r Minus g*

Robert J. Barro

Harvard University

November 2020

Abstract

Long-term data show that the dynamic efficiency condition $r>g$ holds when g is represented by the average growth rate of real GDP if r is the average real rate of return on equity, $E(r^e)$, but not if r is the risk-free rate, r^f. This pattern accords with a simple disaster-risk model calibrated to fit observed equity premia. If Ponzi (chain-letter) finance by private agents and the government are precluded, the equilibrium can feature $r^f \leq E(g)$, a result that does not signal dynamic inefficiency. In contrast, $E(r^e)>E(g)$ is required for dynamic efficiency, implied by the model, and consistent with the data. The model satisfies Ricardian Equivalence because, without Ponzi finance by the government, a rise in safe assets from increased public debt is matched by an increase in the safe (that is, certain) present value of liabilities associated with net taxes.

*I have benefited from suggestions by Marios Angeletos, John Campbell, Xavier Gabaix, and Jon Steinsson.

The familiar dynamic efficiency condition is $r>g$, where r is a real rate of return and g is a rate of economic growth, for example, of real GDP. This condition applies to the steady state of the infinite-horizon neoclassical growth model, as developed in Cass (1965) and Koopmans (1965) and elaborated in Barro and Sala-i-Martin (2004, Ch. 2). The condition rules out excessive saving and investment; specifically, capital is not accumulated in the long run beyond the golden-rule level described in Phelps (1961). In contrast, $r<g$ is possible in the steady state of the overlapping-generations model developed by Diamond (1965), so that reduced saving and investment can be Pareto improving. In this OLG environment, expansions of public debt and enlargement of pay-as-you-go social security systems may be welfare enhancing. The condition $r>g$ was highlighted in Abel, et al. (1989), and it arises in many analyses of wealth accumulation and public debt, such as the recent studies by Piketty (2011) and Blanchard (2019).

Whether $r>g$ applies empirically depends mainly on how one defines r. The bottom line from the available data since 1870 for 14 OECD countries, shown in Table 1, is that the condition holds if r is gauged by the average of the realized real rate of return on equity and does not hold if r equals the average of the realized real rate of return on short-term government bills.[1] The latter variable likely approximates the average of safe real interest rates. Since the $r>g$ condition holds when r is based on risky returns (on equity) but not when r is based on safe returns (approximated by government bills), a key underlying element is the large gap between the expected real rate of return on equity and the safe real interest rate; that is, the equity

[1]This perspective is consistent with the discussion in Abel, et al. (1989, p.2), who considered estimates of the productivity of capital but did not look at rates of return on equity. The results accord with the returns data presented in Barro and Ursúa (2008, Table 5), which appear in updated form in Table 1. The underlying numbers come mostly from *Global Financial Data*. Longer-term patterns are studied in Homer and Sylla (1996) and Schmelzing (2020).

premium. This substantial premium provides a lot of space in which *g* can fit, so that *g* can be simultaneously below the risky rate and above the safe rate.

The *r-g* condition depends also on the definition of *g*, which is typically based on the long-run growth rate of real GDP. In the usual version of the neoclassical growth model, the relevant transversality condition involves the growth rate of the level of real macroeconomic aggregates, not quantities per capita. This result emerges from a specification in which individuals currently alive are connected as parents to members of future generations; for example, through altruistic linkages. In this case, current households consider the asymptotic present value of all future real income, whether growing per capita or because of population growth.

To be more specific on the long-term empirical patterns, Table 1 shows the means since 1870 (or a more recent year when earlier data are missing) of real rates of return and growth rates for 14 OECD countries with available data. The returns refer to averages of realized arithmetic real rates of return on stocks (based on broad indices such as the S&P 500), short-term bills (analogous to Treasury Bills), and government bonds (around 10-year maturity). The growth rates refer to real GDP per capita, real personal consumer expenditure per capita, and population (which enables calculations of growth rates of levels of real GDP and consumer expenditure). When averaging over the 14 countries, the average annual real rates of return were 7.0% on stocks, 1.1% on bills, and 2.6% on bonds. Averages for annual growth rates were 1.9% for per capita GDP, 1.7% for per capita consumer expenditure, and 0.9% for population. Hence, for levels, the average annual growth rates were 2.8% for GDP and 2.6% for consumer expenditure.[2]

---

[2]Results are similar if samples start in 1960, rather than 1870. In the 1960 case, the averages over the 14 countries were 7.8% for stock returns, 1.4% for bill returns, 3.5% for bond returns, 2.1% for per capita GDP, 2.0% for per capita consumer expenditure, and 0.7% for population.

These numbers show that, over the long term, the growth rates of GDP and consumer expenditure were clearly below the real rate of return on equity and above that on bills. In contrast, real rates of return on bonds were close to the growth rates of GDP and consumer expenditure.

To understand what measure of *r-g* matters, one needs a theoretical model that can explain a large equity premium. The standard neoclassical growth model is unsatisfactory for this purpose because, in its deterministic setting, there is a single real rate of return and no equity premium. A satisfactory framework requires uncertainty that is sufficient to generate a large equity premium; that is, to resolve the equity-premium puzzle of Mehra and Prescott (1985). I use a simple representative-agent model with disaster risk, following Barro (2009), which built on Rietz (1988) and Barro (2006). The analysis could also be pursued in alternative frameworks that can generate a large equity premium. Prominent possibilities here are the long-run risks framework of Bansal and Yaron (2004)[3] and the heterogeneous-consumers model of Constantinides and Duffie (1996).

I. Fruit-tree Model with Stochastic Depreciation

I use the simplest representative-agent model I have thought of that has sufficient aggregate uncertainty to generate a realistic equity premium and that also determines endogenously the rate of economic growth and the investment/saving decision. The model is an "AK" version of the Lucas fruit-tree model:

$$Y_t = AK_t, \tag{1}$$

[3]This model features variations in the mean and volatility of long-run growth rates. Other models, such as Gabaix (2012), allow for volatility of disaster risk.

where Y_t is output (of fruit or seeds) and K_t is the capital stock in the form of trees. The model is constructed in discrete time but the length of the period is allowed to approach zero. The timing is specified so that K_t represents the capital stock available at the start of period t and Y_t the output produced during period t. The productivity level, $A>0$, is assumed constant but productivity shocks can be introduced. These shocks would be analogous to the stochastic depreciation introduced below. A simplifying assumption is that the marginal product of capital (equal to the average product) does not diminish as capital is accumulated. This absence of diminishing returns seems most plausible when capital stock and capital services are interpreted broadly to encompass human capital, household durables, intangible capital, etc.

