1 Introduction & Financial Crises

Basic plan of the course

• Covers selected core topics in international macroeconomics
  – Large number of people in macro end up working in international macro

• Many of the topics involve departures from standard complete markets RBC model
  – The problem doing departures from complete markets is that you don't have the elegance that you get from Arrow-Debreu.
    * You have to make some assumptions. They start becoming strategic. You can't solve things very well.
      · I.e. the overlapping generations model as way of getting at fact that future agents aren’t in market. Barro equivalence, but questions about how people take into account future generations.
      · The complete markets has your marginal conditions holding at every margin. Reaching outside of this is problematic, though not unique to macro.
      · Keynes posited labor markets don’t clear. No one knows to this day the microfoundations of why that is.
      · Recent financial crisis will have impact on how we model asset markets.

Rough Outline of topics

• Lecture 1: Growth Crisis/Financial Crisis
  – Today is quite different: talk about some work of his own; financial crises
  – But also frame issue of incomplete markets
    * There’s some debate about this, but profession is very slow-moving. Certain people get hold of journals or departments, and takes time to move. But will make the strong prediction that within 10 years this (complete markets) will have been wiped out as a philosophical approach. Still, will teach it as a frame of reference, especially in 2nd lecture.

• Lecture 2: Complete markets model revisited
  – With complete markets it is much easier to aggregate up to the global economy
• Lectures 3-5: explore various approaches to understanding one of the most profound economic puzzles of our time:

- The extremely low and negative level of global interest rates (even 500 year old records for low nominal interest rate are being broken)
  * Range of methodological things, but issue to focus on is why are interest rates so low.
  * The market that determines the interest rate is the biggest market in the world.
    - This is a gigantic market: basically supply and demand.
    - Yet governments have a big role.
      On a world-wide weighted average, governments probably control 45-50% of GDP. Governments do maximize something, but not necessarily what we look at in building up these macro models. And half the world is like this. But also they are part of the market. China does not set interest rate, neither does Ben Bernanke. So really is a remarkable puzzle.
  * Rogoff’s general at Yale: asked to explain how real interest rates could be negative. Said that they could be expecting a war, so output would go way down. People have tried looking at this:
    * We’ll look at a few explanations but no real consensus on how real interest rates can get so low
      - Complete Markets & uncertainty: Complete markets explanation: most elegant part around uncertainty. Precautionary saving can drive rate this way. Barro has a few ‘beautiful’ papers on this: rare disasters model.
        There are other candidate explanations in complete markets model, but they are not all that convincing. [Even Barro gets his interest rates down, but not negative, or as low as they’ve been in recent years]
      - Global Imbalances: Another explanation: global imbalances, where China & a few fast-growing emerging markets are generating huge pools of savings ⇒ driving down interest rates. We’ll look at a couple models of this, but not in great detail as it gets messy quickly to calibrate (given incomplete markets)
      - Financial Repression: Government intervention writ large driving it. Financial repression: for example, the interest rate coming out of the US until late 1970s, there was a cap that you could pay on bank accounts. There were periods where there was a cap on interest rates the treasury made. [Amazing how many people do time-series analysis on this without knowing the history]
        China today has a lot of repression. Not our market, but generating savings that comes to our market.
        Less model-driven, so won’t talk about it as much.

One measure of the real interest rate: there’s a market for inflation-indexed bonds.

- Not necessarily the most precise measure (since tax issues and other things).
But closest thing we have to actually observing it: otherwise we need \( r_t = i_t - \mathbb{E}_t [\pi_{t+1}] \).

But we don’t really observe expectations (can do a survey, but have this gigantic market governing what expectations are)

- Don’t have long data sets on these because relatively modern invention. (Summers introduced when at Treasury)

- Lecture 3: Look at efforts to explain the low interest puzzle using complete markets model
  - For example, Barro disaster model

- Lecture 4: Look at global imbalances as an explanation
  - Flood of money from fast growing emerging markets, and “financial repression”

- Lecture 5: Expand analysis to include monetary factors and revisit liquidity trap

- Lectures 6-7: explore the most fundamental imperfection in global financial markets, the difficulty enforcing debt contracts across borders

  Long history of countries defaulting on foreign debt. This is one of the things that is special about international macro (there is trade as well, but also between NY and California). Very hard to do quantitatively.

  - Lecture 6: Basic models of Sovereign Lending
  - Lecture 7: Applications of a particular model

- Lectures 8-11: Exchange rates

  * Major shift in gears. All last lectures will look in one way or another at exchange rates. [Above are all real topics, remaining are monetary topics]

    - Lecture 8: Speculative attacks on exchange rates, including an example of global games (methodology has applications to many problems in macro)
    - Lecture 9: Empirical work on currency unions (not that important, but fun)
    - Lecture 10: Explore how exchange rates and prices are connected (or not) across countries

      * Interesting because if you go back to Keynes problem of why are prices sticky, this tells us a lot.

        - Since prices don’t move that fast in US, Canada, hard to separate real and monetary factors

        - But if you look at international trade, the exchange rate moves a lot. Has a dramatic effect on what you pay for different things. So this is a monetary variable having a major effect. If you have a deep theory about why prices are rigid, should look at exchange rates.

    - Lecture 11: Fiscal Devaluations (Gita Gopinath)

      * How you can create changes in relative prices within a currency unit by using fiscal policy.

- Lecture 12: The Great Depression and Review
Financial Crises

- Contradiction with complete markets model
  
  - One of the most stunning **contradictions** between the complete markets model and the data is the fact that roughly 75% of all traded global financial assets are of plain vanilla non-contingent debt (McKinsey Global Institute)

* We learned in complete markets model that it’s not just stocks and bonds; every conceivable contingency has to be diversified away. You should have a complete market for what job you will get.

* Another counterargument: we may not have complete markets, but we behave as if we have complete markets.

  - **Digression on the “philosophy of RBC models”**
    
    If you study RBC models, “they don’t really test them, in fact they just parameterize them. They say well we can explain this variance and that variance, and the variances themselves are very hokey numbers that go through a lot of filters. I think if you look closely at a lot of this literature—which I regard as pretty close to pure philosophy—and not really really something that it implied (though it does have important philosophical implications), it’s all parameterizations. You take a model, and you’re taking the best guess at the parameters. They are never tested, they are never estimated.”

    - Ex: Nobel prize winners Sargent, Sims and Prescott at Minnesota at same time. Prescott upset because Sims was an econometrician, Prescott is a macroeconomist. Sims was giving Prescott a hard time about this—“this is a joke the way you are testing your model.” Prescott, in a power move in the department, had it made that no econometrics was taught in 1st year of the PhD sequence. Sims left and went to Princeton... There is a question about where is the line between philosophy and practice.

  - One point would like to make that is so philosophical: in financial crises, as in all macro, **most things we care about are highly nonlinear events.** Models we learn, solve, estimate, we either directly linearize them to get closed form solutions (or if you look at the matlab codes, they basically imply linearizations). This poses a few problems: dealing with nonlinearities is harder; there are a lot more assumptions. Another problem is that these nonlinear events are fairly infrequent. Even if you have the beautiful right model, you have to ask yourself the question do I have the data to know about these events that drive what I want to know. When you look at papers in macro testing the model, and you do it yourself and know how it works, a ton of the explanatory power is driven by the fact that the data is highly serially correlated. If you guess that tomorrow is similar to today, that’s a pretty good guess for most macroeconomic variables; except some once and awhile that don’t. Some filters take out the big movements. A lot of models that seem to do well...

  - Summers pointed out to Prescott that RBC doesn’t explain asset prices. Prescott said ‘who cares?’ He then goes to say “isn’t it an amazing puzzle that the asset market doesn’t fit the model?” That’s just totally unbelievably because gosh we know that this model is true”

  - “If you look at virtually any model after the financial crisis, the conventional models performed miserably.” They were wrong about what happened next. It is incredibly well documented (since central banks, private sector, academics, AEA mtgs, everyone that was using these conventional models, was just way wrong by orders of magnitude and in important ways). Basically they were consistently way too optimistic.
• Sorting out implications of 2008 crisis will take a generation or more, but
  
  – Debt cycles are highly nonlinear events. Debt crisis underscores this argument. These events often involve massive and sustained real economic consequences.
  
  – Frequency and nonlinearity of debt crises pose challenges to conventional methods
  
  – One can argue that conventional methods only “work” when there is little volatility in the data.
    
    * High serial correlation and low volatility in the data will make models seem to track decently when they in fact explain little
  
• Another approach: using historical data
  
  – Can’t study a flood that happens every 100 years with 25 years of data
    
    * Our idea is that if these are rare events, you have to put together a historical dataset
  
  – Use of older data has many methodological issues, but footprints of financial crises are very pronounced

• Reinhart Rogoff (2009)
  
  – Most unique thing is the first historical series on public debt (central government)
    
    * For most countries, you couldn’t get anything that was pre-2000
    
    * Most of what you could get was deficit. Even Barro equivalence (about debt) had to be tested with deficit data (accumulated).
      
      – Countries lie about their debt all the time
  
  – Data set on international housing prices
  
  – Data is annual: this includes the crisis dates
  
  – Varieties of crises: banking, currency, debt: domestic, debt: external, inflation, hyperinflation

• Recent clusters of banking crises in advanced economies
  
  – 1893, 1907, 1914, 1931, 2008
    
    * So nothing happened between 1931 and 2008. This is because 1931 was so bad that it took the world a long time to build up enough steam. After the financial crisis, things were so bad. Banks were heavily regulated during WWI and after.

• Paper on Peak to Trough in Severe financial crisis
  
  – AER paper using small part of dataset
  
  – Much different than ordinary recession:
    
    * Duration: rare to be over 8-9 months
    
    * After a typical recession, whatever you lost you get back 6-12 months after you start growing again
      
      – Typical NBER recession definition is “fuzz headed”
    
    – After a financial crises, takes ~5 years to get back to the level of per capita GDP; longer to trend (Great Depression is double that)

• Work had become known from a year earlier since US had markers of severe financial crisis
• GDP is hard to estimate
  – Look at how small firms compare to large firms, do statistics and extrapolate from what big firms do.
  – Trouble is during financial crisis, large firms can tap financial markets, small firms can only tap banking sector (which has died)
    * So small firms fall much more than large firm (thesis argued by Bernanke, others)
• This makes point about non-linearities and trying to parameterize models: challenge to future generations

![Past Experience with Severe Financial Crisis, Peak to Trough Changes](image)

<table>
<thead>
<tr>
<th></th>
<th>Cumulative % change</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Housing prices</td>
<td>-36%</td>
<td>5 years</td>
</tr>
<tr>
<td>Equity prices</td>
<td>-56%</td>
<td>3.4 years</td>
</tr>
<tr>
<td>Unemployment</td>
<td>7%</td>
<td>4.8 years</td>
</tr>
<tr>
<td>Real GDP per capita</td>
<td>-9.3%</td>
<td>1.7 years</td>
</tr>
</tbody>
</table>

• What happens to government debt after crises
  – The average increase is 86% – near doubling of debt
  – This is one of the legacies of financial crises
    * Some from legacy of fiscal stimulus
    * Most of it is because
      1. tax revenues fall
      2. most countries have an implicit debt that is not on the books.
        (a) In the case of the US, these two big mortgage guarantee agencies. Gradually being absorbed for ~$5trillion
        (b) So some of the increase in debt is not because of an immediate increase, but things were compounded in books that weren’t recognized

• Gross national government debt as a percentage of GDP
  – Emerging markets actually deleverage because they had a lot of financial crises 10 years earlier
  – Explosion of debt in advanced countries

• Net government debt - takes out social security
  – Also changed by what central bank does—-but this is a joke. Ben Bernanke has bought 3trillion in government debt, and issued deposit (another debt). This hasn’t done anything to US debt.
- [This is why we look at gross debt. Though for most of history the distinction has been very small]

- Larry Kotlikoff - US actuarial debt is $200 trillion

DIRECT EFFECT: Key Legacy Severe Financial Crises is a Huge Buildup in Government Debt

- External debt: developed countries, major increase
- For reasons that mystify, stock prices respond quickly and go higher

Real per capita GDP (levels) aftermath of systemic banking crises in the US, 1873-2011

Average annual unemployment rate in aftermath of systemic banking crises in the US, 1892-2011

Source: Reinhart and Rogoff, 2012
• Looked at when US had banking crisis
  – Looking at red line (taking out the Great Depression), pretty similar on GDP. Also not too different on employment.

• Sovereign Defaults
  – When you have a wave of banking crises, you almost always get a few years later a wave of sovereign defaults
  – Graph from book that gives weighted average of how much of the world was in default on external debt at any given point in time
    * WWII was giant where almost 40% of the world was in default
  – When you go back to study economic theory, it is as if a lot of this can’t happen
    * If you look at fiscal theories of the price—compute price level based on total obligations
    * The fact is that there is a huge amount of debt that doesn’t get paid
  – A lot of people never realize that their country defaulted
    * US has never defaulted on external debt (partly because we’ve never issued much external debt), but we’ve defaulted on debt issued under domestic law at least a couple times. During great depression, if you were a foreigner and held US debt, you could get paid in gold. FDR said we’ll pay gold, but no longer $20/oz, now $35/oz.

  ![Sovereign Default Cycles 1800-2009](image)

• A lot of what creates crises is that property rights are not well defined. Not clear who should pay how much.

• One of the things in complete contract model, is that you’ll obey contract. One of the things that makes financial markets complicated to model, when stock markets go up and down, there’s not a lot of debate. Debt: important problem is complex emotional view of debt. Ambivalent feeling if someone owes a debt and can’t pay it. A lot of what creates financial crises and debt problems is that property rights are not well-defined at debt crisis.

Other theories that are cause of slowdown

• Demographics:
  – Stock and Watson, Brookings paper: big unexplained shock, which is not too convincing. But one of the variables that they have in is demographics.

• Innovation
Gordon, Peter Theil (Facebook founder), Gary Kasparov (world chess champion): argue that innovation has radically slowed, and that the lead country hasn’t

- No real innovation or growth until 1750
  - First industrial revolution 1750-1830 (steam engine, RR)
  - Second industrial revolution 1870-1900 (electricity, internal combustion, petroleum, running water, ...)
  - Third industrial revolution 1960s-1990s (computers, cell phone)

- Thesis: long period of human history. Don’t extrapolate from a short period what it will be like.

- In growth: Technology and natural resources drive growth
  - What do we know about technology? Solow-exogenous. Romer—education is important.
  - Where is technology going? What about artificial intelligence, biomedicine

- We are going to look at equity premium puzzle, and why interest rates are so low
  - Argument: we’ve gotten much more uncertain about the future. This creates long-term uncertainty. Parameterized right, this would have a big effect.

• Debt Overhangs

  - Should we care about debt? Reinhart and Rogoff argue that having these large overhangs of debt do have effects.
  - Very non-linear. Find that it only has an effect when it is huge.
    - So almost no effect when debt is low (consistent with Barro, Stevenson)
    - But if you carve out this small segment of episodes, it is a striking phenomenon.

• Next time: look at complete markets, then puzzles in international macro, lead into puzzles about why interest rates are so low.

  - Winding up slowly towards harder material. Really need 4x time

2 Consumption Correlation and Low Interest Puzzles, Aggregation, Gains from Risk Sharing

Issue of how you aggregate over agents. We have micro foundations, and this makes us sure we can estimate parameters (based on good micro data). But depends on being able to aggregate from micro to macro.

Lucas, others: we don’t know macro parameters, but we do know micro parameters. If we can get our macro parameters to depend on micro parameters, then we’re onto something.

But problems: i.e. Warren Buffet doesn’t have same composition of consumption as other individuals. Increase in inequality over past 30 years adds further strain to aggregation assumptions.

Cowles Foundation tried to look at aggregation and basically gave up. Lucas, etc showed that you could do it under very strict assumptions, then ran with it. For a lot of RBC, modern macro, this is the crucial assumption (the rest is minor detail). How do you make this claim to build up to a macro model (or a car market).

As you’ll see, there are two basic elements to this trick:
1. Complete Markets
   (a) Equating marginal rates of substitution across everything.
       • This is not enough: we don’t have data on marginal rates of substitution, but on levels

2. If MRS are equal, then you can aggregate to get levels
   (a) Certain utility functions, such as CRRA

“It is ridiculous, but until a better horse comes along, that’s the one we’re going to ride”

There have been some mild departures from this, but I’d underscore the word *mild*.

Outline
“Call things a puzzle that the model doesn’t explain”

1. Basic complete markets model (introducing notation)
2. International consumption correlations puzzle
   (a) Model predicts all countries in the world will have common growth rates.
       i. If we have complete markets, we’ve shared risk—that’s what they do. When there’s a
          shock, our consumption rates go up the same way. Why? Because we can only really
          aggregate for models where that happens.
   (b) “You truly need to be socialized into economics to think this is a puzzle”
   (c) Outputs are more correlated across countries than consumption, which is opposite of what
       you’d think if there was risk-sharing

3. Shiller Securities
   (a) Why certain kinds of securities don’t exist. Shiller proposes things we aught to trade.
       i. [All this plain vanilla debt. Models suggest that we should have securities that are
          indexed to different things]

4. Equity Premium and Low Interest Rate Puzzle [Monday]
5. Cost of Capital market exclusion (first pass)

- Shiller Security
  - Can only trade firm’s profits, but not labor income. But labor income was 58% of GDP
    * Why not base security on GDP? i.e. you owned a share of the US. US would issue a
      security, and if GDP goes up by a certain amount, pay a dividend.
  - Why don’t we have this? “Don’t have a concrete answer.”
    * GDP is hard to pin down; it’s a statistical survey. (Estimates are ±15%)
    * No objective referee (i.e. Argentina issuing inflation-indexed bonds, then claiming its
      π rate to be 8%, when most academics said 30%)
Calculate Shiller securities using GDP

### Table 5.4 Measures of $V^w$, the Securitized Value of a Claim to a Country's Entire Future GDP, 1992 (billions of U.S. dollars)

<table>
<thead>
<tr>
<th>Country</th>
<th>$V^w$</th>
<th>Std($\sigma^w$)</th>
<th>Country</th>
<th>$V^w$</th>
<th>Std($\sigma^w$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>2.460</td>
<td>9.86</td>
<td>Nigeria</td>
<td>2.019</td>
<td>10.06</td>
</tr>
<tr>
<td>Australia</td>
<td>4.340</td>
<td>3.88</td>
<td>Pakistan</td>
<td>2.894</td>
<td>2.45</td>
</tr>
<tr>
<td>Brazil</td>
<td>10.032</td>
<td>8.88</td>
<td>Philippines</td>
<td>1.602</td>
<td>3.68</td>
</tr>
<tr>
<td>Canada</td>
<td>7.663</td>
<td>4.22</td>
<td>South Africa</td>
<td>1.722</td>
<td>8.98</td>
</tr>
<tr>
<td>France</td>
<td>12.901</td>
<td>5.38</td>
<td>Spain</td>
<td>6.721</td>
<td>6.30</td>
</tr>
<tr>
<td>Germany (West)</td>
<td>16.796</td>
<td>4.47</td>
<td>Sweden</td>
<td>1.972</td>
<td>5.70</td>
</tr>
<tr>
<td>India</td>
<td>20.378</td>
<td>4.32</td>
<td>Switzerland</td>
<td>1.911</td>
<td>5.30</td>
</tr>
<tr>
<td>Italy</td>
<td>11.540</td>
<td>4.68</td>
<td>Thailand</td>
<td>4.007</td>
<td>3.99</td>
</tr>
<tr>
<td>Japan</td>
<td>31.762</td>
<td>8.41</td>
<td>Turkey</td>
<td>3.868</td>
<td>3.38</td>
</tr>
<tr>
<td>Kenya</td>
<td>418</td>
<td>4.34</td>
<td>United Kingdom</td>
<td>13.495</td>
<td>1.46</td>
</tr>
<tr>
<td>Mexico</td>
<td>9.583</td>
<td>5.33</td>
<td>United States</td>
<td>82.075</td>
<td>2.03</td>
</tr>
<tr>
<td>Netherlands</td>
<td>3.607</td>
<td>4.68</td>
<td>Venezuela</td>
<td>2.501</td>
<td>6.87</td>
</tr>
</tbody>
</table>

Source: Methodology is based on Shiller (1993, ch. 4). Underlying annual real GDP data are from Penn World Table, section 5.6. Standard deviations are on annual return (income plus appreciation) of a perpetual claim to GDP.

