>
<tr>
<td>$ln(t_2)$</td>
<td>0</td>
<td>.5 1 .25</td>
</tr>
<tr>
<td>$ln(t_3)$</td>
<td>0</td>
<td>.25 .25 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expected Revenue</th>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
<th>Auctioneer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4691</td>
<td>0.4691</td>
<td>0.5008</td>
<td>1.4190</td>
<td></td>
</tr>
<tr>
<td>0.3283</td>
<td>0.3283</td>
<td>0.3434</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 9: Private values first price auction where the bidders have positively correlated values.

Next, we can consider pure “common values” auctions, where each player sees a private signal about the common value. Although we do not have a theorem guaranteeing that the single crossing property holds, we can numerically verify that it holds in the neighborhood of the equilibrium we

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14 See Wilson (1996) for a theoretical study of equilibrium in oral auctions with log-normal values.
compute for a particular game; the numerical analysis indicates that it does hold in all of the auctions we consider here. Let $z$ be the common value. Though our numerical algorithms allow for general variance-covariance structures, it is perhaps more intuitive to consider a particular example of a signal structure. Suppose each player $i$ receives a signal $\ln(s_i) = \ln(z) + \ln(e_i)$, where $\text{cov}(z,e_i)=0$. The assumption of a common value implies that signals will be positively correlated, and further, the errors $e_i$ may also be correlated. We can vary the informativeness of each player’s signal as well as the correlation of the signals. An interesting observation is that our result of private values auctions, that the “more independent” bidder (bidder 3) does better, is no longer true. While the “more correlated” bidders (bidders 1 and 2) still compete more aggressively, this aggressiveness has a new effect for bidder 3 which does not arise in the private value auction. In the private value auction, the competition between bidders 1 and 2 was most disadvantageous to those two bidders; bidder 3 won less often, but received higher expected payoffs upon winning due to his less aggressive bidding. But in the common value auction, having two more aggressive opponents is especially bad news due to the winner’s curse. Across a range of parameter values, the expected revenue for the bidder with an independent signal was close to that of the two bidders with correlated signals. Following is an example where bidder 3 does worse.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Variance Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln(e_1)$</td>
<td>0</td>
<td>.75 .1 0 0</td>
</tr>
<tr>
<td>$\ln(e_2)$</td>
<td>0</td>
<td>.1  .75 0 0</td>
</tr>
<tr>
<td>$\ln(e_3)$</td>
<td>0</td>
<td>0  0 0 .75 0</td>
</tr>
<tr>
<td>$\ln(z)$</td>
<td>0</td>
<td>0 0 0 0 .25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expected Revenue</th>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
<th>Auctioneer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0315</td>
<td>0.0315</td>
<td>0.019</td>
<td>1.005</td>
<td></td>
</tr>
</tbody>
</table>

| Prob. of Winning | 0.378 | 0.378 | 0.242 |

**Figure 10:** Common values first price auction where two bidders received positively correlated signals of the true value, and a third bidder receives an independent signal.

Another set of experiments we conducted concerned the effect of the precision of each player’s signal about the true value of the object. We first examined the case of two bidders, one of whom receives a signal which is more precise than the others; we then looked at the effect of a third bidder on equilibrium strategies on expected revenue.

Notice in this auction that the player with the more precise signal, player 1, bids slightly less aggressively. However, since her beliefs are more sensitive to her signals, her conditional
expected value is more variable than that of player 2, and so she is more likely to see high expected values. Thus, she wins just over half of the auctions. Despite the fact that she wins just over half, she achieves much higher revenue in each auction. It is perhaps surprising that player 2, despite having less precise information, still wins so frequently.\footnote{We also examined the effect of lessening the importance of the winner’s curse in this example. We allowed the bidders to have different values for the object, $z_1$ and $z_2$, where $\text{cov}(\ln(z_1),\ln(z_2))=.5$, and all other parameters of the model are the same as the latter example. The qualitative nature of the equilibrium remained unchanged, with player 1 bidding less aggressively and winning just over half the auctions. However, expected revenue went up for each bidder. Player 1 has expected revenue of .65, while player 2 has expected revenue of .56. There are two effects here: the bidders’ values are less correlated with one another, so they do not expect as much competition. Second, the winner’s curse has less bite. For the auctioneer, the first effect is most important: the auctioneer receives expected revenue of .99, less than in the pure common values auction.}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Variable} & \textbf{Mean} & \textbf{Variance} & \textbf{Matrix} \\
\hline
$\ln(e_1)$ & 0 & .1 & 0 0 \\
$\ln(e_2)$ & 0 & 0 & .3 0 \\
$\ln(z)$ & 0 & 0 & 0 1 \\
\hline
\end{tabular}
\caption{Common value auction where one bidder receives a signal which is more precise than the other bidder’s signal.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{Common value auction: Two Bidders Whose Signals Have Different Precisions}
\end{figure}

Returning to the pure common values model, we consider a final experiment, where we add a third player with a more precise signal ($\text{var}(e_3)=.1$). The strategies of the players look very different than in the two-bidder model. Now, the two well-informed players bid more aggressively than the player with the noisier signal; this exacerbates the winner’s curse for player 2. In addition, the bidder with the less precise signal wins less often and receives very little expected revenue relative to the two-bidder example.

In general, we can perform numerical computations for a variety of auctions. By changing the covariance structure between signals and values, we can vary the importance of the winner’s curse, the informativeness of the signals, and the mean values. Further characterizations of mineral rights auctions with heterogeneous bidders await future research.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Variance Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>ln(e₁)</td>
<td>0</td>
<td>0.1 0 0 0</td>
</tr>
<tr>
<td>ln(e₂)</td>
<td>0</td>
<td>0 0.3 0 0</td>
</tr>
<tr>
<td>ln(e₃)</td>
<td>0</td>
<td>0 0 0.1 0</td>
</tr>
<tr>
<td>ln(z)</td>
<td>0</td>
<td>0 0 0 0 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expected Revenue</th>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
<th>Auctioneer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.130</td>
<td>0.030</td>
<td>0.130</td>
<td>1.335</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Prob. of Winning</th>
<th>Bidders 1 and 3</th>
<th>Bidder 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.369</td>
<td>0.260</td>
</tr>
</tbody>
</table>

**Figure 12**: Common values auction where two bidders receive more precise signals than the third bidder.

## 6. Conclusions

This paper has introduced a restriction on a class of games called the Single Crossing Condition for games of incomplete information, where in response to nondecreasing strategies by opponents, players choose their actions as nondecreasing functions of their types. We have shown that pure strategy Nash equilibria will exist in such games when the set of available actions is finite, and further, with appropriate continuity or in auction games, a sequence of equilibria of finite-action games will have a subsequence which converges to an equilibrium. We have further established that similar results can be obtained for games which satisfy our “Limited Complexity Condition.” The formulation of games of incomplete information developed in this paper has the following advantages: (1) existence of pure strategy Nash equilibria can be verified by checking general and economically interpretable conditions, (2) the results for finite-action games require very few regularity assumptions, and (3) the equilibria are straightforward to numerically calculate for finite-action games, and these approximate the continuous equilibria for continuous games and auctions. The application of these results to first price auction games led to a generalization of the existing literature on the existence of equilibria in auctions with heterogeneous bidders with correlated signals and/or common values. The condition for monotonicity of strategies, the single crossing property, can be characterized for many commonly studied games using the results of Athey (1995, 1996). Finally, numerical analysis can be used to analyze behavior in auction games whose properties have not been fully characterized in the existing literature.