During period t, output can be consumed as fruit, C_t, or invested as seed, I_t, so that

(2) $$C_t = Y_t - I_t = AK_t - I_t.$$

The creation of new trees through planting seeds (that is, gross investment) is assumed to be rapid enough so that, as in the conventional one-sector production framework, the fruit price of a unit of capital is pegged at a price normalized to one.[4] Therefore, in the simplest setting, the price, P_t, of equity—in the sense of a claim on all of the trees—always equals K_t. This result corresponds to "Tobin's q" equaling one. In terms of changes over time, the equity price ratio, P_{t+1}/P_t, would equal the capital-stock ratio, K_{t+1}/K_t, which equals the output ratio, Y_{t+1}/Y_t. Therefore, the volatilities of equity prices and output would be the same. In contrast, empirically observed equity prices are far more volatile than output, gauged by real GDP.

In practice, there are many reasons that equity valuations for businesses would depart from the reproduction cost of the underlying capital stock. As an example, the price of goods sold might be a variable markup, $1+\mu_t$, of the underlying cost of production, and variations in μ_t

[4]Irreversibility of investment would affect this result if the constraint that gross investment cannot be negative is sometimes binding. However, in the present model, gross investment is always chosen to be positive.

would generate fluctuations in P_t independently of those in K_t and Y_t. Another possibility is that the cost of shifting between consumables and new capital goods in the production process does not stay fixed (at unity), and a further idea is that the productivity of old capital might vary stochastically relative to that of new capital.

To capture these types of effects, the price ratio for equity, P_{t+1}/P_t, is allowed to deviate from that for the capital stock by a random term:

$$\frac{P_{t+1}}{P_t} = \frac{K_{t+1}}{K_t} + \varepsilon_{t+1}. \tag{3}$$

The shock ε_{t+1} is assumed to have zero mean, serial independence, and to be distributed independently of K_{t+1}/K_t. This specification means that an epsilon shock, possibly representing a change in markups or a shift in the rate of transformation between consumables and capital on the production side, has a permanent effect on the level of the equity price. For asset-pricing purposes, an important condition is that ε_{t+1}will be distributed independently of consumption growth, C_{t+1}/C_t (because ε_{t+1} is assumed to be independent of capital-stock growth, K_{t+1}/K_t). In this case, the epsilon shock will not influence asset pricing in the model's baseline specification but will allow for the equity price to be more volatile than capital stock and output.

The model allows for rare disasters by having stochastic depreciation of the capital stock; that is, destruction of trees. The capital stock evolves because of gross investment and depreciation, $\delta_{t+1}K_t$:

$$K_{t+1} = K_t + I_t - \delta_{t+1}K_t. \tag{4}$$

The depreciation rate, δ_{t+1}, is stochastic and equal to

$$\delta_{t+1} = \delta + u_{t+1} + v_{t+1}, \tag{5}$$

where $0<\delta<1$. The u_{t+1} shock, normally distributed with mean 0, variance σ^2, and serial independence, represents normal economic fluctuations. This shock has a permanent effect on depreciation and, therefore, on the level of the capital stock. (Since u_{t+1} can be negative, it is possible that δ_{t+1} would be negative.) The v_{t+1} shock represents rare disasters, modeled as large-scale destruction of trees. With probability 1-p, v_{t+1}=0, and with probability p, v_{t+1}=-d_{t+1}; that is, the fraction d_{t+1} (0< d_{t+1}<1) of the trees is (permanently) destroyed in a disaster event. The disaster size, d_{t+1}, is subject to a time-invariant probability distribution. (This disaster shock is one-sided because there are no bonanzas associated with the sudden appearance of new trees.) A natural extension, as in Gabaix (2012), would allow for stochastic variations in disaster probability, p, or in the distribution of disaster sizes, d.

One part of tree output is paid out as dividends, which correspond to consumption of fruit by owners. The other part is retained within the tree business to finance investment in new trees. From the standpoint of equity holders, the first part of the return on equity is the dividend yield, which equals the ratio of consumption, C_t, to the capital stock, K_t.[5] The second part consists of equity price appreciation, P_{t+1}/P_t, given in equation (3). Therefore, using equation (2), the gross one-period return on equity, R_t^e, is

(6) $$R_t^e = \frac{C_t}{K_t} + \frac{K_{t+1}}{K_t} + \varepsilon_{t+1} = A - \frac{I_t}{K_t} + \frac{K_{t+1}}{K_t} + \varepsilon_{t+1}.$$

Equations (4) and (5) imply that the growth of the capital stock is given by

(7) $$\frac{K_{t+1}}{K_t} = 1 + \frac{I_t}{K_t} - \delta - u_{t+1} - v_{t+1}.$$

Substitution of equation (7) into the right-hand side of equation (6) yields:

(8) $$R_t^e = 1 + A - \delta - u_{t+1} - v_{t+1} + \varepsilon_{t+1},$$

so that the net rate of return, $r_t^e = R_t^e - 1$, is

(9) $$r_t^e = A - \delta - u_{t+1} - v_{t+1} + \varepsilon_{t+1}.$$

[5]With a markup ratio $1+\mu$, the dividend payout in value terms is $(1+\mu)\cdot C_t$ and the equity price is $(1+\mu)\cdot K_t$. Therefore, a change in μ does not affect the dividend yield.

In equation (9), the shocks u_{t+1} and ε_{t+1} have zero mean, and the disaster shock v_{t+1} has mean $p \cdot E(d)$. Therefore, the expectation of the one-period net rate of return is

$$E(r_t^e) = A - \delta - p \cdot E(d). \tag{10}$$

An assumption is that $A > \delta + p \cdot E(d)$ and, hence, $E(r_t^e) > 0$. Since productivity, A, disaster probability, p, and mean disaster size, $E(d)$, are time invariant, $E(r_t^e)$ is constant over time.