### 2.1 A Simple Two Period, Two State Complete Markets Model

#### 2.1.1 Background:

*To introduce notation*

Basically define Arrow-Debreu securities

- Assume security 1 $[B_2(1)]$ pays 1 unit if state 1 occurs; security 2 $[B_2(2)]$ pays 1 unit if state 2 occurs.
  - $\frac{p(s)}{1+r}$ is the world price of the security for state $s$ [one unit of output in period 2, if state $s$ occurs, denoted in period 1 consumption]
  - Definition of Arrow-Debreu security (for state $s$): Pays 1 unit of output (in this case, on date 2) if state $s$ occurs, and pays nothing in all other states.

- We assume existance of noncontingent securities - riskless bonds that pay $1+r$ in $t=2$, regardless of the state of nature
  - We will see that this market is redundant, as the purchase of $1+r$ state 1 A-D securities and $1+r$ state 2 A-D securities achieves the same thing. This leads us to the:

- **Definition of complete asset markets:** An economy has complete asset markets when people can trade an Arrow-Debreu security corresponding to every state of nature.
  - Of course, A-D securities are not indexed.
  - But later will show that since virtually all assets have state-contingent payoffs, you can sometimes replicate the allocations that arise in complete A-D security trading.

- Population normalized to 1; no population growth. Therefore use representative individual with known $Y_1$ income, and starts with 0 net foreign assets.

#### 2.1.2 Optimization Problem & Budget Constraints

*Optimization Problem*

- Two states, two periods, time separable utility:
\[ U_1 = u(C_1) + \pi(1) \beta u[C_2(1)] + \pi(2) \beta u[C_2(1)] \]

**Budget Constraints**

- Period 1 constraint:
  - where \( p(1) \) is the cost of a bond that pays one unit in period 2, state 1, and \( r \) is the risk-free interest rate:
    \[
    \frac{p(1)}{1 + r} B_2(1) + \frac{p(2)}{1 + r} B_2(2) = Y_1 - C_1
    \]
  - [Value of accumulated assets on date 1 = income - consumption]
  - We need not explicitly consider purchases of bonds, because (as discussed above) riskless bonds are redundant given the 2 A-D securities

- Period 2 budget constraints:
  \[ C_2(s) = Y_2(s) + B_2(s) \quad s = 1, 2 \]

- Present value budget constraint:
  \[
  C_1 + \frac{p(1)}{1 + r} C_2(1) + \frac{p(2)}{1 + r} C_2(2) = Y_1 + \frac{p(1)}{1 + r} Y_2(1) + \frac{p(2)}{1 + r} Y_2(2)
  \]
  - Intuition: Date 1 present value of the country’s uncertain consumption must = date 1 present value of country’s uncertain income [where contingent commodities are evaluated at A-D prices]
    * International markets allow countries to smooth consumption not only across time, but across states of nature

[Algebra:]

Substituting \( B_2(s) \) from Period 2 constraints into Period 1:
\[
B_2(s) = C_2(s) - Y_2(s)
\]
\[
\frac{p(1)}{1 + r} (C_2(1) - Y_2(1)) + \frac{p(2)}{1 + r} (C_2(2) - Y_2(2)) = Y_1 - C_1
\]

Now just distribute to get consumption on RHS:
\[
C_1 + \frac{p(1)}{1 + r} (C_2(1)) + \frac{p(2)}{1 + r} (C_2(2)) = Y_1 + \frac{p(1)}{1 + r} (Y_2(1)) + \frac{p(2)}{1 + r} (Y_2(2)) \quad \text{(which is same as above)}
\]

- Unconstrained Maximization Problem

\[
\max_{\{B_2(1), B_2(2)\}} u(C_1) + \pi(1) \beta u[C_2(1)] + \pi(2) \beta u[C_2(1)]
\]

s.t.
\[
\frac{p(1)}{1 + r} B_2(1) + \frac{p(2)}{1 + r} B_2(2) = Y_1 - C_1
\]
\[
C_2(s) = Y_2(s) + B_2(s) \quad s = 1, 2
\]

- Unconstrained Maximization Problem, **re-written**
Substitute budget constraints for each period:

\[ U_1 = u \left( Y_1 - \frac{p(1)}{1+r} B_2(1) - \frac{p(2)}{1+r} B_2(2) \right) + \pi(1) \beta u [Y_2(1) + B_2(1)] + \pi(2) \beta u [Y_2(2) + B_2(2)] \]

Or:

\[ U_1 = u \left( Y_1 - \frac{p(1)}{1+r} B_2(1) - \frac{p(2)}{1+r} B_2(2) \right) + \sum_{s=1}^{2} \pi(s) \beta u [Y_2(s) + B_2(s)] \]

2.1.3 Optimal Behavior = Euler Equation

- **First Order Euler Equations:**

  - *First Order Conditions are the Euler Equations* (for each \( B_2(s) \)).
    
    * [Don’t need to do FOC, Envelope like in Bellman]

    \[ \frac{p(s)}{1+r} u'(C_1) = \pi(s) \beta u'(C_2(s)) \quad s = 1, 2 \]

  - Intuition: LHS is the cost, in terms of date 1 marginal utility, of acquiring the A-D security for state \( s \). The RHS is the *expected, discounted benefit* of having an additional unit of consumption in state \( s \), date 2.

  - Usefulness: We can use this intertemporal Euler condition for more complex securities that pay off in more than one state of nature. [See riskless bond below]

    * Derivation:
      
      FOCs from re-written optimization problem. Choice variables are \( B_2(1) \) and \( B_2(2) \):

      - \( \text{FOC}_{B_2(1)}: u'(C_1) \left( -\frac{p(1)}{1+r} \right) + \pi(1) \beta u'(C_2(1)) = 0 \)

        Rearranging: \( \frac{p(1)}{1+r} u'(C_1) = \pi(1) \beta u'(C_2(1)) \) as above.

2.1.4 Rearranging Euler to Risk-Free Bond Euler

- Deriving Euler Equation for Riskless Bonds (from Euler Equation above)

  - Use the fact that:

    \[ \sum_{s=1}^{S} p(s) = 1 \]

    * This follows simply from the fact that you can make a “riskless” portfolio in state 1 through buying: \( (1+r) B_1 \) and \( (1+r) B_2 \). This pays of 1 no matter what the state. The price will be is seen from \( (1+r) \frac{p(1)}{(1+r)} + (1+r) \frac{p(2)}{(1+r)} = 1 \). buying the “riskless” portfolio of 1 bond paying \( (1+r) \) in for each state 2. This must have the same price (=1) of a bond paying \( 1+r \) output units next period [that is, it must cost 1 output unit]. Hence \( \sum_{s=1}^{S} p(s) = 1 \).

    * Ex, for \( s = 2 \): \( (1+r) \frac{p(1)}{(1+r)} + (1+r) \frac{p(2)}{(1+r)} = 1 \)

- **Stochastic Euler Equation for Riskless Bonds**

  - Adding Euler equation for each state \( s \) [Algebra below]
\[ u'(C_1) = (1 + r) \beta E_1 \{ u'(C_2) \} \]

- *We see that we can get a riskless bond* (that pays a unit in all states of nature—“plain vanilla debt”) *by buying a unit of each bond for all s states. A “synthetic, riskless bond.”*
  - *Historical note:* banning riskless bonds didn’t work, as people borrowed in different currencies, etc. Could make a synthetic riskless bond.
- Then we get standard Euler condition that we’re used to seeing.
- Rearranging:
  \[
  \frac{\beta E_1 \{ u'(C_2) \}}{u'(C_1)} = \frac{1}{1 + r}
  \]
  - *The expected marginal rate of substitution of present for future consumption = \( \frac{1}{(1 + r)} \)*

[Algebra: straight forward]

1. \( \frac{\pi(s) u'(C_2)}{u'(C_1)} = (1 + r) \pi(s) \beta u'(C_2) \) (From Euler, rearrange)
2. \( p(1) u'(C_1) + p(2) u'(C_1) = (1 + r) \pi(1) \beta u'(C_2(1)) + (1 + r) \pi(2) \beta u'(C_2(2)) \) (adding over \( s \))
3. \( [p(1) + p(2)] u'(C_1) = (1 + r) \beta \{ \pi(1) u'(C_2(1)) + \pi(2) u'(C_2(2)) \} = (1 + r) \beta E_1 \{ u'(C_2) \} \) (factoring, rewriting with expectation)
4. \( u'(C_1) = (1 + r) \beta E_1 \{ u'(C_2) \} \) (using \( \sum_{s=1}^{S} p(s) = 1 \))

2.1.5 Rearranging Euler to Show Relative Prices

- **Euler: MRS between \( C_1 \) and \( C_2(s) \) equals price**
  - Just divide original FOC/Euler by \( u'(C_1) \)
    \[
    \frac{\pi(s) \beta u'(C_2)}{u'(C_1)} = \frac{p(s)}{(1 + r)} \quad s = 1, 2
    \]
  - *Date 1 Arrow-Debreu prices are the marginal rates of substitution of present consumption for future consumption in various states of nature*

- **Euler: MRS between state \( s \) and state \( s' \) (in time 2) equals price ratio:**
  \[
  \frac{\pi(s) \beta u'(C_2)}{\pi(s') \beta u'(C_2)} = \frac{p(s)}{p(s')}
  \]
  - *The marginal rate of substitution between state \( s \) and state \( s' \) in time 2 is equal to the relative price of state \( s \) consumption to the price of state \( s' \) consumption*

- **Actuarial Fairness**
  - Observe that only when
    \[
    \frac{p(1)}{p(2)} = \frac{\pi(1)}{\pi(2)}
    \]
    * does the above MRS of \( s \) to \( s' \) consumption Euler imply that \( C_2(1) = C_2(2) \).
* The A-D securities are *actuarially fair* when this happens

- **Interpretation**
  1. *If prices are actuarially fair*, a country trading in complete asset markets will fully insure against all future consumption fluctuations
  2. *If prices are not actuarially fair*, the country will choose to "tilt" its consumption across states
    * Given two equiprobable states, the country will plan for relatively lower consumption in the state for which insurance is more expensive
      [Can see this by looking at \( \pi(s)\beta u'[C_2(s)] \) \( \frac{p(s)}{\pi(s')\beta u'[C_2(s')]} = \frac{p(s)}{p(s')} \), and noticing that if \( p(s') \uparrow \) while \( p(s) \) remains unchanged, we must have \( \uparrow \) in \( u'(C_2(s')) \) which implies \( C_2(s') \downarrow \)]

### 2.2 Two Country, Two Period Model, Many States

Start moving towards aggregation theorem. From 2 states to \( s \) states, everything is pretty similar.

#### 2.2.1 Defining Variables

Background: world economy consists of two countries, Home and Foreign (*), with output that fluctuates across \( S \) states of nature.

Foreign and Home’s necessary Euler equations are the same as the above analysis [though allowing for \( S > 2 \)].

Home and Foreign have same degree of risk aversion.

- **Market Clearing:**
  - Global equilibrium requires supply = demand in period 1
    \[
    C_1 + C_1^* = Y_1 + Y_1^* 
    \]
  - And in all possible period 2 states
    \[
    C_2(s) + C_2^*(s) = Y_2(s) + Y_2^*(s) \quad s = 1, 2, \ldots, S
    \]
  - Define: \( Y^W = Y + Y^* \)

- Same Euler condition from 2-state model
  \[
  u'(C_1) = \frac{1 + r}{p(s)} \pi(s) \beta u'[C_2(s)]
  \]

#### 2.2.2 CRRA Utility to aid Aggregation

- **CRRA Utility:**
  \[
  U(C) = \frac{C_1^{1-\rho}}{1 - \rho}
  \]

- **FOCs:**
  \[
  C_1^{-\rho} = \frac{1 + r}{p(s)} \pi(s) \beta C_2^{-\rho}(s)
  \]
Rearranging:
\[ C_2^0 \left( s \right) C_1^{-\rho} = \frac{1 + r}{p(s)} \pi \left( s \right) \beta \Rightarrow C_2^0 \left( s \right) = \frac{1 + r}{p(s)} \pi \left( s \right) \beta C_1^0 \Rightarrow \]
\[ C_2 \left( s \right) = \left[ \frac{1 + r}{p(s)} \pi \left( s \right) \beta \right]^{\frac{1}{\rho}} C_1 \]

- So I get a **linear equation** between \( C_2 \) and \( C_1 \). This is the whole reason we're able to aggregate.

* Went from equation that was a general non-linear mess, but for certain utility functions (CRRA is most common example), I can reduce to a linear equation. And if I make some further assumptions (everyone in the world has the same utility function), and then assumption (perfectly integrated capital markets, and all facing the same prices)—then I'll just have lots of these and can just add them up.

* Not talking about marginal conditions anymore, but things that can actually be observed.

### 2.2.3 Equilibrium Prices

- Add home and foreign Euler equations with “market clearing” to get:

\[ Y_2^W \left( s \right) = \left[ \frac{1 + r}{p(s)} \pi \left( s \right) \beta \right]^{\frac{1}{\rho}} Y_1^W \]

- Summed across all agents. Can substitute consumption \( C \) for output \( Y \) from ‘market clearing’ (since no investment or government consumption).

[Just by \( C_t \left( s \right) + C_t^* \left( s \right) = Y_t \left( s \right) + Y_t^* \left( s \right) = Y_t^W \left( s \right) \ s = 1, 2, \ldots, S \)]

- **Equilibrium Security Price**

  - (at date 1 for security \( s \)): Just rearrange summed Euler:

\[ \frac{p(s)}{1 + r} = \pi \left( s \right) \beta \left[ \frac{Y_2^W \left( s \right)}{Y_1^W} \right]^{-\rho} \ s = 1, 2, \ldots, S \]

  - This gives us the date 1 price of the state \( s \) contingent security

[Algebra]: \[ \frac{Y_2^W \left( s \right)}{Y_1^W} = \left[ \frac{1 + r}{p(s)} \pi \left( s \right) \beta \right]^{\frac{1}{\rho}} \Rightarrow \left[ \frac{Y_2^W \left( s \right)}{Y_1^W} \right]^{\rho} = \frac{1 + r}{p(s)} \pi \left( s \right) \beta \Rightarrow \frac{p(s)}{1 + r} = \pi \left( s \right) \beta \left[ \frac{Y_2^W \left( s \right)}{Y_1^W} \right]^{-\rho} \]

  - Note the key role of the assumption that Home and Foreign have the same risk aversion coefficient \( \rho \)

- **Conditions for Actuarially Fair Prices**

  - Relative prices: divide Security Price equations for two states

\[ \frac{p(s)}{p(s')} = \pi \left( s \right) \left[ \frac{Y_2^W \left( s \right)}{Y_2^W \left( s' \right)} \right]^{-\rho} \]
1. Actuarial Fairness: For $\rho > 0$, contingent claims’ prices will be actuarially fair \[ \text{that is } p(s) = \frac{\pi(s)}{\pi'(s)} \]
\[ \iff \text{world output is the same in all states of nature.} \]
\[ (a) \text{ Hence the requirement for actuarial fairness is the absence of uncertainty at the aggregate level.} \]

2. Non-Actuarial Fairness: If world output in $s' >$ output in $s$, then we can see from the above equation $p(s)$ must rise. [Economically, people are consuming more in $s'$, so price goes down]. State $s$ consumption is now at a premium, while $s'$ sells at a discount.

[2.2.4 Interest Rate]

- Interest rate: Solve from Security Price equation:
\[ 1 + r = \frac{(Y_1^W)^{-\rho}}{\beta \sum_{s=1}^S \pi(s) [Y_2^W(s)]^{-\rho}} \]

- Note that the riskless bond is redundant with complete A-D securities

- Intuition of equation:
* higher output at date 1 lowers real interest rate (raises price of date 2 consumption relative to date 1 consumption)
* higher expected output at date 2 [higher future output in any state] raises real interest rate.

- Note this relies on fact that:
* $\sum_{s=1}^S p(s) = 1$. If I buy an Arrow-Debreu asset in every state, it has to sum to 1.

[Algebra]

1. First use $\sum_{s=1}^S p(s) = 1$ to plug into Relative Prices equation \[ \frac{p(s)}{p(s')} = \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho} \Rightarrow p(s) = \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^\rho \]
to solve for $p(s') = 1 - \sum_{s \neq s'} p(s) = 1 - \sum_{s \neq s'} \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^\rho p(s')$.

2. Solve for $p(s')$:
\[ (1 + \sum_{s \neq s'} \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho}) p(s') = 1. \]
\[ \text{More clearly: } (1 + \sum_{s \neq s'} \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho}) p(s') = 1 = \left( \pi(s') \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho} + \sum_{s \neq s'} \pi(s) \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho} \right) p(s') = 1 = \left( \pi(s') \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho} \right) p(s') \]
\[ \left( \sum_{s=1}^S \frac{\pi(s)}{\pi(s')} \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho} \right) p(s') = 1 \] (collecting terms into $\sum$)
\[ p(s') = \frac{\pi(s') \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho}}{\sum_{s=1}^S \pi(s) \left[ \frac{Y_2^W(s)}{Y_2^W(s')} \right]^{-\rho}} \]

3. Plug result for $p(s')$ [as $p(s)$] in Security Price equation \[ \frac{p(s)}{1 + r} = \pi(s) \beta \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^{-\rho} \]
\[ 1 + r = p(s) \frac{1}{\pi(s)} \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^\rho = \frac{\pi(s) \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^{-\rho}}{\sum_{s=1}^S \pi(s) \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^{-\rho}} \cdot \frac{1}{\beta \sum_{s=1}^S \pi(s) \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^{-\rho}} \]
\[ 1 + r = \frac{\left[ \frac{Y_1^W}{Y_2^W(s)} \right]^\rho}{\beta \sum_{s=1}^S \pi(s) \left[ \frac{Y_2^W(s)}{Y_1^W} \right]^{-\rho}} \] (simplifying to get the desired expression)
2.3 Equilibrium Consumption Levels

The complete markets model has strong implications concerning correlations in international consumption levels (using CRRA) across time and across states of nature.

- These strong predictions arise because complete markets allow all individuals in Home and Foreign to equate their marginal rates of substitution between current consumption and state-contingent future consumption to the prices (state-contingent security prices).

We’ll then look at some empirical implications.

2.3.1 Equating MRS in consumption across states [between Home and Foreign]

- With many states, the Euler conditions can be rearranged to: (just as before)

\[
\frac{\pi(s) \beta u'[C_2(s)]}{u'(C_1)} = \frac{p(s)}{(1 + r)} = \frac{\pi(s) \beta u'[C_2^*(s)]}{u'(C_1^*)}
\]

- Dividing though by another state \( s' \)

\[
\frac{\pi(s) u'[C_2(s)]}{\pi(s') u'[C_2(s')]} = \frac{p(s)}{p(s')} = \frac{\pi(s) u'[C_2^*(s)]}{\pi(s') u'[C_2^*(s')]} \]

- You will recognize these as fundamental necessary conditions for efficient resource allocation: all individuals’ marginal rates of substitution in consumption—over time and across states—are equal, so no potential gains from trade remain to be exploited.

- These equations relate only marginal utilities of consumption: below we make functional form assumptions which have implications for consumption levels (CRRA):

2.3.2 Consumption Levels as a Constant Fraction of World Output; Equal Growth Rates (CRRA)

- Using CRRA Utility \( U(C) = \frac{C^{1-\rho}}{1-\rho} \)

\[
\frac{C_2(s)}{C_2(s')} = \frac{C_2^*(s)}{C_2^*(s')} = \frac{Y_2^W(s)}{Y_2^W(s')}
\]

- Why does this fall out? Because we could get down to this linear equation \( Y_2^W(s) = \left[ \frac{1+r}{\pi(s)} \right]^{\frac{1}{\rho}} Y_1^W \),

  this 'mess' \( \left[ \frac{1+r}{\pi(s)} \right]^{\frac{1}{\rho}} \) is the same for everybody.

  * It says that in equilibrium, the ratio of \( C_2(s) \) to \( C_1 \) is the same for everybody. So whatever happens in 2nd period, your consumption will grow.

  * This one thing does not require people having the same wealth.

- which gives us: [cross multiplying]

\[
\frac{C_2(s)}{Y_2^W(s)} = \frac{C_2(s')}{Y_2^W(s')} = \mu \quad \text{and} \quad \frac{C_2^*(s)}{Y_2^W(s)} = \frac{C_2^*(s')}{Y_2^W(s')} = 1 - \mu
\]

  * which implies that home consumption is a constant fraction \( \mu \) of world output

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- and

\[
\frac{C_2(s)}{C_1} = \frac{C^*_2(s)}{C^*_1} = \frac{Y^W_2(s)}{Y^W_1}
\]

which says that **growth rates are the same across countries** in every state and equal to the growth rate of world output.

[Algebra:] To get \(\frac{C_2(s)}{C^*_1} = \frac{C^*_2(s)}{C^*_1} = \frac{Y^W_2(s)}{Y^W_1}\)

Rearrange Security Price equation: 

\[
p(s) = \pi(s') \beta \left[ \frac{\pi^W_2(s')}{\pi^W_1(s')} \right]^{-\rho} \Rightarrow Y^W_1 = \left[ \frac{\pi^W_2(s')}{\pi^W_1(s')} \right]^{-\frac{1}{\rho}} Y^W_2(s')
\]

\[
\Rightarrow \left[ \frac{1 + r}{\pi^W_2(s') \pi(s') \beta} \right]^{\frac{1}{2}} Y^W_1 = Y^W_2(s') = \left[ \frac{1 + r}{\pi^W_1(s') \pi(s') \beta} \right]^{\frac{1}{2}} C^W_1 = \left[ \frac{1 + r}{\pi^W_1(s') \pi(s') \beta} \right]^{\frac{1}{2}} (C_1 + C^*_1)
\]

- this gives us:

\[
\frac{C_2(s)}{Y^W_2(s)} = \mu = C_1 / Y^W_1 \quad \text{and} \quad \frac{C^*_2(s)}{Y^W_2(s)} = 1 - \mu = C^*_1 / Y^W_1
\]

[from cross multiplying the above equation]

* **Countries’ date 1 consumption shares in world output are the same as their date 2 shares**

2.3.3 Different relative risk aversion

- For countries with different relative risk aversion, and different subjective discount factors, we can still get the generalization:

\[
\ln \left[ \frac{c^n_2(s)}{c^n_1} \right] = \left( \frac{\rho_m}{\rho_n} \right) \ln \left[ \frac{c^n_2(s)}{c^n_1} \right] + \frac{1}{\rho_n} \ln \left( \frac{\beta_n}{\beta_m} \right)
\]

- This equation shows that any two countries’ ex post consumption growth rates, although individually random, are perfectly statistically correlated (they have a correlation coefficient of 1)

- Which unfortunately fails miserably:

- Derivation: (asked on generals last year)

  - FOC/Euler relationship is same as above 
    \[
    \left[ \frac{\pi(s) \beta'(C_2(s))}{u'(c^m_1)} \right] = \frac{p(s) \rho_n}{(1 + r)} = \frac{\pi(s) \beta'(C^*_2(s))}{u'(c^m_1)}
    \]
    (just FOC, rearrange)

  \[
  \frac{\pi(s) \beta_n (c^n_2(s))^{-\rho_n}}{(c^n_1)^{-\rho_n}} = \frac{p(s)}{(1 + r)} = \frac{\pi(s) \beta_m (c^m_2(s))^{-\rho_m}}{(c^m_1)^{-\rho_m}}
  \]

  - Solve for 
    \[
    \frac{\beta_n (c^n_2(s))^{-\rho_n}}{(c^n_1)^{-\rho_n}} = \frac{\beta_m (c^m_2(s))^{-\rho_m}}{(c^m_1)^{-\rho_m}} \Rightarrow \left( \frac{c^n_2(s)}{c^n_1} \right)^{-\rho_n} = \frac{\beta_m (c^m_2(s))^{-\rho_m}}{(c^m_1)^{-\rho_m}} \Rightarrow (-\rho_n) \ln \left( \frac{c^n_2(s)}{c^n_1} \right) = (-\rho_m) \ln \left( \frac{c^m_2(s)}{c^m_1} \right)
    \]

  \[
  \ln \left( \frac{c^n_2(s)}{c^n_1} \right) = \left( \frac{\rho_m}{\rho_n} \right) \ln \left( \frac{c^m_2(s)}{c^m_1} \right) + \left( -\frac{1}{\rho_n} \right) \ln \left( \frac{\beta_n}{\beta_m} \right) \Rightarrow \ln \left( \frac{c^n_2(s)}{c^n_1} \right) = \left( \frac{\rho_m}{\rho_n} \right) \ln \left( \frac{c^m_2(s)}{c^m_1} \right) + \left( \frac{1}{\rho_n} \right) \ln \left( \frac{\beta_m}{\beta_m} \right)
  \]

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2.3.4 Testing Empirical Predictions

- One of the most powerful predictions of the one-good complete markets model is that different countries' per capita consumption growth rates should be highly correlated even if growth rates in per capita output are not.
  
  - This is true with identical CRRA preferences, as well as the case with different coefficients of risk aversion $\rho$ as we just saw:
    
    $$
    \ln \left( \frac{c_n^2(s)}{c_i^2} \right) = \left( \frac{\rho_n}{\rho_i} \right) \ln \left( \frac{c_m^2(s)}{c_i^2} \right) + \frac{1}{\rho_i} \ln \left( \beta_n / \beta_m \right)
    $$

- Does this hold empirically? No
  
  - “Paradox”: in general, the correlation between country and world consumption growth is lower than the correlation between country and world output growth [see chart below]

- [From Lecture:]
  
  - Implementation [Equalized growth rates]
    
    - If you try to implement this practically, there are a lot of issues: there are a lot of elements of consumption that are not traded, so when you generalize to get this result, you need separable utility, so that the marginal utilities of the things you can trade is separable. [You can depart from this a bit, but need some strong assumptions]
      
      - Rob Townsend – this doesn’t hold in aggregate, but maybe in villages where you can create these state-contingent contracts, could be true.
    
    - If it was true, it is something simple you can observe. You can’t observe Arrow-Debreu contracts. So it merited some attention.
  
  - Generalizations: different coefficients of relative risk aversion, different wealth: result is that they grow slightly differently (not necessarily identical)
    
    - This has been tested 1000 ways
      
    - First Column: correlation of consumption rates with world consumption
      
    - Second Column: correlation of income with world income
      
      - Paradox here. Though if you didn’t have complete markets, this is not very hard to get.
2.4 Rationale for the Representative-Agent Assumption

KR: “Have to socialize yourself into accepting it.”

2.4.1 Motivation

- We’ve often maintained the convenient fiction that each country is inhabited by a single representative agent. Having now introduced an explicit stochastic general equilibrium model, we are able to present a deeper rationale for this approach.

- This subsection shows that if markets are complete and agents face the same prices, then (for a broad class of utility functions) prices and aggregate per capita consumption behave as if there were a single representative agent [despite substantive differences across individuals]

- In this example: complete markets, \( I \) agents with possibly different wealth levels who have identical discount factors and (separable) period utility that is CRRA
  
  - Basically just show 2 examples. First is a slight generalization of the CRRA, by adding constant \( a_0 \). Will do the complete markets, take the FOCs.
  
  - **Key thing is that FOC will give linear equation.**

- Emphasize however that aggregation is usually impossible absent a complete-markets allocation

2.4.2 Aggregate Euler Equation: Agents with Identical CRRA Utility
• First assume agents have identical generalized CRRA utility but different wealth levels:

$$u(c^i) = \frac{(a_0 + a_1 c^i)^{1-\rho}}{1-\rho}$$

• FOC/Euler equations:

- Making use of our earlier bond Euler equations, we get:

$$\left(a_0 + a_1 c^i_2\right)^{-\rho} = \frac{1 + r}{p(s)} \pi(s) \beta \left(a_0 + a_1 c^i_2(s)\right)^{-\rho} \quad i = 1, 2, \ldots, I$$

- or

$$a_0 + a_1 c^i_2 = \left[\frac{1 + r}{p(s)} \pi(s) \beta \right]^{-\frac{1}{\rho}} \left(a_0 + a_1 c^i_2(s)\right) \quad i = 1, 2, \ldots, I$$

  * Linear version

  - All the stuff you can’t see is summarized in the term: \(\left[\frac{1 + r}{p(s)} \pi(s) \beta \right]^{-\frac{1}{\rho}}\)

  - All the things that are in principal observable: \(a_0 + a_1 c^i_2(s)\)

• Aggregate

- Sum over all agents, divide by \(I\) (number of agents) and raise result to the power \(\frac{1}{\rho}\):

$$\left(a_0 + a_1 c_2\right)^{-\rho} = \left[\frac{1 + r}{p(s)} \pi(s) \beta \right]^{-\frac{1}{\rho}} \left(a_0 + a_1 c_2(s)\right)^{-\rho}$$

  * Get an aggregate Euler equation.

- Didn’t need agents to have the same wealth, and economy acts as if it has a representative agent.

  This implies that the economy’s per capita consumption behaves as if chosen by a representative individual who owns the economy’s endowment.

  * However, aggregation is usually impossible absent a complete-markets allocation

- Can we relax this assumption? What if agents differ in the \(\rho_i\)?

2.4.3 Agents with Different CRRA (different \(\rho\)): Harmonic Mean

• Same utility function, different \(\rho_i\)

  - Turns out that you can’t add equations, but can multiply (to get harmonic mean)

• When agents do not have identical utility functions, we can still sometimes aggregate using harmonic mean

  - We will define the representative consumer to have wealth and consumption equal, respectively, to the geometric averages of all individuals’ wealth and consumption levels

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- The (unweighted) geometric average of individual consumptions is:
\[
\hat{c} = \prod_{i=1}^{I} (c_i)^{\frac{1}{I}}
\]

[The geometric mean is defined as the \(n\)-th root (where \(n\) is the count of numbers) of the product of the numbers]

- The Euler conditions for agent \(i\) are given by

- Identical to the above for the CRRA agent, just indexed by \(i\):

\[
c_i^2(s) = \left[ \frac{1 + r}{p(s)} \pi(s) \beta \right] \frac{1}{\rho_i} c_i^1
\]

- Taking both sides to the power \(\frac{1}{I}\) and multiplying over all agents gives us:

\[
\prod_{i=1}^{I} \left[ c_i^2(s) \right]^{\frac{1}{I}} = \prod_{i=1}^{I} \left[ \frac{1 + r}{p(s)} \pi(s) \beta \right] \frac{1}{\rho_i} (c_i^1)^{\frac{1}{I}} \Rightarrow
\]

\[
\hat{c}_2(s) = \prod_{i=1}^{I} \left[ \frac{1 + r}{p(s)} \pi(s) \beta \right] \frac{1}{\rho_i} \hat{c}_1
\]

* or

\[
\frac{p(s)}{1 + r} = \pi(s) \beta \left[ \frac{\hat{c}_2(s)}{\hat{c}_1} \right]^{-\hat{\rho}} \tag{*}
\]

where \(\hat{\rho} = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{\rho_i}\)

- Therefore, we see that the representative’s period utility function \(u(\hat{c})\) therefore is of the CRRA class with risk-aversion coefficient equal to the harmonic mean of the possibly distinct individual coefficients

* Before we were adding across agents (and multiplying by \(\frac{1}{I}\)) to get average. Now we are multiplying across (and taking to the \(\frac{1}{I}\) power) to get harmonic mean.

- Geometric and arithmetic mean are close if things you are averaging up are close.

2.4.4 Intuition, Shortcomings and Extensions

- Intuition

  - Don’t need people to be quite identical: can have different risk preferences
  - But before was taking these complicated expressions into something that I could observe (if measured right)
    * However, don’t observe geometric mean consumption
    * Very wealthy would have gigantic impact

- Shortcomings

  - There are some important situations in which perfect aggregation is infeasible
    * For example, an overlapping generations framework (people who haven’t been born yet can’t participate in markets for future output).
Lucas 1977 Econometrica:

- Guess what: you can get a world of complete markets if you just have traded stocks. Don’t need a complete market necessarily.
- Result

Two important things to mention about complete markets [beginning of Lecture 3]
- Perfect markets cannot adjust to preference shocks
- Assume frictionless trading of insurance assets

3 Consumption-Based Capital Asset Pricing Model (CCAPM), Rare Disasters

3.1 Essentials of Asset Pricing
- Motivation
  - When the set of assets traded is sufficient to produce a Pareto optimal equilibrium allocation, the price of any asset can be calculated as a linear combination of the underlying Arrow-Debreu prices.
    - The calculation is valid even if A-D securities aren’t traded, so long as it behaves as if they did exist
    - We will see: the less useful an asset is as a consumption hedge, the higher the rate of return that asset must offer investors in equilibrium.
- Pricing Country $m$ Mutual Fund ($V^m_1$)
  - Set Up: Extending the two period multi-state, multi-country model.
    - [Since date 1 A-D prices are the MRS between present consumption for future consumption in various states of nature, they can be used to value the totality of state-contingent payoffs that define any particular asset.]
  - *Market price of a mutual fund from country $m$ that pays off $Y^m_2 (s)$ in date 2 state of nature $s$ is
    \[
    V^m_1 = \sum_{s=1}^{S} \frac{p(s)}{1 + r} Y^m_2 (s)
    \]
    - This country $m$ mutual fund can be perfectly replicated by a portfolio containing $Y^m_2 (s)$ of date $s$ A-D securities. Hence the price must be the same (otherwise there would be an arbitrage opportunity). “This is sort of an aggregate of A-D security”
- Re-writing $V^m_1$: 

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Since countries face the same A-D prices, we may substitute any country $n$’s MRS, given by the Euler equation from above $\left[ \frac{\pi(s)\beta u'[C_2(s)]}{u'(C_1)} = \frac{\mu(s)}{1+r} \right]$;

$$V_1^m = \sum_{s=1}^{S} \left\{ \frac{\pi(s)\beta u'[C_2(s)]}{u'(C_1)} \right\} Y_2^m(s)$$

* where consumption can refer to any individual $i$. [Because of complete markets, it does not matter which individual we use to evaluate risky asset $V$.]
* Note that with CRRA, we can replace individual consumption with world consumption.

- We can further re-express, taking expectation over the possible states $s$ in period 2:

$$V_1^m = E_1 \left\{ \frac{\beta u'[C_2(s)]}{u'(C_1)} Y_2^m \right\}$$

### 3.2 Asset Returns

We now derive a formula that explains asset prices in terms of expected payoffs and the riskiness of those payoffs.

- The above formula can be written as:

$$V^m = E_1 \left\{ \frac{\beta u'(C_2)}{u'(C_1)} \right\} E_1 Y_2^m + Cov \left\{ \frac{\beta u'(C_2)}{u'(C_1)}, Y_2^m \right\}$$

- Substituting with the Euler equation for a riskless asset (above) $\left[ E_1 \left\{ \frac{\beta u'(C_2)}{u'(C_1)} \right\} = \frac{1}{1+r} \right]$ we get:

$$V^m = \frac{E_1 Y_2^m}{1+r} + Cov \left\{ \frac{\beta u'(C_2)}{u'(C_1)}, Y_2^m \right\}$$

- Intuition:

  - The date 1 price of country $m$’s uncertain date 2 output is the sum of two terms
    1. The assets expected payout, discounted at the riskless rate of interest
    2. Covariance between the relative marginal utility of date 2 consumption and the payout.

    * Other things being equal, an asset that tends to pay off unexpectedly well on date 2 when MU of consumption is unexpectedly high (i.e. C is low) has a value as a consumption hedge, and therefore command a value above its “risk-neutral” (or actuarially fair) price

### 3.3 CAPM

- Defining share return $r^m$

  - We now define the ex-post (net real rate of) return on the above risky asset (that is, a share in country $m$’s output, $r^m$) as:
\[ r^m = \frac{Y^m_2 - Y^m}{V^m_1} \]

- [For risk-free bond this would be akin to \( r = \frac{(1+r)-1}{1} \)]

**CAPM Equation**

- We can arrive at the standard equation for the consumption based capital asset pricing model “CCAPM”

\[ \mathbb{E}_1 (r^m) - r = -(1 + r) \text{Cov}_1 \left\{ \beta u'(C_2) \frac{u'(C_1)}{u(C_1)}, r^m - r \right\} \]

* Intuition: similar to what we saw in Laibson

- When the Cov term is high, the risk premium on the asset is lower. This is because it is paying out more in states where marginal utility of consumption is high.
- Stated more technically: a positive covariance means that the asset tends to yield unexpectedly high returns in states of nature when the marginal utility of world consumption is unexpectedly low. Because the asset therefore provides a hedge against world output fluctuations, it offers an expected rate of return below the riskless rate of interest in equilibrium.

* Things that pay a lot in states of nature where you value it a lot, pay more

* This FOC is the root of the equity premium puzzle. We’ll ask what should the stock market pay; what does a riskless asset pay?

* [Algebra:]

1. Multiply both sides of above equation by \( \frac{1+r}{V^m_1} \)

\[ V^m_1 = \mathbb{E}_1 Y^m_2 + \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, Y^m_2 \right\} \Rightarrow 1+r = \frac{1}{V^m_1} \left( \mathbb{E}_1 Y^m_2 + (1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, Y^m_2 \right\} \right) \]

2. Rearrange, then divide by \( V^m_1 \)

\[ \Rightarrow \mathbb{E}_1 Y^m_2 - V^m_1 = (1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, Y^m_2 \right\} \Rightarrow \mathbb{E}_1 \{ r^m \} - r = -\frac{1}{V^m_1} (1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, Y^m_2 \right\} \]

3. Use Covariance property: \( c \cdot \text{Cov} (X, Y) = \text{Cov} (cX, kY) \)

\[ \Rightarrow \mathbb{E}_1 \{ r^m \} - r = - (1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, Y^m_2 \right\} \Rightarrow \mathbb{E}_1 \{ r^m \} - r = -(1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, r^m + 1 \right\} \]

4. Use Covariance property: \( \text{Cov} (X, Y) = \text{Cov} (X + c, Y + k) \)

\[ \Rightarrow \mathbb{E}_1 \{ r^m \} - r = - (1 + r) \text{Cov} \left\{ \frac{\beta u'(C_2)}{u(C_1)}, r^m - r \right\} \]

---

**Post-War Financial Repression?**

Real US Stock, Bond Returns: 1802-1992

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>Stocks</th>
<th>Short Bonds</th>
<th>Long Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1802-1992</td>
<td>6.7</td>
<td>2.9</td>
<td>5.4</td>
</tr>
<tr>
<td>1871-1992</td>
<td>6.6</td>
<td>1.7</td>
<td>2.6</td>
</tr>
<tr>
<td>1802-1970</td>
<td>7.0</td>
<td>5.1</td>
<td>4.8</td>
</tr>
<tr>
<td>1871-1925</td>
<td>6.6</td>
<td>3.2</td>
<td>3.7</td>
</tr>
<tr>
<td>1926-1992</td>
<td>6.6</td>
<td>0.5</td>
<td>1.7</td>
</tr>
<tr>
<td>1946-1992</td>
<td>6.6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

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If you can wait for 10 year period, stocks will always outperform (except the past decade)

3.2 Equity Premium Puzzle (310-315)

Mehra and Prescott pointed out that the CCAPM fails very badly to explain the excess return on stocks over bonds measured over long periods.

• Assume CRRA utility into preceding CCAPM equation:

$$E_1 (r^m) - r = -(1 + r) \text{Cov}_1 \left\{ \beta \left( \frac{C_2}{C_1} \right)^{-\rho}, r^m - r \right\}$$

• Linearize: To do a simple empirical calculation, we approximate the following function (2nd order Taylor Expansion):

$$G \left( \frac{C_2}{C_1}, r^m \right) \equiv \beta \left( \frac{C_2}{C_1} \right)^{-\rho} (r^m - E_1 r^m)$$

- In the neighborhood of $\frac{C_2}{C_1} = 1$ and $r^m = E_1 r^m$.
  * [Note: there are many approaches to this approximation, though all yield the same general approach]

$$G \left( \frac{C_2}{C_1}, r^m \right) \approx \beta (r^m - E_1 r^m) - \beta \rho \left( \frac{C_2}{C_1} - 1 \right) (r^m - E_1 r^m)$$

- Taking conditional (period 1) expectations of both sides yields:

$$E_1 G \left( \frac{C_2}{C_1}, r^m \right) = \text{Cov}_1 \left\{ \beta \left( \frac{C_2}{C_1} \right)^{-\rho}, r^m - r \right\} \approx -\beta \rho \text{Cov}_1 \left\{ \frac{C_2}{C_1} - 1, r^m - r \right\}$$

• With CRRA utility, we can thus approximate the CCAPM equation as:

$$E_1 (r^m) - r = (1 + r) \beta \rho \text{Cov}_1 \left\{ \frac{C_2}{C_1} - 1, r^m - r \right\}$$

$$= (1 + r) \beta \rho \kappa \text{Std}_1 \left\{ \frac{C_2}{C_1} - 1 \right\} \text{Std}_1 \{r^m - r\}$$

- where $\kappa \equiv \text{Corr}_1 \left( \frac{C_2}{C_1}, r^m - r \right)$ is the conditional correlation coefficient between consumption growth and the excess return on equity
- $\text{Std}_1$ is the standard deviation of consumption growth

• Empirical estimate: $\rho = 26$

  - Mankiw and Zeldes (1991) use data from Mehra-Prescott dataset, 1889-1978 equity returns
Find $\kappa = 0.4, \text{Std}_1 = 0.036$ per year, and the standard deviation on excess returns to equity were 0.167 per year.