The next step concerns the determination of the ratio of gross investment to the capital stock, I_t/K_t. Gross investment equals gross saving (for a closed economy with no government sector) and can be determined from dynamic conditions for consumer optimization.

Barro (2009) uses a specification of preferences for the representative consumer based on the analysis of Epstein and Zin (1989) and Weil (1990), described as EZW preferences. This specification, corresponding to Barro (2009, equation [10]), features constant values of the rate of time preference, ρ, the coefficient of relative risk aversion, γ, and the intertemporal elasticity of substitution (*IES*), $1/\theta$, where $\theta=\gamma$ holds in the standard power-utility formulation. The condition $\gamma>1$ is required for the model to have a chance of generating an equity premium in the neighborhood of that observed empirically. In that case, the standard representation would require $\theta>1$ and, hence, $IES<1$. Barro (2009) followed Bansal and Yaron (2004) to argue that the property $IES<1$ generates counter-factual predictions regarding equity pricing. In the present analysis, $\gamma>1$ is crucial, but $\theta<1$ is less important.

In general, EZW preferences do not allow for simple formulas for pricing assets. However, when the underlying shocks are i.i.d., as in the present model, the analysis simplifies dramatically. Specifically, Barro (2009, equations [12] and [13]) used results from Giovannini and Weil (1989, appendix) to show that a standard-looking asset-pricing condition applies:

$$C_t^{-\gamma} = \left(\frac{1}{1+\rho^*}\right) \cdot E_t(R_t \cdot C_{t+1}^{-\gamma}), \tag{11}$$

where R_t is the gross return on any asset between dates t and $t+1$. Two features that differ from those in the standard power-utility model (where $\gamma=\theta$) are worth noting. First, the exponents on C_t and C_{t+1} in equation (11) involve γ, the coefficient of relative risk aversion, not θ, which is the reciprocal of the *IES*. Second, the effective rate of time preference, ρ^*, differs from ρ when γ and θ diverge. The formula for ρ^* is, if $\gamma\neq 1$,

$$(12)\quad \rho* = \rho - (\gamma-\theta)\cdot\left\{E(g_t) - \left(\tfrac{1}{2}\right)\cdot\gamma\sigma^2 - \left(\tfrac{p}{\gamma-1}\right)\cdot[E(1-d)^{1-\gamma} - 1 - (\gamma-1)\cdot Ed\right\}.$$

The term $E(g_t)$ is the economy's expected growth rate and is derived below.

Because the shocks u_{t+1} and v_{t+1} are i.i.d. (permanent to the levels of capital stock and output), the ratio of gross saving to the capital stock will be optimally chosen as a constant, denoted by v. The value of v can be determined by applying the consumption-based asset-pricing formula in equation (11) to the gross return on equity, given in equation (8). The result (applying as the length of the period approaches zero) is:[6]

$$(13)\quad \upsilon = \delta + \left(\tfrac{1}{\theta}\right)\left\{A - \delta - \rho - \left(\tfrac{1}{2}\right)\gamma(1-\theta)\cdot\sigma^2 - p\cdot\left(\tfrac{1-\theta}{\gamma-1}\right)[E(1-d)^{1-\gamma} - 1]\right\}.$$

Equation (13) implies that, if $\gamma>0$, the sign of the effect of uncertainty (σ, p, or the distribution of d) on the gross saving ratio, v, depends on the *IES*, $1/\theta$, not the degree of risk aversion, γ. If $\theta<1$, so that the *IES* exceeds *1*, the "substitution effect" dominates, and more uncertainty (higher σ or p or an outward shift of the d-distribution) decreases v. For the parameter values used subsequently to calibrate the model (in Table 2), the main effect from

---

[6]For equation (13) to be valid, the associated level of consumption has to be positive, corresponding to υ<A. Equation (13) implies that the condition $\upsilon<A$ can be expressed as:

$$\rho > (1-\theta)\left\{A - \delta - \left(\tfrac{1}{2}\right)\gamma\sigma^2 - \left(\tfrac{p}{\gamma-1}\right)[E(1-d)^{1-\gamma} - 1]\right\}.$$

If this inequality does not hold, the optimization problem is not well defined because the attainable expected utility is unbounded. The inequality is satisfied for the parameter values assumed later. (An analogous inequality condition applies in the standard deterministic neoclassical growth model, where $\sigma^2=p=0$.)

uncertainty on the saving ratio comes from disaster risk—the term involving p on the far right of equation (13). The term involving σ^2 is quantitatively unimportant.

In the present model, the growth rates of the macroeconomic variables—capital stock, output, and consumption—are always the same. This common growth rate, denoted by g_t, is given from equation (7) as

$$g_t = \frac{K_{t+1}}{K_t} - 1 = \upsilon - \delta - u_{t+1} - v_{t+1}, \tag{14}$$

where recall that u_{t+1} is the normal shock and v_{t+1} is the disaster shock. A higher gross saving ratio, υ, in equation (13) implies a higher growth rate in equation (14).

The expected growth rate is constant and given from equation (14) by

$$E(g_t) = \upsilon - \delta - p \cdot E(d). \tag{15}$$

An assumption is that the parameters imply $E(g_t)>0$, given that υ-δ is determined from equation (13).

The relationship between the expected rate of return on equity, $E(r_t^e)$ from equation (10), and the expected growth rate, $E(g_t)$ from equation (15), is given by

$$E(r_t^e) - E(g_t) = A - \upsilon. \tag{16}$$

Therefore, if $A>\upsilon$—meaning that output exceeds gross investment and, hence, that consumption is positive (see n. 6)—the expected rate of return on equity exceeds the expected growth rate.[7] In other words, the $r>g$ condition holds in the model when the r applies to the expected rate of return on equity.