- Assuming $(1 + r) \beta = 1$, find the model first the observed excess quantity of return of
  \[ \mathbb{E}\{r^m\} - r = 0.0698 - 0.008 = 0.0618 \]

- if $\rho = 26$, seemingly implausible estimate.

- They also note that equity premium puzzle is even more severe in U.S. post-war, with levels as high as $\rho = 100$ to justify

Explanations for Equity Premium Puzzle

• Habit Persistence

  - “richer” utility function, and a simple one. Like Epstein-Zinn:
  \[ u(C_t, D_t) = \frac{(C_t, D_t)^{1-\rho}}{1 - \rho} \]
  - where $D_t = (1 - \delta) D_{t-1} + \delta \zeta C_{t-1}$, $0 < \zeta, \delta < 1$.  

  - This approach helps explain equity premium since individuals are very averse to big changes in consumption.

    - With habit persistence, accumulated consumption experience $D_t$ tends to make $C_t - D_t$ a small number if $\zeta$ is close to 1. This allows realistic consumption fluctuations to have a very large effect on the marginal rate of substitution between current and future consumption even when $\rho$ is not large.

  - But then the MRS of consumption would be very volatile and the risk free interest rate would be counterfactually volatile

• “Keeping up with the Joneses”

  - Your utility probably depends on what other people are doing

    - Typical models we look at: you don’t care about what someone else’s utility is

  - Lucas, etc: you should care about where you’ll be in 50 years (trend), not the small fluctuations until then

  - “This is hard to do”

• Transactions costs

  - [Aiyagari and Gertler 1991] transaction costs on stock trades may be higher than on bond trades.

    - The cost difference leads to a liquidity premium on bonds that can explain the high equity premium.

    - However, transaction costs must be very high to explain a significant portion of it

• Non-diversifiable consumption

  - 2/3 of income is not tradeable. [I believe this is same as labor income – “nondiversifiable labor income”]

• Tail risk (Barro-Reitz)
• Uncertainty about Future Trend Growth
  – Big literature in finance: trying to model shocks to the trend of long-run growth

History and Equity Premium Puzzle

• The equity premium puzzle seems to have arisen in several countries despite disparate economic circumstances
  – i.e. it’s not just that the US has been great success story. Britain, “hardly an example of unalloyed economic dynamism” and Germany (WWII destruction) also see same trend

• Could be anomaly of post-Depression period. Blandard (1993) contends equity premium has become much smaller since 1980.

• “Survival bias”. Looking over very long time periods may erase puzzle; shorter time periods introduce “survival bias.” Brown, Goetzman and Ross (1995) note that in 1792, could buy stocks in Holland, France, Austria, Germany, US and Britain. But US and Britain are only markets that have operated continuously since then.
  – They argue when market causalities are taken into account, equity premium become much smaller.

3.3 Low Riskless Interest Rate Puzzle

A related asset-pricing puzzle, raised by Mehra and Prescott (1985)

• Parallel calculation with the interest rate (Mehra and Prescott, 1985)

• With CRRA utility, the stochastic Euler equation can be written:

\[
1 + r = \frac{1}{\mathbb{E}_t \left\{ \beta \left( \frac{C_{t+1}}{C_t} \right)^\rho \right\}}
\]

  – Linear approximation:

\[
r \approx \ln (1 + r) = \rho \mathbb{E}_t \left\{ \ln \left( \frac{C_{t+1}}{C_t} \right) \right\} - \frac{\rho^2}{2} \text{Var} \left\{ \ln \left( \frac{C_{t+1}}{C_t} \right) \right\} - \ln \beta
\]

  – Mehra Prescott sample: mean per capita consumption growth is 0.018 per year, variance 0.0013. With \( \rho = 2 \) and \( \beta = 1 \):
    * Model predicts riskless rate is 3.34, but sample mean is only 0.8

• Some of the same factors that help to explain the equity premium puzzle also imply low riskless rate
  – For example, transaction costs and uninsured labor income

• Some eras have erased this puzzle
  – In the early 1980s, interest rates rose globally, largely erasing the conundrum from recent data.
3.4 Ramsey Model with Infinitely Lived Dynasty

- Steady-State Return on Capital
  - [This is very similar to Solow model with “efficiency units” we saw in Aghion]
    \[ f'(\bar{k}E) \frac{(1 + g)^\rho}{\beta} - 1 \]
    - is the steady state return on capital where \( \bar{k}_E \equiv \frac{K}{E_t L_t} \) where \( E \) is technology progress (augmenting labor), and \( g = \dot{E} E \) is technological change.
    - Note that if \( g = 0 \), then of course \( f'(\bar{k}E) \) is simply equal to the rate of time preference.

- Variant of Neoclassical model with a growing population
  - One variant of the neoclassical model that is widely utilized assumes individuals are weighted inversely according to the size of their cohort:
    \[ f'(\bar{k}E) \frac{(1 + g)^\rho}{\beta(1+n)} - 1 \]
    - [so a rise in population growth has the same effect as an increase in the rate of time preference]

- Inefficiency in overlapping generations model where future generations not in utility
  - In overlapping generations models where the welfare of future generations is not incorporated into utility, it is possible to get dynamic inefficiency where:
    \[ f'(\bar{k}E) < (1 + g) (1 + n) \]
the growth rate of the economy, which in the steady state implies that the economy invests more than 100% of profits to maintain an inefficiently high capital stock.

Abel, Mankiw, Summers, Zeckhauser (1991) test this implication and argue that it does not hold in the data, though there remains some controversy.

In a large part of the global economy, a very large share of economic activity is controlled by governments, whose motives are very far from the standard neoclassical assumptions.

### 3.5 Barro Rare Disasters

- **Model**

\[
\log(Y_{t+1}) = \log(Y_t) + g + u_{t+1} + v_{t+1}
\]

- \(g\) is the deterministic part of the growth process
- \(u\) is the i.i.d. shock
- \(v_{t+1}\) is the disaster

\[\begin{align*}
\text{Probability } 1 - p, & \quad v_{t+1} = 0 \\
\text{Probability } p, & \quad v_{t+1} = \log(1 - b), \quad 0 < b < 1
\end{align*}\]

- \(p\), the probability of disaster, is small, but \(b\) is large

- **Calibration**

  - Barro 2006 QJE paper calibrates \(b\) and \(p\) by looking at disasters where \(b > .15\) using cumulative loss in output over the potentially multyear catastrophe.
  - Barro’s list of the main economic crises of the 20th century:

**ROBERT BARROS list of the Main Economic Crises of 20th Century**

For real per capita consumer expenditure:
- WWII: 23 cases, average 34%
- WWI: 20 cases, average 24%
- Great Depression: 18 cases, average 21%
- 1920s (influenza): 11 cases, average 18%
- Post-WWII: 38 cases, average 18% (only 9 in tranquil OECD)
- Pre-1914: 21 cases, average 16%
• (31:00) Neoclassical growth model benchmarks
• Going back to what Philippe taught:
  – Ramsey:
• Barro: 37:06
  – [but note that if you go too far back in time, get small amount of
• Precautionary savings effect: stock market actually goes up, since volatility increases

Epstein Zinn utility function (47:00)

\[ U_t = \left\{ (1 - \beta) C_t^{1 - \rho} + \beta E_t \left( U_{t+1}^{1 - \rho} \right)^{\frac{\theta}{\rho}} \right\}^{\frac{\sigma}{\theta}} \]

\[ \theta = \frac{1 - \rho}{1 - \frac{1}{\sigma}} \]

[There are certain instances in which the restriction from CRRA that \( \frac{1}{\rho} = \sigma \) is too constricting.]

(Stepping aside: Selected slides on Liquidation of Government Debt)
• Recent take: interest rate is the big problem. But is the interest rate really the interest rate? Government intervention?
  – Term is called “financial repression” – France, Italy; didn’t have floating interest rates until the 90s
• In equity premium calculation: they use 1946-1992, which is comingling this period where real interest rates were very high
  – But in different period, interest rates would be much lower
• Was the equity premium puzzle just that we weren’t observing market interest rates in that period?
• Sociology of profession: not really ask whu interest rate isn’t changing over time

3.4 How Large are Gains to International Risk Sharing? (329-332)
Main interest in covering this topic is wan’t to look at international debt markets. This is a calculation that is a building block.

Note: Calculation makes extensive use of the fact that if \( X \sim N(\mu_x, \sigma^2_x) \) when \( \exp(X) \) is distributed lognormally with mean: \( \mathbb{E}\exp(X) = \exp(\mu_x + 1/2 \sigma^2_x) \).

• Representative Agent
  – Assume utility function:
    \[
    U_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \frac{C_s^{1-\rho}}{1-\rho} \right\}
    \]
  * where consumption grows at this exogenous rate \( g \) and \( C_s = (1+g)^{s-t} \bar{C}e^{(\epsilon_s - \frac{1}{2} \text{Var}(\epsilon_s))} \)
  and \( \epsilon_s \) is normal iid with mean zero.
  – By direct calculation:
    \[
    \mathbb{E}_t \left\{ C_s^{1-\rho} \right\} = (1 + g)^{(1-\rho)(s-t)} \bar{C} \exp\left(-\frac{1}{2} (1-\rho) \rho \text{Var}(\epsilon) \right)
    \]
  * where we have used the fact that \( \exp((1-\rho) \epsilon_s) \) is lognormally distributed with mean \( \exp\left\{ \frac{1}{2} (1-\rho)^2 \text{Var}(\epsilon) \right\} \)
  – Thus prior to observing \( \epsilon_s \):
    \[
    U_t = \bar{C}^{1-\rho} \left[ \frac{1}{1-\beta (1+g)^{1-\rho}} \right] \exp\left[ \frac{1}{2} (1-\rho) \rho \text{Var}(\epsilon) \right]
    \]
  – assuming \( \beta (1 + g)^{1-\rho} < 1 \) important assumption
• Lucas: Imagine consumption uncertainty is eliminated
  – We’d get: \( \bar{C}_s \equiv \mathbb{E}_t C_s = (1 + g)^{s-t} \bar{C} \)
  – Lifetime utility would then be
    \[
    \bar{U}_t = \bar{C}^{1-\rho} \left[ \frac{1}{1-\beta (1+g)^{1-\rho}} \right]
    \]
• Solve for tax \( \tau \) (subsidy) that would make consumers equally well off
We can ask what permanent subsidy $\tau$ to consumption would make consumers equally well off:

$$\frac{(1 + \tau) \bar{C}}{1 - \rho} \exp \left[ \frac{1}{2} (1 - \rho) \rho \text{Var} (\epsilon) \right] = \frac{\bar{C}}{1 - \rho}$$

[Algebra: just set $U_t[\text{with tax}] = \bar{U}_t$:

$$\frac{(1 + \tau) \bar{C}}{1 - \rho} \left[ \frac{1}{1 - \beta (1 + g)^{-\tau}} \right] \exp \left[ \frac{1}{2} (1 - \rho) \rho \text{Var} (\epsilon) \right] = \frac{\bar{C}}{1 - \rho} \left[ \frac{1}{1 - \beta (1 + g)^{-\tau}} \right]$$

[divide by $\frac{1}{1 - \beta (1 + g)^{-\tau}}$ to get above expession]

which can be solved to yield

$$\tau = \left\{ \exp \left[ \frac{1}{2} (1 - \rho) \rho \text{Var} (\epsilon) \right] \right\}^{\frac{1}{1 - \rho}} - 1$$

- Approximation of tax:

  - A first-order Taylor approximation in the neighborhood of $\text{Var} (\epsilon) = 0$ yields:

  $$\tau \approx \frac{1}{2} \rho \text{Var} (\epsilon)$$

- Comparison with empirical measures

  - On US 1950-1990 data: $\text{Var} (\epsilon) = 0.000708$, which corresponds to a standard deviation of consumption of 2.7%.

  - Interpretation: Even if $\rho = 10$ (generally presumed an unreasonably high number for the risk parameter), eliminating all consumption risk would only be worth one third a percent per year of consumption (!)

- Some objections:

  - Fluctuation around fixed trend
  - US had atypically low standard deviation of consumption
  - Lucas calculation does not allow for individual heterogeneity

- Next time: global analysis, focusing on question of why interest rates are so low. Brings together precautionary saving, etc

4  Global Imbalances and the Low Interest Rate Puzzle

- The evidence is overwhelming that the low interest rate puzzle is a global phenomenon, and cannot be understood with the US in isolation

  - Figure: Remarkable Correlation of Long-Term Government Bond Yields [Advanced country yield on 10-year bonds tracking together, from 7% in 2000 to 2% in 2013]
- Figure: World Real Interest Rates (1980-2012): negative in 2008-2012, as well as (short-term bond) 2002-2004

- **Capital flows from emerging markets** to advanced markets are likely a significant piece of the puzzle

  - Figure: Current Accounts as % of Global GDP 1996-2016 [US massively negative, getting down to -1.5%]. Everyone else above.
  - Current Account: \[\text{[net exports - net imports]} + \text{[net lending - net borrowing]}\]

- Global Current Accounts:
  - Sum all national 2004 CA deficits: -$965 billion
  - Sum all national 2004 CA surplusses: +$888 billion
    - Global discrepancy: -$77 billion
  - US deficit: -$666 billion
    - Hence, US accounts for 75% of measured surpluses, 69% of deficits

**Meltzer Diagram**

- US \(S(r), I(r)\) curves intersect to make a higher interest rate than the rest of the world.
  - Hence, when move to world \(r^w\), US will save less and invest more than US equilibrium, creating current account deficit
  - For ROW, \(r^w\) is greater than equilibrium \(r\), hence there will be more saving and less investment than in equilibrium. This creates a positive current account balance.

- Many explanations of low rates play off Meltzer diagram
  - Savings in emerging markets ↑ due to precautionary savings (from poor capital markets)
- Investment \downarrow in emerging markets due to poorly developed capital markets
- Fast growth of emerging markets magnifies their weight

• Strong arguments going the other way
  - Emerging markets should have higher interest rate under autarky, since they are growing faster (6-7%, vs 2%)
  - If Labor/Capital ratio is higher in emerging markets, then in free trade, rates of return to capital should rise in emerging markets and fall in advanced countries.
  - Most of the surpluses of emerging markets are explained by government surpluses.

• Also theories that point to developed country policies
  - *Let Them Eat Credit*, Rajan *Fault Lines* 2010, “Inequality, Leverage and Crises” by Kumhof and Ranciere

• Figure: Income Inequality + Household Leverage, 1983-2008
  - Rough trend upward in both income inequality and household leverage.

• Evaluating flow of funds from poor to rich
  - A lot of debate centers around whether this is a good thing.
  - There is an academic literature showing that it can be mutually beneficial.
    * This is opposite to intuition from standard growth models which have flow going the other way.

• Modelling Fund Flow from Poor to Rich
  - Gertler-Rogoff (1987, 1990)
    * Standard private information borrowing model \Rightarrow international version in which capital markets are endogenously weaker in developing countries due to less collateral
      - Shows how capital can flow poor \rightarrow rich.
  - Lucas (1990) argues capital flow to rich countries due to increasing returns (in contrast G-R “conventional” view).
  - Several models on capital market asymmetries
    * Farhi et al 2008: Capital market imperfections (especially shortage of safe assets) cause capital poor \rightarrow rich.
    * Mendoza et al 2009: Poor capital markets in EMs can induce high precautionary savings.
  - *Public flows*
    * These patterns emphasize private capital flows, but public flow seems much more relevant for China, Oil countries.

• US: “Exorbitant Privilege” / *A very successful hedge fund*
  - Recent empirical work suggests flows from emerging markets are good for US.
* Figure: US Net International Investment Position. Becomes sharply negative after early 80s, to around -25% of GDP

* Figure: Cumulated US current account deficit and Net Foreign Assets.
  - While Cumulated CA gets to -40%, Net foreign asset position drops to only a few % negative.
  - This trend is not nearly as striking for other G7 countries, where Cumulated CA and net foreign asset position track more closely

  “Exorbitant Privilege” - Giscard D’Estaing:

  * US gets *superior return on assets abroad* than it pays on its liabilities to foreigners
    - It therefore can sustain growing trade balance deficit by “grossing up.” [And seems that this is exactly what it has done]

  * Estimate of Exorbitant Privilege [Gourchias and Rey 2008]
    - Argue that US exorbitant privilege has averaged 3.32% per year
    - US returns on equity exceed foreign returns by 6.32% per year. US returns on bonds exceed foreign returns by 3.72% per year.
    - Another paper offers a revised estimate of 1%, based mostly on FDI. Case weakened but still significant. (Curcuru et al 2007)

  Intuition of argument: huge rise in gross flows offsets net flows

  * US runs big deficits but makes up for it by essentially being a very successful hedge fund

  Figures: G7 Cross-Border Assets and Liabilities (G7 massively increasing to 200%, BRIC still below 25%)

  * US Gross External Assets (rising to 120% GDP) and US Gross External Liabilities (rising to 150% GDP)

  Reasons for higher Return on US FDI

  Classic argument: US FDI is older, more mature; in dividend paying phase

  “Dark Matter” – US FDI is larger than measured (Hausman and Sturzenegger 2006)

  * McGrattan and Prescott – Dark matter is intangible capital, explains 1/2 differential

  Under & over invoicing: *cost-switching and tax breaks*

  * Gros (2006): Gap between US and foreign returns on FDI is largely a statistical illusion, due to *cost switching by multinationals*
    - Half US-global trade is intra-firm

  * Bosworth et al (2007): US multinationals face higher taxes at home than abroad
• **Tax Havens**

  - Europe’s net external wealth likely greatly understated by tax avoidance.

  ![Excess Return on US FDI](image)

  **Figure 1:** Unrecorded Assets Held in Tax Havens Are Double the Recorded Net Debt of the Rich World

  ![My estimates of households’ unrecorded assets held in tax havens](image)

  **Source:** Zucman, QJE, 2013

**Policy Literature**

  - **Bretton Woods II**

    - Dooley, Folkerts-Landau, Garber: Todays developing countries simply following the strategy of Europe after WWII
      1. Maintain low fixed exchange rates
      2. Export-led growth to soak up excess labor
      3. Buy lots of US treasuries to hedge weak financial systems

    - Bernanke’s famous 2005 “Global Savings Glut” speech is a refinement of this

    - Holes in BWII:
      * Japan and Germany are large surplus regions, hardly have surplus labor
      * Oil Rich Middle East Countries: Do have massive surplus labor problems, but policy not directed towards putting people back to work
But idea that US has superior financial system makes a lot of sense

* Although ubiquitous assumption in literature that US financial market is perfect, is perhaps an overstatement that can sometimes lead to exaggerated or even wrong conclusions

Rogoff Model: Moral Hazard and North-South Capital Flows

• Set Up
  – Small country, populated by “entrepreneurs.”
  – World interest rate is \( r \)
  – Agents consume only in period 2
  – Utility: simply linear in \( C_2 \):
    \[
    U_1 = U(C_1, C_2) = C_2
    \]
  – Production decision
    * Agents are endowed with \( Y_1 \). They can:
      1. Buy Bonds abroad
      2. Invest in the Family firm:
        \[
        Y_2 = \begin{cases} 
        Z & \text{with probability } \pi(I) \\
        0 & \text{with probability } 1 - \pi(I) 
        \end{cases}
        \]
        * \( \pi' > 0, \pi'' < 0 \). Doesn’t look like a “normal” production function, but it is.
        * Probability increases with how much you invest. [Introducing this so that we can go to asymmetric information, where you don’t observe whether the person invested or not. Only observe output]

• Optimization Under Full Information
  – Maximization:
    * Under full information, the agent chooses to maximize expected profits:
      \[
      \max_I -I + \frac{\pi(I) Z}{1 + r}
      \]
      * Maximize: Opportunity cost of borrowing \(-I\) (if they had money at \( Y_1 \), they could lend it abroad) plus expected value of output \( \left( \frac{1 + r}{1 + \pi(I)} \right) \)
    * FOC\(_I\):
      \[
      \pi'(\bar{I}) Z = 1 + r
      \]
      * Expected return to investment [Marginal gain in probability from more investment
        * what you gain if you win] = \( 1 + r \)

• Assume Borrowing and Lending
To make problem interesting, we assume investors need to borrow: \( \bar{I} > Y_1 \) (i.e. investment more than they have)

First period budget constraint:

\[
I + L = Y_1 + D, \quad L \geq 0, \quad D \geq 0
\]

where \( L = \text{gross lending [money they put abroad]} \); \( D = \text{gross borrowing} \).