To get perspective on equation (16), note that the expected net rate of return on equity is

$$E(r_t^e) = \frac{C_t}{P_t} + E\left(\frac{P_{t+1}}{P_t} - 1\right) = \frac{C_t}{P_t} + E(g_t), \tag{17}$$

[7]The condition $A>\upsilon$ corresponds to the criterion for dynamic efficiency proposed in Abel, et al. (1989, p.2): " … an economy is dynamically efficient if it invests less than the return to capital …" In the present model, A equals the marginal (and average) product of capital and, therefore, corresponds to the gross return on capital.

where C_t/P_t is the dividend yield and $E(g_t)$ is the expected rate of capital gain. Rearranging equation (17), assuming $E(r_t^e) \neq E(g_t)$, yields a formula for the price-dividend ratio of an equity claim:

$$\frac{P_t}{C_t} = \frac{1}{E(r_t^e) - E(g_t)} \,. \tag{18}$$

Equation (18) is usually called the "Gordon growth formula."[8] This formula relates the price-dividend ratio for equity to the reciprocal of the difference between the expected net rate of return on equity and the expected growth rate (of dividends; that is, of C_t in the present model). Equation (18) implies that P_t/C_t will be positive and finite only if $E(r_t^e) > E(g_t)$, as already noted in equation (16).

Another way to think about the result is that claims on capital give owners the rights to a stream of dividends (fruit) that is expected to rise at the rate $E(g_t)$. If this expected growth rate were at least as high as the discount rate—which corresponds to the expected rate of return on the asset, $E(r_t^e)$—then the present value of the claim would be infinite. This result is inconsistent with an equilibrium in which consumption is chosen optimally over time. However, this inconsistency does not arise if $E(r_t^e) > E(g_t)$.

Consider now the risk-free real interest rate, denoted by r^f. In the model, risk-free claims are internal instruments, amounting to private bonds that represent loans from one agent to another.[9] The aggregate of risk-free assets always equals zero; that is, these assets are in zero net supply. Section III considers government bonds, which may be in positive net supply.

Although the aggregate quantity of risk-free assets will be zero, the model still determines a shadow real interest rate, r^f, applying to these assets. The value of r^f, which is

[8]See Campbell (2018, p. 130). The reference to Gordon refers to Gordon and Shapiro (1956, equation [7]).
[9]The assumption is that there do not exist storable goods or other real assets that yield a risk-free real rate of return above the equilibrium r^f that arises in the model.

constant, can be determined from the asset-pricing condition in equation (11) along with the result that C_{t+1}/C_t equals the value g_t given in equation (14). The solution is

$$r^f = A - \delta - \gamma\sigma^2 - p \cdot E[d \cdot (1-d)^{-\gamma}]. \tag{19}$$

More uncertainty—higher σ or p or an outward shift of the d-distribution—decreases r^f.[10] For the parameter values used to calibrate the model (in Table 2), the main effect from uncertainty on r^f comes from the disaster term, which involves p on the far right of equation (19). With these parameter values, including $\gamma=3.5$, and for the observed histogram of disaster sizes, the term $E[d \cdot (1-d)^{-\gamma}]$ equals 1.7, implying that a (once-and-for-all) rise in disaster probability, p, by 0.010 per year lowers r^f by a substantial 0.017 per year. The effect associated with the normal shock, which involves σ^2, is quantitatively unimportant. Reasonable parameter values accord with a value of r^f close to zero in equation (19). For the values specified in Table 2, the equilibrium r^f is 0.001, essentially zero.[11] By comparison, long-term averages for OECD countries of realized real rates of return on government bills, as shown in Table 1, are around 0.01 per year.

The equity premium is given from equations (10) and (19) by

$$E(r_t^e) - r^f = \gamma\sigma^2 + p \cdot E\{d \cdot [(1-d)^{-\gamma} - 1]\}. \tag{20}$$

For the parameter values specified in Table 2 (which include $\gamma=3.5$), the predicted equity premium from equation (20) is around 0.06 per year. This result accords with measured equity

[10]Since the model has i.i.d. shocks, the term structure of risk-free rates is flat; that is, r^f is the short-term and long-term risk-free rate. Therefore, the model does not explain why, in Table 1, the average real rate of return on 10-year government bonds exceeds that on short-term government bills.

[11]This low value of r^f reflects the fatness of the disaster tail, not the skewness, which arises from the exclusion of bonanzas. Skewness would not apply if there were a bonanza probability, q, that equaled p and if the distribution of bonanza sizes were the same as that for disasters. In this case, in turns out that the calibrated model's equilibrium value of r^f becomes 0.005, rather than 0.001. To put it another way, with a small rise in $p=q$, which turns out to be from 0.040 to 0.042, the equilibrium r^f would still be 0.001, despite the lack of skewness. Skewness is unimportant because, with the substantial "diminishing marginal utility" implied by $\gamma=3.5$, the positive tail from bonanzas counts little for the equilibrium r^f.

premia, gauged by the gap shown in Table 1 between long-run averages of realized real rates of return on equity and short-term government bills.[12] The dominant part of the equity premium in the model reflects disaster risk, which appears in the term in equation (20) that contains the disaster probability, p. The other part, $\gamma\sigma^2$, is quantitatively unimportant, as in the related analysis of the equity-premium puzzle in Mehra and Prescott (1985). Note that, in the model, the positive effect of disaster risk on the equity premium reflects the negative impact on r^f in equation (19), not an effect on the expected rate of return on equity, $E(r_t^e)$, in equation (10).[13]

The gap between r^f and the expected growth rate, $E(g_t)$, is given from equations (19) and (15) by

$$r^f - E(g_t) = A - v - \gamma\sigma^2 - p \cdot E\{d \cdot [(1-d)^{-\gamma} - 1]\}. \tag{21}$$

The parameter values considered below suggest that this gap is likely to be negative (because the model's predicted value for r^f in equation (19) is likely to be close to zero). In any event, the model does not require the $r>g$ condition to hold when the r refers to the risk-free rate. Thus, there is no conflict between the theory and the empirical observation that this condition fails to hold when r is measured by the average real rate of return on government bills.