Zero Profit condition for lenders:

* "Promised state-contingent payments must yield world-market (expected) return = 0"

\[
\pi (\bar{I}) P (Z) = (1 + r) (\bar{I} - Y_1)
\]

\[
\pi (I) P (Z) + [1 - \pi (I)] P (0) = (1 + r) D.
\]

\( P (Z) \) is what you pay in good state of nature; \( P (0) = 0 \) is what you pay in bad state. \( Z = \text{payment when } Y_2 = Z, \ D = (\bar{I} - Y_1) \) under full information

* “You have to make payments so that the expected rate of return is 0 to lenders.” We’ll later introduce the concept of asymmetric information, where they can’t observe what you invest, or what you put abroad. So they lent you some money, but they don’t know what you invested.

Algebra

* Asymmetric information: Outside lenders do not observe \( I \) or \( L \):

\[
EC_2 = \pi (I) [Z - P (Z)] - [1 - \pi (I)] P (0) + (1 + r) L
\]

* Individual no longer maximizes expected profits.

\[
= \pi (I) [Z - P (Z)] - [1 - \pi (I)] P (0) + (1 + r) (Y_1 + D - I)
\]

- FOC

* “Skipping a lot of things, the FOC looks like this.”

\[
\pi' (I) \{Z - [P (Z) - P (0)]\} = 1 + r
\]

* Now there is a “wedge” in your FOC. \( \text{You end up investing less.} \) You invest very little compared to what you would do otherwise. But it is a function of your net worth, so if your income goes up, then you don’t have to borrow as much, so the problem is ameliorated, so you invest more. [This is how Kiyotaki-Moore, Bernanke Gerhler]

  - In 2-country version, where one country has more net wealth than the other, you can get the poor country lending to the rich country because of the asymmetry.
  - Expected 2nd period consumption. No longer maximizing 2nd period consumption because there is a tax to pay \( P \): once you borrow money, need to pay a portion of it if project goes well (operates like a tax)–that’s different than if you paid it all to yourself.

5 A Simple Illustrative Model of Liquidity Traps with and without capital goods and capital market frictions
Liquidity Trap Model

- **Set Up** (Follows Krugman, 1998)
  - Closed economy, endowment flow $Y_t$, representative agent.
  - Money is modelled through *cash-in-advance*
  - Representative agent’s utility is:
    \[ U = \sum_{s=t}^{\infty} \beta^{s-t} \frac{C_{t+1} - \rho}{1 - \rho} \]
  - Cash-in-advance constraint:
    \[ P_t C_t \leq M_t \]
      - Financial markets meet at the beginning of period $t$. Producers here take any loss on money due to inflation between periods.
  - Goods market equilibrium condition:
    \[ C_t = Y_t \]
  - Euler Equation
    - Consumption Euler equation ("IS" curve of modern macro)
      \[ u'(C_t) = \beta (1 + r_{t+1}) u'(C_{t+1}) \]
    - with functional form:
      \[ C_t^{-\rho} = \beta (1 + r_{t+1}) C_{t+1}^{-\rho} \]
      - Derivation algebra below. This implied budget constraint appears to be:
        \[ C_t + M_t = Y_t + (1 + r_t) M_{t-1} \]
        \[ \max(M_t, C_t) \sum_{s=t}^{\infty} \left( \frac{C_{t+1} - \rho}{1 - \rho} \beta^{s-t} + \lambda_t \left[ -C_t - M_t + Y_t + (1 + r_t) M_{t-1} \right] \right) \]
        FOC$_{C_t}$: $C_t^{-\rho} \beta^{-t} = \lambda_t$
        FOC$_{M_t}$: $\lambda_t = (1 + r_{t+1}) \lambda_{t+1}$
        Substituting for $\lambda_t$ and $\lambda_{t+1}$ from FOC$_{C_t}$: $C_t^{-\rho} \beta^{-t} = (1 + r_{t+1}) C_{t+1}^{-\rho} \beta^{-t+1}$ \( \Rightarrow C_t^{-\rho} = \beta (1 + r_{t+1}) C_{t+1}^{-\rho} \)
      - Rewriting Euler with goods market equilibrium:
        \[ 1 + r_{t+1} = \beta^{-1} \left( \frac{Y_{t+1}}{Y_t} \right)^{-\rho} \]
      - Assuming a nominal bond, arbitrage implies (ignoring risk issues):
        \[ 1 + i_{t+1} = (1 + r_{t+1}) \frac{P_{t+1}}{P_t} \]
          - This looks to be an identity, adjusting for inflation.
      - The nominal Euler Equation:
\[ 1 + i_{t+1} = \frac{1}{\beta} \frac{P_{t+1}}{P_t} \left( \frac{Y_{t+1}}{Y_t} \right) \]

5.1 Flexible Prices

- **Set Up**
  
  - Price level \( \bar{P} \):
    * Assume central bank policy is absolutely fixed and cannot change people’s assumptions. Also assume money supply and endowments are fixed after time \( t \).
    \[ P_{t+k} = \frac{\bar{M}}{\bar{Y}} = \bar{P} \]
    * where bar superscripts denote the long run (period \( t + 1 \) and beyond).
  
  - Long-run interest rates (nominal and real)
    \[ 1 + r_{t+k} = 1 + \bar{i}_{t+k} = \frac{1}{\beta} \equiv 1 + \bar{i} \]
  
  - Short term interest rate
    * Denoted \( i \) (no subscript). Then the equivalent of the IS curve:
    \[ 1 + i = \frac{\bar{P}}{\beta P} \left( \frac{\bar{Y}}{Y_t} \right)^\rho \]
    \( \text{CC}_{fp} \)
  
  - MM Curve:
    \[ P = \frac{M_t}{Y_t} \quad \text{MM}_{fp} \]
  
  - ZLB restraint on price
    * Because the nominal interest rate cannot go below zero, then under flexible prices with fixed output, the maximum current-period price is:
    \[ P^A = \frac{\bar{P}}{\beta} \left( \frac{\bar{Y}}{Y_t} \right)^\rho \]
    [Algebra: just set \( i = 0 \) and rearrange \( \text{CC}_{fp} \)]

- **Suppose negative \( r \)**
  
  - Suppose that real interest rate is negative due (for now) to an adverse expected future shock
  
  - That is, suppose real Euler from above is less than 1:
    \[ 1 + r_{t+1} = \beta^{-1} \left( \frac{\bar{Y}}{Y_t} \right)^{-\rho} < 1 \]

- **Graph:**
5.2 Sticky Prices
Flexible Prices and Demand-Determined Output

- MM Curve becomes
  \[ Y \equiv Y_t = \frac{M_t}{P_t} \quad \text{MM}_{sp} \]

- CC Curve becomes
  \[ 1 + i = \frac{\bar{P}}{\beta P_t} \left( \frac{\bar{Y}}{Y_t} \right)^{\rho} \quad \text{CC}_{sp} \]
  
  - [Nominal Euler with steady state definitions]

- ZLB maximum level of output:
  \[ Y^A = \bar{Y} \left( \frac{\bar{P}}{\beta P_t} \right)^{\frac{1}{\rho}} \]
  
  [Algebra: just set \( i = 0 \) and rearrange \( \text{CC}_{sp} \)]

- Graph: [Same as above, but horizontal axis becomes \( Y \)]

5.3 Temporary Increase in Government Spending

- Government Spending Identity
  \[ C + G = Y \]
• The IS CC curve becomes

\[ 1 + i = \frac{\bar{P}}{\beta P_t} \left( \frac{\bar{Y} - \bar{G}}{\bar{Y}_t - G_t} \right)^\rho \text{ CC}_{gs1} \]

- Just add in government spending to CC\text{sp}. \( \bar{G} \) is long-term, \( G_t \) is short-term.
- Intuition: \( \uparrow G_t \Rightarrow \uparrow i \). [An increase in temporary government spending can increase nominal interest rates]
  * A permanent rise in government spending has little or no effect because it hits short-run and long-run spending proportionately
  * To be effective, the government must credibly commit to only a temporary increase in spending
- Note*: If the government had a different technology, then government spending would also affect the MM curve.

• Government Cash-in-advance constraint (same as consumers):

\[ PG \leq M^G \]

• The MM curve is given by

\[ P (G + C) = PY \leq M = M^G + M^P \]

• Graph:

5.4 Preference Shock

Revisiting negative real interest rates in a closed economy

• Assume monetary policy cannot credibly raise inflation expectation [Eggertsson (not to be confused with E-K below) 2010, others]
Shock to preferences: (real Euler)

\[ \varepsilon u'(C_t) = (1 + r_{t+1}) \beta u'(C_{t+1}) \quad 0 < \varepsilon < 1 \]

- Nominal bond Euler:

\[ \varepsilon u'(C_t) = (1 + i_{t+1}) \frac{P_t}{P_{t+1}} \beta u'(C_{t+1}) \quad 0 < \varepsilon < 1 \]

Situation at ZLB for \( i \):

- Expectations of future price level are fixed [from above assumption]
- Hence, there must be deflation [see above equation]
- If current prices are also sticky, then current consumption must fall to raise marginal utility of current consumption to offset the shock to \( \varepsilon \) because prices cannot move

Eggertsson observes that if the government spending is permanent, it can still have an effect by raising long-run inflationary expectations (assuming monetary policy doesn’t react)

\[
1 + i = \frac{\varepsilon}{\beta} \left( \frac{P_{t+1}}{P_t} \right) \left( \frac{\bar{Y} - \bar{G}}{\bar{Y}_t - G_t} \right)^\rho \quad \text{CC}
\]

Caveat: Does government spending work to get out of liquidity trap?

- Eggertsson’s model suggests it is a key driver. If so then employment should rise.
- However, total hours worked in the Great Depression didn’t recover that much with stimulus (going from 71% to 83% from ’34-’37)

**Eggertsson-Krugman**

Eggertson-Krugman gives a deeper rationale for why the short-term marginal utility of consumption might fall, at least for the class of consumers whose behavior is most relevant for determining the interest rate.

- Two types of agents: borrowers and savers. \( \beta (s) > \beta (b) \)
  - Equal number of savers and borrowers
  - Both have log utility (limiting case of CRRA where \( \rho = 1 \))
  
  \[
  \mathbb{E}_t \sum_{z=t}^{\infty} \beta^{z-t} (i) \ln c_z (i)
  \]

  - Model ignores money, but implicitly assumed that central bank can influence the price level

- Debt accumulation (Budget constraint):

\[
D_{t+1} (i) = (1 + r_{t+1}) D_t (i) - \frac{1}{2} Y_t + c_t (i)
\]
- The $\frac{1}{2}Y_t$ is from equal shares of output going to each of the two types

- **Steady State**
  - In the steady state, borrowers will asymptote to borrowing entire NPV of life income

**Single Debt Limit**

- **Debt Limit**
  - Assume exogenously determined debt limit in normal times takes on “high value” $D^{high}$, so debt must obey the constraint:
    \[(1 + r_{t+1}) D_t (i) \leq D^{high} > 0\]
    * Will hold with equality in equilibrium

- **Initial Set Up**
  - $Y$ is constant. Borrowers do not recognize possibility that debt limit might change.

- **Steady state**
  - Borrowers will borrow up to the limit:
    \[c^b_t = \frac{1}{2} Y - \frac{r}{1 + r} D^{high}\]
    * [the borrower must leave each period with no more than $\frac{1}{1 + r} D^{high}$ in debt, so he does not arrive in next period with more than $D^{high}$ due, given interest due]
  - Savers consume endowment plus interest
    \[c^s_t = \frac{1}{2} Y + \frac{r}{1 + r} D^{high}\]

- **Euler equation:** Determined by savers
  - Euler equation is determined by savers, since they are the ones in the economy that are not constrained
    \[\frac{1}{c_t} = (1 + r_{t+1}) \beta E_t \frac{1}{c_{t+1}^s}\]
    - $[\beta = \beta (s)$ for simplicity$]$  

- **Interest rate in steady state**
  - Set $c^s_t = c_{t+1}^s$
    
    \[r = \frac{1 - \beta}{\beta}\]

Algebra: in steady state, we have $c^s = c^s_t = c_{t+1}^s$
\[\frac{1}{c^s_t} = (1 + r_{t+1}) \beta E_t \frac{1}{c^s_{t+1}} \Rightarrow 1 = (1 + r_{t+1}) \beta = \beta + r_{t+1} \beta \Rightarrow -r_{t+1} \beta = \beta - 1 \Rightarrow r_{t+1} = \frac{1 - \beta}{\beta}\]
Deleveraging Experiment

- **Debt limit suddenly drops to** $D_{\text{low}} < D_{\text{high}}$
  - Assume entire adjustment must be made in one period
- **Borrowers’ Consumption:**
  - Short run (adjustment period)
    \[ c^b_t = \frac{1}{2} Y + \frac{D_{\text{low}}}{1 + r_{t+1}} - D_{\text{high}} \]
    
    * Needs to go from old to new debt ceiling in one period. Entered period at full $D_{\text{high}}$, and needs to get down to $D_{\text{low}}$ including room for next period’s interest $\left( \frac{1}{1 + r_{t+1}} \right)$
  - Long run
    \[ c^b = \frac{1}{2} Y - \frac{r}{1 + r} D_{\text{low}} = \frac{1}{2} Y - (1 - \beta) D_{\text{low}} \]
    
    [Algebra: from above: $r = \frac{1 - \beta}{\beta}$. So $\frac{r}{1 + r} = \frac{1 - \beta}{\beta + 1 - \beta} = 1 - \beta$]

- **Savers**
- **Euler**
  - General: determined by savers as before:
    \[ \frac{1}{c^s_t} = (1 + r_{t+1}) \beta \mathbb{E}_t \frac{1}{c^s_{t+1}} \]
  - Savers $c^s$ in period $t$:
    \[ c^s_t = Y - c^b_t = \frac{1}{2} Y - \frac{D_{\text{low}}}{1 + r_{t+1}} + D_{\text{high}} \]
    
    \[ c^s_{t+1} = Y - c^b_{t+1} = \frac{1}{2} Y - (1 - \beta) D_{\text{low}} \]
  - Substituting for savers’ consumption
    \[ \frac{1}{2} Y + (1 - \beta) D_{\text{low}} = (1 + r_{t+1}) \beta \left\{ \frac{1}{2} Y - \frac{D_{\text{low}}}{1 + r_{t+1}} + D_{\text{high}} \right\} \]
    
    * This uses the same logic as above, where the savers’ consumption path is directly related to the borrowers’. Signs on $D$ values get flipped, given their creditor status.
  - Just substitute into slightly rearrange Euler: $\mathbb{E}_t c^s_{t+1} = (1 + r_{t+1}) \beta c^s_t$
  - **Simplified Euler:**
    \[ 1 + r_{t+1} = \frac{\frac{1}{2} Y + D_{\text{low}}}{\beta \frac{1}{2} Y + \beta D_{\text{high}}} \]
• Negative Real Interest Rate
  - This interest rate can be less than 1 if $\beta D^{high} - D^{low}$ is large enough – that is, if deleveraging is large enough.

• Contracting Terms:
  - Deleveraging problem can be further exacerbated if debt is contracted in nominal terms but deleveraging is in real terms.
    * [Denoting nominal debt as $B$]
    \[
    \frac{B^{high}}{P_t} - \frac{D^{low}}{1 + r_{t+1}}
    \]
    * In this case, the short terms real interest rate also depends on short term prices so:
    \[
    1 + r_{t+1} = \frac{\frac{1}{2}Y + D^{low}}{\beta \frac{1}{2}Y + \beta \frac{B^{high}}{P}}
    \]
  - Again, government spending is very powerful if it can increase prices more in the long run than in the short run, even if permanent.

Some Qualifications

• How does the efficacy of government spending change when looking at open economy? Even US is at most 25% of world economy.

• How serious is the assumed failure of monetary policies to be able to credibly promise inflation, certainly a rare problem historically?

• What if there are a large class of agents who are not technically liquidity constrained but choosing to save more because of heightened perception of output and liquidity risk?

6 Sovereign Debt I: Reputation and Risk Sharing with Incomplete International Enforcement

Key Ideas on Sovereign Debt

• Two ways to think about debt:
  1. Consumption Smoothing (access to insurance)
  2. Financing Investment ($K_2$)

• Two frictions we add to examine possibility of sovereign debt
1. Threat of exclusion from future asset trading
   ⇒ Analyzed in context of Consumption Smoothing (Insurance)
   [Bulow Rogoff 1989]

2. Threat of punishment through direct sanctions ($\eta$)
   ⇒ Analyzed in context of Investment ($K_2$)

- Main Results from Models

1. Threat of exclusion from future asset trading (Bulow Rogoff 1989)
   (a) Insurance ($\equiv$ debt) can be sustained in equilibrium if the punishment for default is complete exclusion from international capital markets
      i. However, following Lucas (1987), calculation of the gains from risk sharing do not support much debt
   (b) Insurance ($\equiv$ debt) cannot be sustained in equilibrium if collateralized borrowing (i.e. net lending) is allowed. [Punishment for default is just inability to borrow uncollateralized loans]

2. Threat of punishment through direct sanctions
   (a) Investment increases with borrowing up to a debt ceiling $\bar{D}$
      i. At $\bar{D}$, an additional dollar of debt leads to default and a discrete fall in investment
   (b) Debt overhang: Pre-existing debt can depress investment relative to first-best
      i. Debt Laffer curve generated, where a reduction in the the face value of debt can actually increase its market value
      ii. Debt buyback is not justified, as it is purchased at average cost which is below marginal cost

---

Sovereign Debt: Some analysis and intuition from lecture

Look at major events, not tremors
We don’t know why anyone pays their debts across countries.
In any debt contract, property rights are not well defined.
Used to have debtors prisons.
No international bankruptcy court. New York, London have some treaties. But not very strong or enforceable.
Theories today: connection to reality very thin, but elegant models.
Debt:GDP the year country defaulted: over half occur with less than 60%
Countries hardly ever pay more than a few percent of GDP to other countries. Belgium paid down from 125% to 80%, paying back 5% per year.
Defaults normally occur when things are bad. But when you think about international markets deeply, “it’s all insurance.” At least according to model. But nothing looks like this (default when things aren’t bad).
Countries don’t default the same way people do. For people: 10cents/dollar. For countries: 70cents/dollar.
Russian debt: tore it up in 1917. But when reentered international markets in 1970s, had to renegotiate.
In negotiations, often able to get 3rd parties to make side payments (i.e. IMF, US bailing out Mexico)

The big question in understanding international lending is why do countries pay at all.
We start with the most popular model (because it is the cleanest and simplest) where the threat is future exclusion from international capital markets. Theoretically elegant, empirically much less clear.
6.1 Eaton and Gersovitz

Empirics and Background:
Intuition: [Pascal]

- Here, once you default, you are cut off of world markets forever for both borrowing and saving.
- So to make something incentive compatible, have to have high enough $\beta$ such that benefit from default is less than cost from default

\[^{\text{Gain}}(\bar{\epsilon}) \leq \text{Cost}\]

\[u(\bar{Y} - \epsilon_t) - u(\bar{Y}) \leq \frac{\beta}{1 - \beta} [u(\bar{Y}) - \mathbb{E}_t u(\bar{Y} + \epsilon)]\]

6.1.1 Small Country Model of Reputation and Debt

- Set up
  - Small country inhabited by an infinitely lived representative agent
  \[U_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \right\}\]
  - Country endowments: constant stream with stochastic component:
  \[Y_t = \bar{Y} + \epsilon_t\]
  \* where $\epsilon$ is iid with mean 0.
  \* country will want to insure against this shock. This is diversifiable risk, so that in GE you shouldn’t have to bear this risk.
  - Budget constraint: [we can ignore this]
  \[B_{s+1} = (1 + r) B_s + \bar{Y} + \epsilon_s - C_s - P_s(\epsilon_s)\]
  \* where $B$ are holdings of riskless real bonds and $P_s(\epsilon_s)$ is a stable contingent one-period insurance contract.
  \* $\pi(\epsilon_i)$ is the probability of state $\epsilon_i$
  - International insurance market [this allows for possibility that insurance can happen]
  \* Assuming the country faces competitive international insurers, any insurance contract must obey:
  \[\sum_{i=1}^{N} \pi(\epsilon_i) P_s(\epsilon_i) = 0\]
  \* No rent to the insurer. $P$ positive means you get something, negative means you pay something. $N$ states of nature. Expected payment has to be 0.
  \* Using $\epsilon$ notation to allow later for possibility of partial insurance. Why? Because might not be enough punishment for full insurance.
  - Full insurance
* Full insurance payment would have $P_s(\epsilon_t) = -\epsilon_t$, so that $Y_t = \bar{Y}$ in all periods.
  - So if $\epsilon$ is negative, you get something, and if $\epsilon$ is positive, you pay insurer.