In terms of portfolio allocation, the representative agent in period t holds claims on all of the capital stock, K_t, at price, P_t, and holds a zero net position in risk-free assets. That is, in the equilibrium—for the distribution of r_t^e implied by equation (9) and for the value of r^f given in equation (19)—the agent is willing at all points in time to hold 100% of assets in risky capital and 0% in risk-free claims.

[12]The observed stock returns from Table 1 should be adjusted downward because of leverage associated with bond financing to match up with the r_t^e in the model. However, the computation of the expected net marginal product of capital requires an upward adjustment to account for taxation of earnings at the corporate level.
[13]Equation (10) implies a small negative effect from the disaster probability, p, on $E(r_t^e)$. This effect would not arise if bonanzas were treated symmetrically with disasters, as in n. 11.

Because the representative agent ends up with a zero net position in risk-free assets, the equilibrium would be the same if risk-free assets did not exist. However, the existence of these assets would matter if households were heterogeneous; differing, for example, by coefficients of relative risk aversion, γ. In that case, agents with relatively high γ would tend to be net positive in safe assets, thereby lending funds on a risk-free basis to those with relatively low γ.

Consider now whether an outcome with $r^f \leq E(g_t)$ violates a transversality condition. To assess this possibility, consider the position of the representative agent, who, in equilibrium, holds all assets as risky equity with a zero net position in safe assets. Suppose that this agent in period t perturbs his or her position by issuing the quantity B_t of safe bonds and using the proceeds to raise C_t. Suppose further that the agent plans never to repay this debt; that is, the debt is rolled over in perpetuity, so that the stock of debt grows at the rate r^f. All other parts of the plan for consumption and portfolio allocation subsequent to date t are assumed to be unchanged (and r^f and the time path of r_t^e are unaffected by an individual's perturbation). If this new plan were feasible, the agent's expected overall utility would rise, corresponding to the increase in C_t and unchanged values of C_{t+1}, … That is, the agent would not actually have been optimizing, and the proposed equilibrium would be invalid.

The key element is the agent's debt, which, in the proposed perturbation, equals B_t>0 at date t. If this debt is rolled over in perpetuity at the rate r^f, the debt at date t+N is $B_{t+N} = B_t(1 + r^f)^N$. The present value, computed at the rate r^f, is, therefore, B_t>0; hence, the present value does not go to zero as N tends to infinity. In order for this perturbation to be feasible, other agents would have to be willing to accumulate risk-free assets so that the present value remains positive as N tends to infinity. However, no agents would be willing to accumulate risk-free assets in this manner because they would be better off using the assets at some date in finite time to raise consumption. To reconcile this inconsistency, the credit market has to require each individual to choose a time path of debt so that the present value, $B_{t+N}/(1+r^f)^N$, approaches zero as N tends to infinity. Importantly, this condition applies independently of the values of r^f and the expected growth rate, $E(g)$, and the result $r^f \leq E(g)$ is possible.[14]

The key condition needed to validate the equilibrium is the exclusion of a form of Ponzi or chain-letter finance in which an individual borrows at the risk-free rate and then fully rolls over the added debt in perpetuity. Individuals can consider an array of perturbations to the equilibrium in which they change the amount consumed in any period t and then change the amount consumed in any other period, such as $t+1$, while shifting correspondingly the amounts held in risk-free form during the two periods. That is, shifts in the timing of consumption associated with the rate r^f are allowed among any periods within a finite horizon. These perturbations away from the equilibrium are not optimal for the individual because equation (19) guarantees that the first-order conditions for intertemporal choices of consumption (associated with r^f) are all satisfied.

II. Illustrative Calibration of the Model

Table 2 shows how the model works quantitatively for a reasonable set of parameter values. The most important settings in the calibration relate to disaster probability and size distribution. The values for disaster probability, p, and the distribution of disaster sizes, d, come from an updated version of the numbers in Barro and Ursúa (2008). Specifically, peak-to-trough disaster events of size 10% or more for per capita GDP were isolated for 185 cases,

[14]The condition $r^f > E(g_t)$ may hold in the model. For example, if there is no uncertainty, so that $p=\sigma^2=0$, then r^f would equal $E(r_t^e)$, and both returns would exceed the expected growth rate, $E(g_t)$.

corresponding to data for 40 countries going back as far as 1870 and up to 2012.[15] This sample shows an average proportionate disaster size, $E(d)$, of 0.21. Taking account of the duration of each disaster event, the implied disaster probability, p (the chance of entering into a disaster state), is 0.04 per year. The objects $E(1\text{-}d)^{-\gamma}$, $E(1\text{-}d)^{1-\gamma}$, and $E[d((1\text{-}d)^{-\gamma}]$ equal 4.0, 2.3, and 1.7, respectively, for a coefficient of relative risk aversion, γ, of 3.5 and for the observed histogram of disaster sizes.[16] This value of γ, when used in equation (20) (along with the value of p and the size distribution of d), generates an equity premium that accords with empirically observed values around 0.06 per year.[17] The variance of the normal shock, σ^2, is set at 0.0004 per year to accord with the observed annual volatility of real GDP growth. However, in the relevant range, the results on the equity premium and other outcomes are insensitive to the value of σ^2.

The deterministic part of the depreciation rate, δ, is set at 0.05 per year, the average BEA number from 1948 to 2018 for fixed assets (including government assets but excluding consumer durables other than residential housing). The value set for A, 0.12 per year, is chosen to generate realistic levels of the real rates of return, which turn out to be 0.062 per year for $E(r^e)$ and 0.001 per year for r^f. The values of $\rho=0.04$ per year and $\theta=0.5$ determine the gross saving and investment ratio, υ, to be 0.090 per year[18] and the expected growth rate, $E(g)$, to be 0.032 per year (the long-run average U.S. growth rate of real GDP).

---

[15]Colombia is the only one of the 40 countries to experience no GDP-based contractions of size 10% or more. Malaysia and Singapore, which have long-term GDP data, were excluded from the calculations because of the large gaps in data around World War II.

[16]An alternative approach, pursued in Barro and Jin (2011), assumes that disaster sizes follow a power-law distribution. The results are similar to those based on the observed histogram.