- Punishment
  - *The only punishment is permanent exclusion from future international insurance contracts*
  - Assume foreign insurers can perfectly commit to pay any zero profit insurance contract but the country cannot precommit.

- Driving question: under what conditions is the full insurance contract incentive compatible? 
  [that is, when would you never default?]
  - Or: what is the gain to defaulting? Assume punishment is that you get cut off from these contracts forever
  - [Could have other models where you are punished for several periods and come back. But here we just look at permanent]
  - Why are we talking about insurance contract when we’re after lending? Because at a deep level they are trying to sign Arrow-Debreu contracts.
  - Assume no international court, so only reputation can achieve in this purely game theoretic sense.

- Under what conditions would you never default?
  - In order for this to be an equilibrium, has to be in your incentive to pay (your insurance payment) in every state of nature
  - We expect sum of insurance contracts to be 0; this is equilibrium. If there is some state of the world in which you’re not going to pay, then it won’t balance out to zero, and it’s not an equilibrium. [We will later look at partial insurance as a way around]

- Cost/Benefit of Default
  - Suppose the country gets an output shock in period $t$.
  - Gain to defaulting (immediate):
    \[
    \text{Gain} (\epsilon_t) = u (\bar{Y} - \epsilon_t) - u (\bar{Y})
    \]
    - Short term gain is if you get a positive shock and walk away from it.
  - Cost to defaulting (long-run):
    \[
    \text{Cost} = \sum_{s=t+1}^{\infty} \beta^{s-t} u (\bar{Y}) - \sum_{s=t+1}^{\infty} \beta^{s-t} \mathbb{E}_t [u (\bar{Y} + \epsilon_s)]
    \]
    - Cost: present value of your utility loss from having more volatile output. Just like Lucas calculation.
    - Note:
      \[
      \sum_{s=t+1}^{\infty} \beta^{s-t} u (\bar{Y}) - \sum_{s=t+1}^{\infty} \beta^{s-t} \mathbb{E}_t [u (\bar{Y} + \epsilon_s)]
      \]
      - utility under full insurance
      - utility under autarky
    - Cost term reduces to:
\[ \text{Cost} = \frac{\beta}{1 - \beta} [u(Y) - E_t u(Y + \epsilon)] \]

[Algebra: \( \sum_{s=t+1}^{\infty} \beta^{s-t} = \frac{\beta}{1-\beta} \text{ since } \sum_{k=0}^{\infty} a^k = \frac{a}{1-a} \) where \( \sum_{s=t+1}^{\infty} \beta^{s-t} = \sum_{s=1}^{\infty} \beta^s = \beta \sum_{s=0}^{\infty} \beta^s = \beta \left( \frac{1}{1-\beta} \right) \)]

- **Cost/Benefit Condition**
  - Full insurance contract requires that
    \[ \text{Gain} (\bar{\epsilon}) \leq \text{Cost} \]
    * for all \( \epsilon \), including \( \bar{\epsilon} \) (the assumed upper bound to \( \epsilon \)).
  - Substituting from above:
    \[ u(Y - \epsilon_t) - u(Y) \leq \frac{\beta}{1 - \beta} [u(Y) - E_t u(Y + \epsilon)] \]
    **
    * This is what you have in all states of nature if perfect insurance is possible
  - Other issues
    * In general, even where full insurance is not possible, partial insurance is still feasible.
    * “Excusable default” (Bulow-Rogoff and Grossman Van Huyck)
      - Expect that Argentina will default every 18 years. But get high interest payments in return. Expected renegotiation is factored in to the debt contract.
    * Can exclusion be enforced solely by reputation? If exclusion enforced by courts, it is plausible to assume that creditors can make multiple take or leave it offers.

6.1.2 Costs of Exclusion from International Markets
Above we had to bound \( \epsilon \) to \( \bar{\epsilon} \). But here, doing Lucas exclusion calculation, see that utility asymptotes to \( \bar{U} \), so don’t need bound for \( \epsilon \).

- **Driving question:** what would we pay to get rid of uncertainty? (Lucas calculation)

- **Country Stochastic \( Y \)**
  - In analogy with Lucas business cycles calculation, assume that the country has an endowment process
    \[ Y_s = (1 + g)^{s-t} \bar{Y} \exp \left[ \epsilon_s - \frac{1}{2} \text{Var} (\epsilon) \right] \]
    - where \( \epsilon \sim N(\mu_x, \sigma_x^2) \), so that \( \exp(\epsilon) \) is distributed log normally with mean \( E \{\exp(\epsilon)\} = \exp (\mu_x + \frac{1}{2} \sigma_x^2) \)

- **Country Utility**
  - Assume country has CRRA period utility given by \( u(C) = \frac{C^{1-\rho}}{1-\rho} \)
    * Need \( \rho \geq 1 \). Utility will asymptote to upper bound.
- We can then proceed to ask whether full insurance is possible under the threat of capital market exclusion

- Full Insurance Expected Utility

- Future expected utility, after shock realization:

\[
\beta \bar{U}_{t+1} = \frac{1}{1 - \rho} \sum_{s=t+1}^{\infty} \beta^{s-t} (1 + g)^{(1-\rho)(s-t)} \bar{Y}^{1-\rho}
\]

\[
\beta \bar{U}_{t+1} = \frac{\bar{Y}^{1-\rho}}{1 - \rho} \cdot \beta (1 + g)^{1-\rho} 
\]

- where we assume \( \beta (1 + g)^{(1-\rho)} < 1 \)

* Intuition:
  - With this exact utility function, has the property that it asymptotes (with \( \rho \geq 1 \)). So even though shock is unbounded, utility is bounded.

- Autarky Expected Utility

- If instead the country were in future autarky, its expected utility \( U^A \) would be given by:

\[
\beta \bar{E}_t U^A_{t+1} = \frac{1}{1 - \rho} \sum_{s=t+1}^{\infty} \beta^{s-t} \bar{E}_t Y_s^{1-\rho}
\]

\[
\beta \bar{E}_t U^A_{t+1} = \frac{\bar{Y}^{1-\rho}}{1 - \rho} \cdot \beta (1 + g)^{1-\rho} \exp \left[ -\frac{1}{2} \rho (1 - \rho) \text{Var} (\epsilon) \right] < \beta \bar{U}_{t+1}
\]

[Algebra: \( \beta \bar{E}_t U^A_{t+1} = \frac{1}{1 - \rho} \sum_{s=t+1}^{\infty} \beta^{s-t} \bar{E}_t Y_s^{1-\rho} = \frac{\bar{Y}^{1-\rho}}{1 - \rho} \sum_{s=t+1}^{\infty} \beta^{s-t} (1 + g)^{(1-\rho)(s-t)} \exp \left[ (1 - \rho) \left[ \epsilon_s - \frac{1}{2} \text{Var} (\epsilon) \right] \right] \]

\[
= \frac{\bar{Y}^{1-\rho}}{1 - \rho} \cdot \frac{\beta (1 + g)^{1-\rho}}{1 - \rho (1 + g)^1} \text{Var} (\epsilon) < \beta \bar{U}_{t+1}
\]

- Full insurance

- For full insurance, we require again that country would default, even in limiting state:
  - Limit as shock \( \epsilon_t \) goes to \( \infty \). This is where we are approximating to bounded utility.
  - So LHS is “what is the benefit to defaulting in the best state of nature;” RHS is what the gain is.
    - [Looks different than Lucas calculation because have forwarded 1 period, which includes \( \beta \), since don’t start punishment until next period]

\[
\lim_{Y_t \to \infty} [u(Y_t) - u(\bar{Y})] \leq \beta (\bar{U}_{t+1} - \bar{E}_t U^A_{t+1})
\]

\[
\frac{\lim_{\epsilon_t \to \infty} \exp \left\{ (1 - \rho) \left[ \epsilon_t - \frac{1}{2} \text{Var} (\epsilon) \right] \right\} - 1}{1 - \rho} \leq \frac{\beta (1 + g)^{1-\rho}}{1 - \rho (1 + g)^1} \exp \left[ -\frac{1}{2} \rho (1 - \rho) \text{Var} (\epsilon) \right]
\]

- Note that full insurance is never possible if \( \rho \leq 1 \). So \( \rho > 1 \).
Manipulate

* When $\rho > 1$, \( \lim_{t \to \infty} \exp \{ (1 - \rho) [\epsilon_t - \frac{1}{2} \text{Var}(\epsilon)] \} = 0 \). So we get

**Expression of what is required for full insurance to be possible:**

\[
1 \leq \beta (1 + g)^{1-\rho} \exp \left[ \frac{1}{2} \rho (\rho - 1) \text{Var}(\epsilon) \right]
\]

- Comparative Static

  - $\beta$, $\text{Var}(\epsilon)$: higher values of $\beta$ and $\text{Var}(\epsilon)$ make full insurance more likely to be feasible
  - A higher $g$ makes full insurance more difficult. Why?
    * Because utility is asymptoting. So when you go up to a high level, shocks aren’t bothering you as much. [If we adjusted volatility to grow proportionately to $g$, this wouldn’t be true]

\[
u [(1 + \kappa) \bar{Y} - u(\bar{Y})] = \beta (\bar{U}_{t+1} - E_t U_{t+1}^A)
\]

- Table on cost of exclusion from capital markets for different countries

  - For most countries, see 2% to .1% Countries with more volatility in output (Lesotho 5%)
    * Hence, wouldn’t pay much.
  - This is a problem with motivating the model. The model won’t motivate big debts and high interest payments.

- Comment

  - Reputational model is theoretically appealing because it requires little institutional structure.
  - But a number of empirical problems
    1. Difficult to motivate large cost
    2. In practice, debtors seem to have substantial bargaining power
    3. Alternative “direct punishments” model has creditors and country constantly bargaining over flow of debt payments.
      (a) Allows for possibility that interest third parties can get gamed into making side payments (i.e. IMF bailouts, eurozone bailouts)
    4. Multiple equilibria
      (a) Also equilibria where you’re excluded for 5 periods, etc. In reality people aren’t excluded forever.
    5. Literature on default and capital market exclusion gives very unclear results
    6. Hidden debt
      * In external default, additional debts usually exposed.
  - Other ideas: international bankruptcy court?
    * With companies, senior and junior bondholders.
6.2 Bulow and Rogoff: “Sovereign Debt: Is to Forgive to Forget?”

**Intuition:** [Pascal]

- Now, once you default, you are only cut off from uncollaterized borrowing. So you can still:
  - Save at world interest rate \( r \)
    - Makes sense, people will still take your money
    - Purchase collateralized insurance contract
      - So this is like collateralized borrowing
      - Key assumption is that even if a country has forfeited its reputation for repayment, a foreign insurer will still write a collateralized contract. This is intuitive, because the collateral should cover any risk.
      - Insurer must get global rate of return
  - With these types of options still available, the reputation for repayment contract must contain at least one state of nature where the country prefers to default. This the reputation for repayment contract unravels
  - So I think the “Bulow Rogoff” contract is the incentive compatible and feasible contract that you can write on the saving and/or collateralized insurance, that permits you to default on your reputation for repayment contract.
  - Good because explains default. We actually do see default. Hit to your reputation is not enough to prevent you from paying your debt, because you have access to these other instruments.
- Logic
  - I borrow reputation for repayment
  - Once good state of nature occurs (problem set get 1, in lecture notes and book get \( \bar{c} \)), keep that and don’t pay debt
    - Invest in foreign bond
    - Use that foreign bond as collateral to purchase collateralized insurance contract guaranteeing \( \bar{Y} \) in every state. The collateral is enough because it is equal to the greatest payment you would ever have to make. So if you ever default, the creditor just keeps your \( \bar{c} \).
  - In all future periods, strictly better off because:
    - Get \( \bar{Y} + \text{interest on bond} = \bar{Y} + r\bar{c} > \bar{Y} \)
  - \( \Rightarrow \) Hence since there would be a “Bulow Rogoff” contract that makes you strictly better off when you default, the Eaton and Gersovitz reputational debt contract must unravel.

**Model**

- Driving question: if you have the ability to invest abroad, can you have a reputation contract?
  - Answer will be **no**. Can’t have a reputation contract, through an arbitrage argument.
  - Very general, doesn’t matter what the preferences or technology is.

**Background**
- **Pure reputation, no legal rights approach**
- If country defaults will lose its reputation. But can still invest abroad.
- Remember implicit contract is something you cannot default on. It either exists or doesn’t exist in equilibrium.
- Claim: A “reputation for repayment” cannot support sovereign debt if a country always retains the ability to hold foreign assets. Thus claims must either be enforced by legal restrictions or broader notion of reputation.
- **Note:** The proof is basically an arbitrage argument that requires very little structure on preferences or technology

- The model
  - A small country faces competitive risk-neutral lenders
  - Production
    \[ Y_t = f \left( \vec{\theta}_t, \vec{I}_{t-1} \right) \]
    * where \( \vec{\theta}_t \equiv (\theta_t, \theta_{t-1}, \theta_{t-2}, ...) \) is a random serially independent productivity shock, and
    * \( \vec{I}_{t-1} \equiv (I_{t-1}, I_{t-2}, ...) \) are lagged investments
    * No restrictions are needed on the production function, other than the assumption that production is non-decreasing in all elements of \( \vec{I}_{t-1} \)
  - Net Exports
    \[ X_t = Y_t - I_t - C_t \]
    \[ C, I \geq 0, \quad Y > 0, \quad C + I \leq Y^W \]
    * [where \( Y^W \) is global output]
  - Time of events in period \( t \)
    * \( \theta_t \) is observed: affects production in current and possibly future periods
    * The country decides how to divide \( Y \) between \( I_t, C_t \) and \( X_t \)
    * Net exports \( X \) can either be used to make payments on debt or to increase assets held abroad
    * \( \theta \) and \( I \) can be observed by all; there is no private information about aggregate variables
  - Define:
    * World Market value of a claim to a country’s entire net future output (in a particular equilibrium)
    \[ W_t \left( \vec{\theta}_t \right) = E_t \sum_{s=1}^{\infty} \frac{y^s}{(1 + r)^{s-t}} \]  
  \[ (3) \]
    * Just like a Schiller contract.
    * where \( y \equiv Y - I \). Proofs simpler with \( y > 0 \). Assume constant interest rate.

- Types of Lending Contracts
  1. **Reputation for Repayment Contracts**
These contracts specifies payments that are functions of \( \tilde{\theta}_t \): \( P_t(\tilde{\theta}_t) \) \( \forall t, \tilde{\theta}_t \) [could also make it a function of \( \tilde{I}_{t-1} \) if you wanted]

* **Note:** for the reputation contract to be in equilibrium, it must be in the country’s interest to honor it in **every** state of nature

- For any given implicit reputation contract, one can write the market value of the country’s debt as:

\[
D_t(\tilde{\theta}_t) = E_t \left\{ \sum_{s=t}^{\infty} \frac{P_s}{(1+r)^{s-t}} \right\} \quad (4)
\]

* Like the Schiller contract we were looking at, but this is market value of the debt. [Simplified things to avoid getting into CAPM]

- At any one point in time, if someone could sell their reputation debt
- In previous example, you never accumulated any debt. Books get cleared every period. A-D securities. So before, expectation of future dividend was always 0. Debt was current payment.
- But here debt is present value of present and future payments. Allowing serial correlations.

* Note: \( D_t \) can never exceed \( W_t \) (obvious) the market value of a claim to all the country’s income (since consumption is non-negative)

\[
D_t(\tilde{\theta}_t) \leq k' W_t(\tilde{\theta}_t) \quad (5)
\]

* Let \( k \) be the smallest \( k' \) such that the above equation holds. Need to have \( k' \leq 1 \).

2. **Collateralized Insurance Contracts**

- The country pays \( A_t \) at the end of period \( t \) in exchange for a contract that pays \( G_{t+1}(\tilde{\theta}_{t+1}) \) in period \( t+1 \).

- **Key assumption:** Even if a country has forfeited its reputation for repayment, a foreign insurer will write collateralized contracts

- Insurer must get **global rate of return**:

\[
E_t \left[ G_{t+1}(\tilde{\theta}_{t+1}) \right] = (1+r) A_t \quad (6)
\]

- Country must never be called on to make positive payments:

\[
G_{t+1}(\tilde{\theta}_t) \geq 0 \quad (7)
\]

- **Theorem 1:** In any sequential equilibrium, \( D_t \leq 0 \quad \forall t \)

- **Proof**

  - Suppose \( D_s \geq k(W_s - y_s) \). Then the country can cease payment on its reputation contract and initiate the following sequence of cash-in-advance contracts:

    * For all \( t \geq s \), invest \( A_t \) in return for a payments of \( G_{t+1} \) in the ensuing period where:

\[
A_s(\tilde{\theta}_s) = P_s(\tilde{\theta}_s) + k(W_s - y_s) - D_s \quad (8)
\]
\[ A_t \left( \tilde{\theta}_t \right) = G_t \left( \tilde{\theta}_t \right) + P_t \left( \tilde{\theta}_t \right) - k\gamma_t \quad \forall t > s \quad (9) \]

\[ G_t \left( \tilde{\theta}_t \right) = kW_t \left( \tilde{\theta}_t \right) - D_t \left( \tilde{\theta}_t \right), \quad \forall t > s \quad (10) \]

- Since \( D_t \leq kW_t \), inspection of (10) indicates that condition (7) is satisfied.
  - We note from (3) that \( \mathbb{E}_t[W_{t+1}] = (1 + r)(W_t - \gamma_t) \) \( (11) \)
  - We note from (4) that \( \mathbb{E}_t[D_{t+1}] = (1 + r)(D_t - P_t) \) \( (12) \)

- Substitution of (11) and (12) into (8), (9), (10) yields immediate confirmation of (6)

\[ \mathbb{E}_t \left[ G_{t+1} \left( \tilde{\theta}_{t+1} \right) \right] = (1 + r)A_t \]

- Thus, the sequence of cash-in-advance constraints is feasible. Furthermore, the country must pay only \( A_s \leq P_s \) in period \( s \) and \( P_t - k\gamma_t \leq P_t \) for \( t > s \), with equality holding when \( k = 0 \).
- Thus \( k \) must equal zero and by (5)

\[ D_t \leq 0 \quad \forall t \]

- **Thus any “candidate” implicit reputation contract must contain one state of nature where the country prefers to default: thus any contract unravels.**

- **Conclusions**
  - Major stumbling block for theories of reputation based debt?
  - Problems, open issues
    - English: US states defaulted in 1800s did not suffer any direct sanctions, able to reenter credit market, so sanctions can’t be important
    - Cole and Kehoe: develop generalized model says one must look at broader economic relationships to understand debt

7 Soveriegn Risk and Investment

A number of interactions between sovereigns’ borrowing and investment decisions are most easily understood in a setting without uncertainty.

[Thanks Pascal:]

- This is in the section of “Sovereign Risk and Investment: Further Examples of Financial Fragility.” One of the meta-questions we have been asking is why countries ever repay their debt.
- In Bulow and Rogoff, we saw that in theory any sovereign debt contract where the worst reputational penalty is that no one will lend to them is not sustainable as an equilibrium, because a country will always have an incentive to default in some states, hence the thing unravels.
Here, we will see that if there is some kind of pledgeability mechanism / direct sanctions where a country can put up a fraction of its output as collateral, then we can sustain some positive debt in equilibrium.

- While analyzing that main question, we will also see a few interesting results

  - **Knife-edge equilibrium**
    * In equilibrium, constrained maximization (3-steps) will lead to an outcome where the country is exactly indifferent between repaying and not repaying, and any infinitesimal increase in debt would lead to default.

  - **Inherited debt leads to reduced investment**
    * This is the traditional result from the financial frictions models [Bernanke-Gertler-Gilchrist (costly state verification); Gertler-Rogoff (moral hazard in SOE)], where higher initial debt / lower initial wealth leads to lower investment

  - **Debt Laffer Curve theoretically exists**
    * Under some specifications, creditors can gain by forgiving debt. This is a result of the claim above, that lower debt will increase investment, and hence can actually increase the value of that debt.

  - **Debt buybacks in general are not helpful to creditor countries**
    * Buying back debt will just increase the price of existing debt, and you’ll have to pay that higher price to buy it!

### 7.1 The Role of Investment Under Direct Sanctions

- Rogoff introduced this as a “direct sanctions” model, and following on Bullow and Rogoff it seems like a point is that now, some debt can be sustained in equilibrium, whereas Bullow and Rogoff showed under their specification without such direct sanctions there could not be sovereign debt in equilibrium

- Vania introduced this as a pledgeability constraint, implying that the point is also that this itself is like a financial friction. This will lead to a limit on debt and the knife-edge equilibrium.

### Set Up

- A small country that faces a fixed world interest rate
- Utility: $U_1 = u(C_1) + \beta u(C_2)$
- Output:
  * Period 1: $Y_1$ [endowment utility]
  * Period 2: $Y_2 = F(K_2)$
- Finance Constraint
  * The country can finance investment either by borrowing $D_2$ or by reducing consumption (assume no inherited $K$).
  * Period 1 Finance Constraint: $K_2 = Y_1 + D_2 - C_1$
  * Period 2 Finance Constraint: $C_2 = F(K_2) + K_2 - \Re$
    - $\Re$ is repayments, and assumed capital can be consumed after production
- Repayments

\[ R = \min \{(1 - r)D_2, \eta [F(K_2) + K_2]\} \]

- The first term represents payments if the country does not default.
- The second term represents repayments if the country does default and creditors can extract repayments.

### 7.2 First Best: No possibility of Default

- If they were able to commit (no possibility of default) and there was no pledgeability friction, then the country would choose \( K_2 \) and \( D_2 \) to maximize two-period utility:

\[
\max u(C_1) + \beta u(C_2) = \max_{K_2, D_2} u(Y_1 + D_2 - K_2) + \beta u(F(K_2) + K_2 - (1 + r)D_2)
\]

- Where we have subbed in for \( C_1 \) and \( C_2 \) and assumed repayment is always \((1 + r)D_2\)

  - FOCs:

    \[
    \text{FOC}_{K_2} : \quad u'(C_1) = \beta u'(C_2) (F'(K_2) + 1)
    \]

    \[
    \text{FOC}_{D_2} : \quad u'(C_1) = (1 + r) \beta u'(C_2)
    \]

  - Combining the FOCs, we get the optimality conditions

    \[
    \text{Combined FOCs} : \quad F'(K_2) = r
    \]

    \[
    \text{FOC}_{D_2} : \quad u'(C_1) = (1 + r) \beta u'(C_2)
    \]

    \[
    [\text{Algebra:}] \quad u'(C_1) = \beta u'(C_2) (F'(K_2) + 1) = (1 + r) \beta u'(C_2)
    \]

    \[
    \Rightarrow (F'(K_2) + 1) = (1 + r) \Rightarrow F'(K_2) = r
    \]

### 7.3 Outcome with No Commitment: Debt Ceiling

- Set up

  - If the country cannot commit to repayment, there is an additional constraint

    \[ R \leq \eta [F(K_2) + K_2] \]

    - Creditors will not lend to the point where a country will default in this nonstochastic model.
    - Calculating the debt ceiling is trickier than it appears

- Discretion over investment: Calculating the Debt Ceiling

  - To fully solve the system, need to work backwards in 3 steps:

    1. Determine \( \bar{D} \):

        * Calculate whether SOE chooses to default for a given \( D \), which determines the optimal amount of capital invested in the default and no-default cases, and the maximum level of debt at which the SOE is indifferent between defaulting and not defaulting \( \rightarrow \bar{D} \), where \( D \geq \bar{D} \Rightarrow \text{SOE defaults.} \)
2. Foreigners choose to lend: \( D \leq \bar{D} \)
   * Foreigners choose how much to lend to the SOE, taking into account that the SOE will default if the lend more than \( \bar{D} \), hence they chose not to lend more than \( \bar{D} \), hence constraint becomes \( D \leq \bar{D} \)

3. SOE maximizes no default utility s.t. constraint that \( D \leq \bar{D} \)
   - Calculating
     1. Solve for \( \bar{D} \) as the largest \( D \) such that the country would choose to repay as opposed to default
        \[
        \max_{K_2} u(Y_1 + D_2 - K_2) + \beta u(F(K_2) + K_2 - \min\{(1+r)D_2, \eta(F(K_2 + K_2))\})
        \]
        (a) Derivation is in the textbook, pp. 381-387. Functional form assumed, and then look at \( U_N - U_D = 0 \) as a function of \( \left(\frac{D_2}{Y_1}\right) \) [debt:output ratio at end of period 1], to get indifference condition for \( \bar{D} \).
     2. Foreign lenders impose \( D \leq \bar{D} \)
     3. Constrained maximization by SOE:
        \[
        \max_{K_2, D_2} u(Y_1 + D_2 - K_2) + \beta u(F(K_2) + K_2 - (1+r)D_2) \text{ s.t. } D \leq \bar{D}
        \]
        s.t. \( D \leq \bar{D} \)

We can see from this graph and its discussion that we get two of our main results:

- **1: Can sustain debt in equilibrium**
  - Because of the direct sanctions, the country can now borrow up to the limit of the sanction. Like being able to post collateral

- **2: Knife-edge equilibrium**
  - In equilibrium (if the condition is binding and the direct sanction is less than the SOE would like to borrow), they borrow up to the point where they will be indifferent between paying and not paying in utility terms. But this means even a small increase in debt would push them to default
- If they were at (B) instead of (A), this implies dramatically lower levels of investment. This is because they are choosing to default, and hence don’t keep 100% of the excess profits next period, and will therefore choose to invest less.

- Were the lenders to allow the country’s borrowing to rise above \( \bar{D} \), the point at which it is indifferent between default and repayment, investment would crash discontinuously as the sovereign moved to reduce its vulnerability to the anticipated creditor sanctions.

- An increase in \( D \) causes both \( GNP^D \) and \( GNP^N \) to shift upwards, but there is a bigger shift in \( GNP^D \) because it is flatter.

- Graphs: investment effects of growing debt
  - (A) is the initial utility maximization point, where full repayment is optimal
  - (B): an increase in debt. country indifferent between repayment and default \((U^N = U^D)\). We take tie goes to repayment
  - (C): an \( \epsilon \) increase in debt \( \Rightarrow \) country in default, and causes sharp investment decline from \( K_B \) to \( K_C \)

- Results
  - Discontinuity in investment
  - Important point:
    * At \( B \), \( K^B \) implies that repayment is strictly less costly than default \( \eta [F(K^B) + K^B] > (1 + r) \bar{D} \).
      - [we can see this because \( B \) is to the left of the kink where \( \eta [F(\bar{K}) + \bar{K}] = (1 + r) \bar{D} \).
    * Hence, creditor sanctions appear sufficient to discourage default, yet a penny more of debt will trigger default.
The reason is the catastrophic investment decline that the extra penny of borrowing sets off.

### 7.4 Debt Overhang

Some in 1980s argued that large debt caused slow growth (as opposed to the other way around). The proposed channel is that a legacy of foreign debt effectively generates a tax on investment.

**Set Up**
- Assume a small country that inherits debt $D$
  - Still assume either repay $(1 + r) D = D$ (since $r = 0$) or default and creditor gets $\eta AF(K_2)$
- Normalize $\beta = 1$ and $r = 0$
- Linear Utility: $U_1 = C_1 + E_1(C_2)$
- Income:
  - Period 1: $C_1 = Y_1 - K_2$ [endowment]
  - Period 2: $C_2 = AF(K_2) - \min \{\eta AF(K_2), D\}$ [production technology, minus minimum repayment]
- Productivity $A$
  - Assume $A$ is a random variable with mean $E_1(A) = 1$, distributed over $[A, \bar{A}]$ with pdf $\pi(A)$.

**Maximization Problem**
\[
\max_{K_2} C_1 + E_1(C_2) = \max_{K_2} Y_1 - K_2 + E_1(AF(K_2) - \min \{\eta AF(K_2), D\})
\]
\[
= \max_{K_2} Y_1 - K_2 + F(K_2) - V(D, K_2)
\]
- where $V(D, K_2)$ is the payment that creditors actually expect to receive on date 2.
  - (This is the debt’s market value)

**$V(D, K_2)$**
- The borrower will default when $\eta AF(K_2) < D$.
  - This is rewritten: $A < \frac{D}{\eta F(K_2)}$.
- This gives us:
\[
V(D, K_2) = \eta F(K_2) \int_{\frac{D}{\eta F(K_2)}}^{\bar{A}} A \pi(A) dA + D \int_{\frac{D}{\eta F(K_2)}}^{\bar{A}} \pi(A) dA
\]
* which we can note is:

\[
V(D, K_2) = \eta F(K_2) \int_{\Delta}^{D(K_2)} A\pi(A)\,dA + D \int_{\frac{D}{\hat{D}F(K_2)}}^{\hat{A}} \pi(A)\,dA
\]

payment in default

payment in non-default

- **Optimization:**
  - Solving above problem \( \max_{K_2} Y_1 - K_2 + F(K_2) - V(D, K_2) \) using expression for \( V(D, K_2) \) yields:

\[
F'(K_2) \left[ 1 - \eta \int_{\Delta}^{D(K_2)} A\pi(A)\,dA \right] = 1
\]

* This condition states that the debtor will invest up to a point where (LHS:) expected marginal product of investment, net expected additional penalty payments to creditors, equals the current consumption cost of investing (that is, 1).

- **Result:**

\[
K'(D) < 0
\]

- If 2nd order conditions hold, capital investment is decreasing in inherited (or “overhang”) debt.

### 7.5 Debt Laffer Curve

*Can creditors gain by forgiving debt?*

- Take the expression for \( V(D, K_2) \) and totally differentiate with respect to \( D \)
  - (taking into account the dependence of \( D \) on \( K_2 \))

\[
\frac{dV}{dD}(D, K_2) = \left[ \eta F(K_2) \int_{\Delta}^{D(K_2)} A\pi(A)\,dA \right] K'(D) + \int_{\frac{D}{\hat{D}F(K_2)}}^{\hat{A}} \pi(A)\,dA
\]

- 2nd term is probability of full repayment; clearly non-negative
- 1st term is negative, since a higher face value of debt depresses investment and thus makes default more probable

- **This curve can become decreasing at high enough \( D \)**

- Since second term is negative because higher value of \( D \) depresses investment
7.6 Debt Buybacks

- **Motivation:** Secondary-market debt for developing countries had very low prices during 1980s. Buybacks seemed an attractive option (effectively cancel a dollar of debt by paying much less than a dollar). However, this model suggests that when buybacks are not accompanied by negotiated creditor concessions, they are likely to harm a highly indebted country while helping its creditors.

- **Market price of debt:**

  \[ p = \frac{V(D, K_2)}{D} \]

  - The market price of the country’s debt on date 1, \( p \), is the ratio of total expected repayments to total face value outstanding

- **We assume buybacks are publicized before they are executed**

  - This is important because it means the country will have to pay the *post-buyback* price for every unit of debt repurchased

- **Debt Buyback: Expected Utility**

  - Country uses some of its first-period endowment \( Y_1 \) to buy back an amount \( Q \) of its debt on date 1 at a market price \( p \)
    
    * (where \( p \) is the post-buyback price and incorporates rational expectations of the buyback’s investment effect)

  - Country’s Expected Utility after Buyback:

  \[
  U_1 = Y_1 - pQ - K_2 + F(K_2) - V(D - Q, K_2)
  \]

  \[
  = Y_1 - \frac{V(D - Q, K_2)}{D - Q} Q - K(D - Q) + F(K(D - Q)) - V(D - Q, K(D - Q))
  \]
The last term reflects the optimal dependence of investment on debt implicit in the function $K(D)$.

- **Effects of a Small Buyback**
  
  - Differentiating $U_1$ with respect to $Q$, at $Q = 0$:
    \[
    \frac{dU_1}{DQ}_{|Q=0} = - \left\{ F'(K(D)) - 1 \right\} K'(D) - \left\{ \frac{V(D,K(D))}{D} - \frac{dV(D,K(D))}{dD} \right\}
    \]
  
  - The first term is an unambiguous gain for the country ($F'(K(D)) > 1$, $K'(D) < 0$).
  - Second term is a net loss: this term is the difference between the debt’s average price (what the country pays to repurchase) and the marginal price (the reduction in total expected future debt payments)
    - *Marginal price is less than average price* (seen from concavity of Debt Laffer curve)
      [Note that this loss to the country is a pure gain to creditors]
  
  - The equation reduces to:
    \[
    \frac{dU_1}{DQ}_{|Q=0} = - \frac{\eta F(K_2)}{D} \int_{\Delta}^{\frac{D}{\eta F(K_2)}} A \pi(A) dA < 0
    \]
  
  - The country’s payment goes entirely to expected payments to creditors; on balance the country therefore must *lose* when it repurchases discounted debt.

- **Example: Bolivia Debt Buyback**:
  
  - March 1988: Bolivia spends $34 million ($308 million \times 0.11$/dollar) on the debt reduction.
  - However, the total *market* value of Bolivia’s debt only fell from $40.2 million-$39.8 million = $400,000.
    - Bolivia thus recouped less than 1.2% of the money spent.
  - What happened? Model suggests that with marginal value $V$ near zero, halving $D$ nearly doubled $p$.

<table>
<thead>
<tr>
<th>Table 6.2</th>
<th>Bolivia’s March 1988 Debt Buyback</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prebuyback</strong></td>
<td><strong>Postbuyback</strong></td>
</tr>
<tr>
<td>Face value of debt, $D$</td>
<td>$670$ million</td>
</tr>
<tr>
<td>Price, $p$ (fraction of a dollar)</td>
<td>$0.06$</td>
</tr>
<tr>
<td>Total market value, $p \times D$</td>
<td>$40.2$ million</td>
</tr>
</tbody>
</table>

8 Speculative Attacks I

[Much of this content pulled from Pascal Noel’s notes.]
8.0 Background

Talk about exchange rates (then exchange rate taxes)
Monetary Model exchange rates (23:00)
There’s some accounting in this model that gets swept under rug in NKM.

Money Demand Equation:
- Domestic Country:
  \[
  m - p = -\lambda i + \phi y
  \]
  
  $m$ is log of money
  $p$ is log of price level
  $i$ is level of interest rate
  $y$ is log output

- Foreign Country:
  \[
  m^* - p^* = -\lambda i^* + y^*
  \]

Exchange Rate:
\[
(m - m^*) - (p - p^*) = -\lambda (i - i^*) + \phi (y - y^*)
\]

Exchange Rate Definition: [Purchasing Power Parity]
\[
p = ep^*
\]

$p^*$ = foreign price, $p =$ home price level

$e$ = foreign currency

Nominal Interest Rate:
\[
i - i^* = E_t (e_{t+1} - e_t)
\]

This doesn’t hold empirically. But we’ll assume it does. [31:00]

Exchange Rate:
- Substituting for $i - i^*$:
  \[
  (m - m^*) - (p - p^*) = -E_t (e_{t+1} - e_t) + \phi (y - y^*)
  \]
  
  [Simplest version of monetary model treats output as constant]

\[
\dot{m} = \mu \\
e_t = m_\ell + \eta \mu \\
m - \dot{e} = \mu
\]
8.1 Model
Speculative attacks can happen even though everyone expects it, because “authorities are doing something stupid”

- Set up
  - Look at a completely stochastic world
  - Assume small country that faces fixed world interest rate $r^*$ which we normalize to 0
  - We also assume purchasing power parity, perfectly flexible domestic prices, and normalize the log of world prices $p^*$ to zero
    - PPP (law of one price): $p_t = e_t + p_t^*$
    - $p_t^* = 0$
      - $⇒ p_t = e_t$
  - Assume Cagan-style money demand equation: $m_t - p_t = -\eta i_t + y_t$
    - Where demand for money balances $m_t$ is a function of interest rate $i_t$ and consumption $y_t$
    - UIP (no arbitrage condition): $i_t - i_t^* = \mathbb{E}(\dot{e}_t)$.
      - Normalizing $i_t^*$ to 0, $i_t = \mathbb{E}(\dot{e}_t)$

- Substituting, we get

- Money Demand Equation

$$m_t - e_t = -\eta \dot{e}_t$$

$m$ \hspace{1cm} log of money supply
$e$ \hspace{1cm} log of exchange rate
$\dot{e}$ \hspace{1cm} expected rate of change of exchange rate

- As long as the exchange rates fixed

$$\bar{m} = \bar{e}$$

- Simplified central bank balance sheet given by:

$$M_t = B_{H,t} + \bar{e} B_{F,t}$$

- where variables are in levels (\bar{e} is the level of the fixed exchange rate)
- Can think of this as

  liabilities=assets
  - “Currency in private hands at time $t$ must have been issued by the central bank to buy domestic or foreign currency debt”

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8.2 Domestic credit policy

• Assume that the central government relentlessly expands domestic credit at rate $\mu$ and will not sacrifice this objective for any reason, including exchange rate instability.

• Then domestic credit creation is given by:

$$\frac{\dot{B}_H}{B_H} = \dot{b}_H = \mu$$

• By the equation for the central bank’s balance sheet, we see that this growth in domestic bonds must correspond to selling foreign bonds:

$$\bar{\varepsilon} \dot{B}_F = -\dot{B}_H$$

– This cannot be sustained forever, as foreign reserves will run out.

From Pascal:

• This is what is going to cause an unsustainable situation that precipitates an inevitable attack

• The central bank will have two mandates:

  – Monetize the deficit
    * First, it has to monetize part of the deficit incurred by the government. The government runs an exogenously determined deficit and the central bank is required to monetize part of that fiscal deficit by printing money and giving that money to the government in return for domestic government bonds.

  – Maintain the peg
    * Second, it has to intervene in the money market to maintain the currency peg. As we see in [], this will force it to keep the money supply constant.

• Specifically, we will assume that the government forces the central bank to absorb a relentlessly expanding quantity of domestic bonds, so that holdings expand at rate $\mu$ (remember time derivative of a log is a growth rate of a level variable):

$$\frac{\dot{B}_H}{B_H} = \dot{b}_H = \mu$$

• This requires the CB to print money and buy domestic bonds. This would normally expand the money supply. But in order to keep the peg, the CB needs to maintain a fixed money supply. We can see by looking at its balance sheet [] that it must sell its holdings of foreign debt to soak up the money it just released into the economy. In other words:

$$\dot{M}_t = \dot{B}_{H,t} + \bar{\varepsilon} \dot{B}_{F,t}$$

$$\dot{M} = 0 \Rightarrow \bar{\varepsilon} \dot{B}_F = -\dot{B}_H$$

– So the CB must be reducing its reserves of foreign bonds at an ever increasing pace. Clearly, this is unsustainable because they will eventually run out of reserves.
8.3 Timing the attack

- The previous section showed that if we just let this go on, eventually the CB will run out of foreign reserves. But we will see here is that it can never be rational for holders of domestic debt to allow them to get all the way there.

  - **There would be a discrete jump in e if CB allowed to run out of reserves**

- The key insight of the model is that at some point, *remaining reserves must be drained in a sudden speculative attack*

  - Otherwise there would be a discrete change in the rate of depreciation of the exchange rate (from 0 to \(\mu\)), meaning an anticipated jump in the exchange rate.
    * We will see the equilibrium diagrammatically.