[17]This calculation views each disaster event as permanent for the level of real GDP. Barro, Nakamura, Steinsson, and Ursúa (2013) and Barro and Jin (2020) assume, instead, that disasters are only partly permanent. The estimated eventual recovery fraction is about 50% of the initial contraction. An allowance for the partly temporary nature of macroeconomic disasters raises the value of γ required to match the observed equity premium.

[18]The values of ρ and θ both contribute to υ and $E(g)$. The value of θ would also affect the sensitivity of the gross saving rate to dimensions of uncertainty—p, the size distribution of d, and σ^2. However, in the present analysis, these uncertainty parameters are fixed. Therefore, the results shown for outcome variables in Table 2 could be generated with alternative combinations of ρ and θ.

The implied ratio of gross investment to GDP is $\upsilon/A=0.75$. An interpretation of this high ratio is that, as mentioned before, the underlying *AK* production function should be interpreted in terms of a broad definition of what constitutes capital and investment.

III. Government Bonds

Let B_t^g be the real quantity of government bonds outstanding at the start of period t. These bonds are assumed to be analogous to the safe short-term bonds that private agents were able to issue. The real interest rate applicable to government bonds is assumed to be r^f, the same rate as that for private bonds. This assumption of equal interest rates is not far from reality for the United States if one identifies private bonds with prime corporate obligations. For example, from 1920 (which has the first data on securities comparable to U.S. Treasury Bills) to 2019, the average of annual nominal yields was 0.034 for 3-month Treasury Bills and 0.041 for 90-day commercial paper; that is, the spread was only 0.007. The average annual real rate of return on 10-year U.S. government bonds over this period was 0.029, whereas that on Moody's AAA corporate bonds was 0.036, again a spread of 0.007.

The government collects lump-sum real taxes net of transfers of T_t during period t. The quantity T_t is treated as a random variable. Neglecting government purchases of goods and services (government consumption and investment), the government's budget constraint is:

$$B_{t+1}^g - B_t^g = r^f \cdot B_t^g - T_t. \tag{22}$$

Hence, the budget deficit equals the excess of spending (for interest and transfers) over revenue (from net taxes). The real interest rate r^f is assumed to be constant and known in period t, as in the model worked out before without a government sector. Net taxes, T_{t+j}, in each future period ($j\geq 1$) are stochastic from the perspective of period t, and the evolution of these net taxes determines the future quantities of government bonds, $B^g_{t+2}, B^g_{t+3}, \ldots$ These quantities are random from the perspective of period *t*, but the realizations of these quantities are linked to the realizations of net taxes from the government's budget constraint in equation (22).

A key object for the equilibrium is the present value of net taxes, computed using as a discount rate the safe real interest rate, r^f. Iterating equation (22) into the future for $N \geq 1$ periods yields an intertemporal budget constraint for the government:

(23) $$\sum_{j=1}^{N}[T_{t+j-1}/(1+r^f)^j] = B^g_t - B^g_{t+N}/(1+r^f)^N.$$

Therefore, aside from the final term on the right-hand side, the present value of net taxes, computed as of period *t*, is non-stochastic and pinned down to equal the stock of government bonds, B^g_t, at the start of period *t*.

The previous infinite-horizon model corresponds to *N* tending to infinity. A key issue is how the final term on the right-hand side of equation (23), $B^g_{t+N}/(1+r^f)^N$, behaves as *N* approaches infinity. The asymptotic behavior of this term is analogous to the Ponzi-type borrowing considered before for an individual agent in the model without government. In the previous setting, the term had to approach zero because no private agents would be willing to hold bonds with an asymptotically positive present value. The same argument applies now to government bonds. The term $B^g_{t+N}/(1+r^f)^N$ has to approach zero as *N* approaches infinity because private agents are unwilling to hold bonds with asymptotically positive present value. This result rules out Ponzi borrowing by the government, and equation (23) becomes, when expressed over an infinite horizon:

(24) $$\sum_{j=1}^{\infty}[T_{t+j-1}/(1+r^f)^j] = B^g_t.$$

Equation (24) says that the government can choose the timing of tax collections (and, therefore, budget deficits), but the present value of taxes is pinned down to equal the starting amount of public debt.

The model described before still applies for private agents. The only new elements are that the representative agent holds the initial stock of government bonds, B_t^g, and has to pay the stream of lump-sum net taxes, $T_t, T_{t+1}, \ldots$ However, the present value of the latter over an infinite horizon (evaluated at the rate r^f) is pinned down from equation (24) to equal B_t^g. Therefore, the net wealth of the representative agent is unaffected by the level of B_t^g or the timing of taxes and budget deficits. It follows that the model satisfies Ricardian Equivalence—the equilibrium with respect to real rates of return, investment, and economic growth is invariant with choices related to public debt.

Another way to look at the results is in terms of quantities of safe assets. Government bonds, B_t^g, are a form of safe asset, which pays the safe real interest rate, r^f. However, net taxes owed to the government amount to a safe (that is, certain) liability—although the timing of taxes is uncertain, the overall obligation has a fixed present value when evaluated using the rate r^f. Equation (24) implies that the net of the asset and liability is nil, implying no change in the economy's net quantity of safe assets. In this sense, the presence of public debt does not mean that safe assets are in positive net supply.

The model can be extended to allow for a path of government purchases, G_t. Future values $G_{t+1}, G_{t+2}, \ldots$ are stochastic from the perspective of period t. These choices might reflect war and peace, changing preferences for public programs, and so on. If Ponzi finance by the government is still ruled out, the government's intertemporal budget constraint over an infinite horizon in equation (24) becomes:

(25) $$\sum_{j=1}^{\infty}[T_{t+j-1}/(1+r^f)^j] = B_t^g + \sum_{j=1}^{\infty}[G_{t+j-1}/(1+r^f)^j].$$

Equation (25) implies that the present value of taxes is no longer pegged to equal B_t^g. Instead, as time evolves, the changing present value of taxes (above B_t^g) reflects randomness in the present value of government purchases.

The equilibrium of the economy depends on the realization of G_t, the stochastic process that generates future values G_{t+1}, G_{t+2}, ... and the ways that government purchases affect household utility or production. The equilibrium also takes into account that G_t adds to C_t and I_t as a use of current output, Y_t. However, Ricardian Equivalence concerns effects from changes in public debt and taxes for a given behavior of government purchases. That is, the issue is how the economy responds to variations in B_t^g and the path of taxes (and budget deficits), for a given path of G_t, G_{t+1}, ... In equation (25), the probability distribution of the present value $\sum_{j=1}^{\infty}[G_{t+j-1}/(1+r^f)^j]$ is held fixed in this experiment, and the changes related to public debt still have no impact on the net of B_t^g over the present value of taxes, $\sum_{j=1}^{\infty}[T_{t+j-1}/(1+r^f)^j]$ (although this net term is now random and does not equal zero). Therefore, as in the model that excluded government purchases, these fiscal changes have no effect on the net wealth of the representative agent, and Ricardian Equivalence holds.

V. Summary Observations

The empirical pattern for several OECD countries back as far as 1870 indicates that the familiar dynamic efficient condition $r>g$ holds when g is the average growth rate of real GDP if r is the average real rate of return on equity, r^e. The condition does not hold if r is the risk-free rate, r^f, proxied by the average real rate of return on Treasury Bills. This pattern accords with a simple disaster-risk model calibrated to fit observed equity premia. The model features stochastic depreciation of capital, with the potential for disaster events in which large portions of capital are destroyed.

As long as Ponzi-type finance for private agents and the government are precluded, the equilibrium can feature a risk-free rate, r^f, below the expected growth rate, $E(g)$, and possibly close to zero. (The model assumes that there do not exist real assets that deliver a positive, safe real rate of return.) The result $r^f \leq E(g)$ does not signal dynamic inefficiency. In contrast, the inequality $E(r^e) > E(g)$ is required for dynamic efficiency, implied by the model, and consistent with the data. The model satisfies Ricardian Equivalence for public debt because, with Ponzi finance by the government excluded, a rise in safe assets from increased government bonds is matched by an increase in the safe (that is, certain) present value of liabilities associated with net taxes.

Alternative versions of the model that could be considered feature productivity shocks, shifts in the relative cost of producing new capital goods, changes in the value (productivity) of old capital, and variations in markup ratios. Another extension is the allowance for heterogeneity across agents; for example, with respect to coefficients of relative risk aversion and shocks to labor income. However, the present results for a representative-agent economy provide a baseline setting for models that incorporate heterogeneity.

Table 1

Long-Term Rates of Return and Growth Rates for 14 OECD Countries

Country	Stock return	Bill return	Govt. bond return	Growth rate of:		
				GDP per capita	C per capita	Population
Australia	0.087	0.012	0.033	0.016	0.015 (1902)	0.018
Canada	0.075	0.013 (1900)	0.034	0.020 (1871)	0.018 (1872)	0.016
Denmark	0.058 (1874)	0.029	0.040	0.018	0.015	0.007
France	0.065	-0.008	0.008	0.018	0.015	0.004
Germany	0.053	-0.012	0.015*	0.021	0.018	0.005
Italy	0.061 (1925)	0.003	0.016	0.019	0.016	0.005
Japan	0.082 (1886)	0.004 (1883)	0.023 (1871)	0.026 (1871)	0.024 (1875)	0.009
Netherlands	0.069 (1900)	0.011	0.028	0.018	0.018	0.011
New Zealand	0.076	0.023 (1923)	0.033	0.014	0.012 (1879)	0.016
Norway	0.061** (1915)	0.016	0.030	0.021	0.018	0.007
Sweden	0.080 (1871)	0.023	0.028	0.022	0.020	0.006
Switzerland	0.068 (1914)	0.010 (1895)	0.020 (1900)	0.014	0.014	0.008
United Kingdom	0.065	0.016	0.026	0.015	0.014	0.005
United States	0.083	0.012	0.028	0.021	0.018	0.014
Means	0.070	0.011	0.026	0.019	0.017	0.009

Notes to Table 1

Stock return refers to broad stock-market indexes and includes dividends. Bill return refers to short-term securities analogous to 3-month Treasury Bills. Bond return refers to government bonds, typically with maturity around ten years. *C* refers to personal consumer expenditure.

Sample periods are 1870-2019 for annual real rates of return, 1870-2017 for annual growth rates of per capita GDP and consumer expenditure, and 1880-2019 for population growth, unless a different starting date for a variable is indicated in parentheses. Rates of return are calculated arithmetically from nominal total returns divided by consumer price indexes. Data on total nominal returns (including dividends paid on stocks) and consumer price indexes are mostly from *Global Financial Data*. See the discussion of an earlier version of these data in Barro and Ursúa (2008, Table 5). The long-term data on macroeconomic variables are updated versions of those described in Barro and Ursúa (2008), available at scholar.harvard.edu/barro.

The samples are based on availability of long-term data for rates of return and macroeconomic variables. Countries with long-term information that could not be used include Belgium and Finland (missing information on consumer price indexes), Portugal (missing data on stock returns and consumer price indexes), and Spain (missing data on stock returns, government bond returns, and consumer price indexes).

For U.K. consols, the average real rate of return for 1870-2015 was 0.035. For U.S. high-grade corporate bonds, the average real rate of return for 1870-2019 was 0.038. These results are based on information in *Global Financial Data*.

*Excludes 1923.
**Based on stock-market index and estimated dividend yield, rather than a total-return index.

Table 2

Baseline Calibration of Model

Parameter	**Value**
Coefficient of relative risk aversion, γ	3.5
Reciprocal of *IES*, θ	0.5
Rate of time preference, ρ (per year)	0.04
Productivity, A (per year)	0.12
Depreciation rate, δ (per year)	0.05
$E(d)$ (mean of disaster size for depreciation)	0.21
$E(1-d)^{-\gamma}$ (based on observed histogram of disaster sizes)	4.0
$E(1-d)^{1-\gamma}$ (based on observed histogram of disaster sizes)	2.3
$E[d((1-d)^{-\gamma}]$ (based on observed histogram of disaster sizes)	1.7
p (disaster probability for depreciation, per year)	0.04
σ^2 (variance of normal shock to depreciation, per year)	0.0004
Implied outcome variables in model	
$E(r^e)$ (expected unlevered rate of return on equity, per year)	0.062
r^f (risk-free real interest rate, per year)	0.001
υ (gross saving ratio, I/K, per year)	0.090
$E(g)$ (expected growth rate, per year)	0.032

Notes:

The disaster probability, p, and the distribution of disaster sizes, d, come from an updated version of the numbers in Barro and Ursúa (2008). Peak-to-trough contractions of real per capita GDP of 10% or more were isolated for 185 cases, corresponding to data for 40 countries going back as far as 1870 and up to 2012. For this sample, the average proportionate disaster size, $E(d)$, is 0.21 and the disaster probability, p (the chance of entering into a disaster state), is 0.040 per year. The objects $E(1-d)^{-\gamma}$, $E(1-d)^{1-\gamma}$, and $E[d((1-d)^{-\gamma}]$ equal 4.0, 2.3, and 1.7, respectively, based on the histogram for observed disaster sizes and for a coefficient of relative risk aversion, γ, of 3.5. This value of γ generates an equity premium of 0.061 per year, close to that observed empirically. The variance of the normal shock, σ^2, is set at 0.0004 per year to accord with the observed annual volatility of real GDP growth. The deterministic part of the depreciation rate, δ, equals 0.05 per year, the average BEA number for the depreciation rate from 1948 to 2018 for fixed assets (including government assets but excluding consumer durables other than residential housing). The value for A, 0.12 per year, generates realistic levels of real rates of return, 0.062 per year for $E(r^e)$ and 0.001 per year for r^f. The values $\rho=0.04$ per year and $\theta=0.5$ determine the gross saving and investment ratio, υ, to be 0.090 per year and the expected growth rate, $E(g)$, to be 0.032 per year (the long-run average growth rate of U.S. real GDP).

References

Abel, Andrew B., N. Gregory Mankiw, Lawrence H. Summers, and Richard J. Zeckhauser (1989). "Assessing Dynamic Efficiency: Theory and Evidence," *Review of Economic Studies* 56 (January), 1-19.

Bansal, Ravi and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset-Pricing Puzzles." *Journal of Finance*, 59 (August), 1481-1509.

Barro, Robert J. (2006). "Rare Disasters and Asset Markets in the Twentieth Century." *Quarterly Journal of Economics*, 121 (August), 823-866.

Barro, Robert J. (2009). "Rare Disasters, Asset Prices, and Welfare Costs." *American Economic Review* 99 (March), 243-264.

Barro, Robert J. and Tao Jin (2011). "On the Size Distribution of Macroeconomic Disasters," *Econometrica* 79 (September), 1567-1589.

Barro, Robert J. and Tao Jin (2020). "Rare Events and Long-Run Risks," forthcoming, *Review of Economic Dynamics*.

Barro, Robert J. and Xavier Sala-i-Martin (2004). *Economic Growth,* 2nd ed., Cambridge MA, MIT Press.

Barro, Robert J. and Jose F. Ursúa. 2008. "Macroeconomic Crises since 1870." *Brookings Papers on Economic Activity*, Spring, 255-350.

Blanchard, Olivier (2019). "Public Debt and Low Interest Rates," *American Economic Review* 109 (April), 1197-1229.

Campbell, John Y. (2018). *Financial Decisions and Markets*, Princeton NJ, Princeton University Press.

Cass, David. 1965. "Optimum Growth in an Aggregative Model of Capital Accumulation." *Review of Economic Studies*, 32, 233-240.

Constantinides, George M. and Darrell Duffie (1996). "Asset Pricing with Heterogeneous Consumers." *Journal of Political Economy* 104 (April), 219-240.

Diamond, Peter (1965). "National Debt in a Neoclassical Growth Model." *American Economic Review* 55 (December), 1126-1150.

Epstein, Larry G. and Stanley E. Zin. (1989). "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica*, 57 (July), 937-969.

Gabaix, Xavier (2012). "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance," *Quarterly Journal of Economics*, 127 (May), 645–700.

Giovannini, Alberto and Philippe Weil. (1989). "Risk Aversion and Intertemporal Substitution in the Capital Asset Pricing Model." National Bureau of Economic Research Working Paper 2824 (January).

Gordon, Myron J. and Eli Shapiro (1956). "Capital Equipment Analysis: The Required Rate of Profit," *Management Science* 3 (October), 102-110.

Homer, Sidney and Richard Sylla. (1996). *A History of Interest Rates*, 3rd ed., New Brunswick NJ, Rutgers University Press.

Koopmans, Tjalling C. 1965. "On the Concept of Optimal Economic Growth," in The Pontifical Academy of Sciences, *The Econometric Approach to Development Planning*, Amsterdam, North Holland.

Mehra, Rajnish and Edward C. Prescott. (1985). "The Equity Premium: A Puzzle." *Journal of Monetary Economics*, 15 (March), 145-161.

Nakamura, Emi, Jon Steinsson, Robert J. Barro, and Jose F. Ursúa (2013). "Crises and Recoveries in an Empirical Model of Consumption Disasters," *American Economic Journal: Macroeconomics*, 5 (July), 35–74.

Phelps, Edmund S. (1961). "The Golden Rule of Accumulation: A Fable for Growthmen," *American Economic Review* 51 (September), 638-643.

Piketty, Thomas (2011). "On the Long-Run Evolution of Inheritance: France 1820-2050." *Quarterly Journal of Economics* 126 (August), 1071-1131.

Rietz, Thomas A. (1988). "The Equity Risk Premium: A Solution." *Journal of Monetary Economics*, 22 (July), 117-131.

Schmelzing, Paul (2020). "Eight Centuries of Global Real Interest Rates, R-G, and the 'Suprasecular' Decline, 1311-2018." Bank of England, staff working paper no. 845, January.

Weil, Philippe. (1990). "Nonexpected Utility in Macroeconomics." *Quarterly Journal of Economics*, 105 (February), 29-42.