- To solve analytically, helpful to define *shadow exchange rate*

  - Shadow exchange rate is the rate obtained if the attack happened today, the central bank lost all its reserves, but the central government kept putting out domestic credit at rate \(\mu\):
    \[ \tilde{e}_t = b_{H,t} + \eta\mu \]
    * “What the exchange rate if everything was cashed out”

- **Derivation & Intuition [Pascal]:**
  * We know at some point the CB must run out of foreign reserves and the peg must be abandoned.
  * At that point, expected exchange rate appreciation will be at the rate \(\mu\).
  * Hence value of the exchange rate will be value required to clear the money market when these conditions are:
    - Normal Money demand: \(m_t - e_t = -\eta\dot{e}_t \Rightarrow e_t = \eta\dot{e}_t + m_t\)
    - Money supply once \(b_F = 0\): \(m_t = b_{H,t}\)
    - Expected depreciation once \(b_F = 0\) and peg abandoned (assuming no speculative bubbles): \(\dot{e}_t = \mu\)
  * Combining these equations to get the shadow rate
    - (exchange rate that would prevail if \(b_F = 0\) and the currency was floating)
    \[ \tilde{e}_t = b_{H,t} + \eta\mu \]

- **Graphically: [Pascal]**
  * We can graph shadow vs. fixed exchange rate
  * **Attack must occur at \(T\) when \(\bar{e} = \tilde{e}_T\)**
    * We have already proven that it can’t happen after \(T\). If it did, then at time \(T + X\) there would be an anticipated discrete jump in the exchange rate to its shadow value, and that can’t happen because speculators attacking just before \(T+X\) would get an instantaneous rate of capital gain.
But similarly, the attack can't happen before $T$. If it occurred at $T - X$, then the currency would have to appreciate ($e \downarrow$) to clear the market once it floated, which would mean that speculators who bought up the CB's foreign bonds would make a capital loss. (They are essentially trading domestic currency for foreign currency, so like $\$5$ for 1 euro, but once they wipe out the CB, the exchange rate falls and now their 1 euro only buys $\$3$, so they've lost money).
**Finding Expression for T**

- We can follow the evolution of domestic credit as:
  \[ b_{H,t} = b_{H,0} + \mu t \]
  - [Since \( B_{H,t} = B_{H,0}e^{\mu t} \)]

- The attack must occur when \( \bar{e} = \bar{e} \), or in other words when
  \[ \bar{e} = b_{H,0} + \mu T + \eta \mu \]
  - Since \( \bar{e} = \bar{e}_t = b_{H,t} + \eta \mu \)
  - [This is the exact \( T \) when \( \bar{e}_T = \bar{e} \)]

- We can solve for the timing of the attack as:
  \[ T = \frac{\bar{e} - b_{H,0} - \eta \mu}{\mu} \]
  - We can further manipulate this expression to yield:
  \[ T = \frac{\ln (B_{H,0} + \bar{e}B_{F,0}) - b_{H,0} - \eta \mu}{\mu} \]
– [Since from above: \( M_t = B_{H,t} + \bar{\varepsilon}B_{F,t} \), and \( \bar{\varepsilon} = \bar{m} = \ln (B_{H,0} + \bar{\varepsilon}B_{F,0}) \)]

This analysis depends critically on the (very dubious) assumption that there is a link between the post crisis exchange rate and the fundamentals. But in practice, this link is very hard to detect.

- **Implications:**
  - The larger the initial foreign debt holdings \( B_{F,0} \), the longer the fixed rate regime will last
  - The faster the CB is monetizing debt \( \mu \), the quicker the attack will happen
    * [since running out of foreign bonds faster]

### 8.4 Bubbles

- **A speculative bubble could make the attack happen earlier, and could even bring down a viable fixed-rate regime** [Pascal]
  - Bring in speculative bubbles in the following way:
    * Before, had ruled out speculative bubbles: \( e_t = m_t + \eta \mu' \)
    * Allowing for speculative bubbles: \( e_t = m_t + \eta \mu' + b_0 e^{\frac{t}{\eta}} \)
      * \( b_0 \) is the gap between the initial price level and its fundamental, no-bubble value (in a sense, a measure of how big the bubble is)
  - Suppose there is a bubble in the post-crisis exchange rate, then:
    \[ \tilde{e}_t = \ln (B_{H,t}) + \frac{\eta \mu + b_T e^{\frac{(t-T)}{\eta}}}{\mu} \]
  - And the **timing of the attack changes as well**:
    \[ T = \frac{\ln (B_{H,0} + B_{F,0}) - b_{H,0} - \eta \mu - b_T}{\mu} \]
    - Remarkably, bubbles in shadow floating rates spill back over into fixed period.
    * Idea that fixed exchange rate regimes are immune to bubbles is not true.
    * \( b_T \) is an arbitrary constant that is a function of the size of the bubble

### 8.5 Limitations

- **Theoretical:** It is asymmetric in treatment of public and private behavior
- Private sector is portrayed as completely rational and capable of fully anticipating events
  - But the monetary and fiscal authorities operate by mechanical rules without any regard for longterm sustainability of the exchange rate
Why wouldn’t a rational government want to smooth its consumption of foreign exchange reserves over a long horizon? Here it effectively “consumes” them all during the finite life of the fixed regime

**Empirical**: Insolvency doesn’t seem to be the ultimate force underlying the collapse of fixed exchange rates

- Virtually all major countries that suffered attacks on their exchange rates during the early 1990s had enough foreign exchange reserves and gold to buy back most of their monetary base

  I think this means that, if you look at the CB balance sheet, they could continue to hold their promise of accepting domestic currency and paying out foreign bonds (or gold) and still not run out. But wouldn’t this still mean that the money supply was falling? I guess they could be buying other domestic assets to “sterilize” these sales.

Empirically, a number of things missing from this model: [1:06]
A lot of countries have a lot of reserves, and still go off fixed exchange rate
Huge collapse in europe in early 1990s
“You don’t just run out of money, this ends in a ball of fire”

9 **Speculative Attacks & Multiple Equilibria (Obstfeld, 1996)**

*These notes give an examples of how an apparently stable fixed exchange rate regime can be vulnerable to multiple equilibria*

- Here we show how speculative attacks can arise
  - Before, we treated the government’s behavior as mechanical
  - Here, we have model with speculative attacks where government’s objectives are spelled out explicitly

- Implications
Fixed exchange rate commitments may do little to improve the credibility of governments with otherwise strong incentives to inflate.

* Indeed, they may give rise to multiple equilibria and the possibility of currency crises with self-fulfilling element.

**Model**

- **Set up**

  - In open economy where PPP holds, so \( P = \varepsilon P^* \), or in log notation with \( P^* \) normalized to 1, \( e = p \).

- **Government Loss Function**

  \[
  \mathcal{L}_t = (y_t - \bar{y})^2 + \chi \pi_t^2 + C(\pi_t)
  \]

- **Notation:**
  
  * \( \pi_t \): corresponds to \( e_t - e_{t-1} \), the realized rate of currency depreciation, as well as to the inflation rate.
  
  * Under a fixed exchange rate, \( \pi_t = 0 \).

  * \( C(\pi) \): verbally, can be viewed as the political cost of reneging on a promise to fix the exchange rate.

  - If \( \pi > 0 \) (\( e \) increases), \( C(\pi_t) = \bar{c} \).
  
  - If \( \pi < 0 \) (\( e \) decreases), \( C(\pi_t) = c \).
  
  - \( C(0) = 0 \).

  * \( \bar{y} \) is the output target.

**Intuition:** Fixed cost to breaking the exchange rate may help reduce expected inflation but may also leave the economy open to speculative attacks.

- The government has adopted a “fixed but adjustable” exchange rate.
  
  * It has placed itself in a position such that any change in \( e \) leads to a cost.

- **Output:**

  - Inflation is described by an expectations-augmented Phillips curve:

    \[
    y = \bar{y} + (\pi - \pi^e) - z
    \]

  - “Output net of the natural rate (\( \bar{y} \)) depends on unexpected currency depreciation and a random supply shock.”
  
  - \( z \) is random supply shock.

  * As in Barro Gordon, central bank has “2nd mover advantage” and takes \( \pi^e \) as given when it sets \( \pi \).

**Equilibria:**

- We focus on the equilibria of the one-shot game.

*First: Ignoring \( C(\pi) \) term.*
1. Ignore $C(\pi)$ term.

(a) Government would choose:

$$\pi = \frac{k + \pi^e + z}{1 + \chi}$$

i. Note: $k = \bar{y} - \tilde{y}$

(b) Substituting optimal $\pi$ into output equation:

$$y = \bar{y} + \frac{k - \chi \pi^e - \chi z}{1 + \chi}$$

(c) Find Inflation Expectations [using (a)]

$$\pi_t^e = \frac{k}{\chi}$$

(d) Therefore, central bank utility (ignoring $C(\pi)$):

$$L^{FLEX} = \frac{\chi}{1 + \chi} (k + \pi^e + z)^2$$

2. On other hand, if exchange rate fixed, government utility will be:

$$L^{FIX} = (k + \pi^e + z)^2 > L^{FLEX}$$

(a) **Lose more under fixed exchange rate** (ignoring fixed cost of inflation)

*Derivation:*

1. Ignore $C(\pi)$ term.

(a) Government would choose:

i. Manipulating welfare loss function (substitution):

$$\L_t = (y_t - \bar{y})^2 + \chi \pi_t^2$$

A. Substituting in $y = \bar{y} + (\pi - \pi^e) - z$:

$$\L_t = (\bar{y} + (\pi_t - \pi^e) - z - \tilde{y})^2 + \chi \pi_t^2$$

B. Where $k = \bar{y} - \tilde{y}$

$$\L_t = ((\pi_t - \pi^e) - z - k)^2 + \chi \pi_t^2$$

ii. Optimize:

$$\max_{\pi_t} \L_t = \max_{\pi} ((\pi_t - \pi^e) - z - k)^2 + \chi \pi_t^2$$

A. FOC $\pi_t$:

$$2((\pi_t - \pi^e) - z - k) + 2\chi \pi_t = 0$$

$$\pi^e + z + k = \chi \pi_t + \pi_t$$
\[ \pi_t = \frac{k + \pi^e + z}{1 + \chi} \]

(b) Substituting into above “inflation augmented phillips curve”:

\[ y = \bar{y} + (\pi - \pi^e) - z \Rightarrow \]

\[ y = \bar{y} + \frac{k + \pi^e + z}{1 + \chi} - \pi^e - z = \bar{y} + \frac{k + \pi^e + z}{1 + \chi} - \frac{(1 + \chi)\pi^e}{1 + \chi} - \frac{(1 + \chi)z}{1 + \chi} \]

\[ y = \bar{y} + \frac{k - \chi\pi^e - \chi z}{1 + \chi} \]

(c) Find Inflation Expectations [using (a)]

i. Rational expectations:

\[ \pi_t^e = E_{t-1}[\pi_t] = E_{t-1}\left[ \frac{\pi_t^e + z_t + k}{1 + \chi} \right] \]

A. We we have that \( E[z_t] = 0 \)

\[ (1 + \chi) \pi_t^e = \pi_t^e + z_t + k \]

\[ \pi_t^e = \frac{k}{\chi} \]

(d) Therefore, central bank utility (ignoring \( C(\pi) \)):

i. Government loss function (and re-write):

\[ \mathcal{L}_t = (y_t - \bar{y})^2 + \chi \pi_t^2 = (y_t - \bar{y} - k)^2 + \chi \pi_t^2 \]

ii. Substituting in for \( y_t \) from (b): \( y = \bar{y} + \frac{k - \chi \pi^e - \chi z}{1 + \chi} \)

\[ \mathcal{L}_t = \left( \bar{y} + \frac{k - \chi \pi^e - \chi z}{1 + \chi} - \bar{y} - k \right)^2 + \chi \pi_t^2 = \left( \frac{k - \chi \pi^e - \chi z}{1 + \chi} - \frac{(1 + \chi)k}{1 + \chi} \right)^2 + \chi \pi_t^2 \]

\[ \mathcal{L}_t = \left( \frac{-\chi k - \chi \pi^e - \chi z}{1 + \chi} \right)^2 + \chi \pi_t^2 \]

iii. Substituting in for \( \pi_t \) from (a): \( \pi_t = \frac{k + \pi^e + z}{1 + \chi} \)

\[ \mathcal{L}_t = \left( \frac{-\chi k - \chi \pi^e - \chi z}{1 + \chi} \right)^2 + \chi \left( \frac{k + \pi^e + z}{1 + \chi} \right)^2 \]

iv. Factoring:

\[ \mathcal{L}_t = \left( \frac{-\chi \left( k + \pi^e + z \right)}{1 + \chi} \right)^2 + \chi \left( \frac{k + \pi^e + z}{1 + \chi} \right)^2 \]

\[ \mathcal{L}_t = \chi^2 \left( \frac{k + \pi^e + z}{1 + \chi} \right)^2 + \chi \left( \frac{k + \pi^e + z}{1 + \chi} \right)^2 \]

\[ \mathcal{L}_t = \frac{\chi^2}{(1 + \chi)^2} \left( k + \pi^e + z \right)^2 + \frac{\chi}{(1 + \chi)^2} \left( k + \pi^e + z \right)^2 \]
\[ L^\text{FLEX} = \frac{\chi}{1 + \chi} (k + \pi^e + z)^2 \]

2. On other hand, if exchange rate fixed, government utility will be:

(a) Government loss function, when \( \pi_t = 0 \) (since fixed exchange rate):
\[ L^\text{FIX} = (y_t - \bar{y})^2 + \chi \pi_t^2 \Rightarrow L^\text{FIX} = (y_t - \bar{y})^2 = (y_t - \bar{y} - k)^2 \]

(b) Substituting in for original output equation: \( y = \bar{y} + (\pi - \pi^e) - z \)

i. Re-write as \( y_t - \bar{y} = \pi_t - \pi^e - z \), which here is \( y_t - \bar{y} = -\pi^e - z \)
\[ L^\text{FIX} = (y_t - \bar{y} - k)^2 = (-\pi^e - z - k)^2 = (-1) (\pi^e + z + k))^2 = (k + \pi^e + z)^2 \]

ii. So we have our result:
\[ L^\text{FIX} = (k + \pi^e + z)^2 > L^\text{FLEX} \]

Now take into account fixed costs of currency realignment \( C(\pi) \):

1. Conditions for changing fixed exchange rate commitment

(a) Authorities will change exchange rate only when shock \( z \) is large enough. That is if:

i. \( L^\text{FIX} - L^\text{FLEX} > \bar{c} \) (in which case the currency is devalued)

ii. \( L^\text{FIX} - L^\text{FLEX} < \bar{c} \) (in which case the currency is revalued)

(b) For a given level of \( \pi^e \), can calculate \( \bar{z} \) and \( \bar{z} \):

i. \( \bar{z} = \sqrt{\frac{\chi}{1 + \chi}} - k - \pi^e \)

ii. \( \bar{z} = -\sqrt{\frac{\chi}{1 + \chi}} - k - \pi^e \)

[Algebra: simply use conditions above, substituting for \( L^\text{FIX} \), \( L^\text{FLEX} \)
\[ L^\text{FIX} + c > L^\text{FLEX} \Rightarrow (k + \pi^e + z)^2 + c > \frac{\chi}{1 + \chi} (k + \pi^e + z)^2 \]
\[ \bar{c} > (\frac{\chi}{1 + \chi} - 1) (k + \pi^e + \bar{z})^2 = \frac{1 - \chi}{1 + \chi} (k + \pi^e + \bar{z})^2 \]
\[ \bar{c} (1 - \chi) = (k + \pi^e + \bar{z})^2 \Rightarrow \sqrt{\bar{c} (1 - \chi)} = k + \pi^e + \bar{z} \]

(c) Hence for \( z \in [\bar{z}, \bar{z}] \), the fixed exchange rate is maintained

Calculate expected inflation to examine multiple equilibria:

1. The rational expectation of inflation (depreciation) \( \pi \) in the next period, given wage setters' expectations \( \pi^e \), is:
\[ E[\pi] = E(\pi|z < \bar{z}) Pr(z < \bar{z}) + E(\pi|z > \bar{z}) Pr(z > \bar{z}) \]

2. Inflation depends on whether \( z > \bar{z} \), but \( \bar{z} \) depends on \( \pi^e \)

(a) Ex:

i. If \( \pi^e \uparrow \Rightarrow \bar{z} \downarrow \) so \( Pr(z > \bar{z}) \uparrow, Pr(\pi) \uparrow \) [self fulfilling]

ii. If \( \pi^e \downarrow \Rightarrow \bar{z} \uparrow \) so \( Pr(z < \bar{z}) \uparrow, Pr(\pi) \uparrow \) [self fulfilling]
(b) Expected inflation $\pi^e$ enters here in determining the inflation rate the government chooses conditional on choosing to realign, and in determining the probability of a realignment.

(c) The fact that ex post inflation depends on $\pi^e$ in a potentially very complicated way gives rise to the possibility that there are multiple equilibrium expected inflation rates (under the "fixed but adjustable" exchange rate scheme)

3. Parametric example multiple equilibria:

$$E[\pi] = \frac{1}{1 + \chi} \left[ \left( 1 - \frac{\bar{z} - \bar{z}}{2Z} \right) (k + \pi^e) - \frac{\bar{z}^2 - \bar{z}^2}{4Z} \right]$$

(a) In equilibrium, expectations must be consistent with the model, so

$$\pi^e = E[\pi]$$

(b) In figure, can see that there are potentially 3 equilibria.

i. In the highest inflation equilibrium, $\pi^e = \frac{k}{\chi}$ as in the case where there is no cost to breaking the no inflation commitment.

ii. The other two equilibria have lower expected rates of inflation/depreciation.

$$\frac{dE\pi}{d\pi^e} = \begin{cases} \frac{1}{1 + \chi} \left[ \frac{1}{2} + \frac{1}{2Z} (k + \pi^e) \right] & (\text{for } z > -Z) \\ \frac{1}{1 + \chi} \left[ \frac{1}{2} + \frac{1}{2Z} (k + \pi^e) \right] & (\text{for } \bar{z} > -Z, \bar{z} > -Z) \\ \frac{1}{1 + \chi} \left[ \frac{1}{2} + \frac{1}{2Z} (k + \pi^e) \right] & (\text{for } \bar{z} = -Z) \end{cases}$$
10 Morris and Shin

PS3 Problem 2

• Intuition
  - If individual spectators have a small amount of private information, then equilibrium is unique.
  - Therefore, in this setting, can speak concretely of an attack occurring, which is not possible in a multiple equilibrium context.
  
  * Importantly, a concrete model of how information affects probability of a speculative attack allows one to look at how variables such as capital controls, increased transparency, affect the likelihood of an attack.

• Model: strategic game between speculators and a government
  - $\theta$ is the exchange rate fundamental. $\theta \in [0, 1]$. Higher $\theta$ means stronger fundamentals.

10.1 a) Equation for size of the attack $A$

a) Show that the size of the attack is given by $A = \int_0^1 I_i di = \Phi \left( \sqrt{\epsilon} \left( x^* - \theta \right) \right)$ and therefore the cutoff fundamental is given by $\theta^* = \Phi \left( \sqrt{\epsilon} \left( x^* - \theta^* \right) \right)$

10.1.1 Size of the Attack

1. LLN to get $A$ through $P(x < x^*|\theta)$

   (a) By the LLN, we have that the fraction of the population in $A$ will be given by the probability that $x_i < x^*$ given $\theta$. That is:

   $$A(\theta) = P(x < x^*|\theta)$$

   (b) Use the definition of $x_i$ to get the $P(\cdot)$ in terms of $\epsilon$

   Since $x_i = \theta + \epsilon$, we have that

   $$A(\theta) = P(x < x^*|\theta) = P(\theta + \epsilon < x^*|\theta)$$

   $$= P(\epsilon < x^* - \theta|\theta)$$

2. Use the normal distribution $Z-$ statistic to get a formula for $P(\epsilon < x^* - \theta|\theta)$:

   (a) $Z-$ statistic formula:

   We are given that $\epsilon \sim N(0, \sigma_{\epsilon}^2)$

   For a normal distribution, the probability that $\epsilon < x^* - \theta$ is given by

   $$= \Phi \left( \frac{(x^* - \theta) - 0}{\sigma_{\epsilon}} \right) = \Phi \left( \frac{x^* - \theta}{\sigma_{\epsilon}} \right)$$

   (Since $\Phi \left( \frac{Z-\mu(\epsilon)}{\sigma(\epsilon)} \right)$ is probability $\epsilon < Z$, where $\sigma(\epsilon)$ is root of the variance of $\epsilon$)
(b) Re-writing for $P(\varepsilon < x^* - \theta|\theta)$

Given that $\lambda_\varepsilon = \sigma_\varepsilon^{-2} \Rightarrow \sigma_\varepsilon = \lambda_\varepsilon^{-\frac{1}{2}}$ we get that:

$$\Phi \left( \frac{x^* - \theta}{\lambda_\varepsilon^{\frac{1}{2}}} \right) = \Phi \left( \sqrt{\lambda_\varepsilon} (x^* - \theta) \right)$$

(c) Therefore we conclude that the size of the attack is deterministic, given by the LLN

$$A(\theta) = P(\varepsilon < x^*|\theta) = \Phi \left( \sqrt{\lambda_\varepsilon} (x^* - \theta) \right)$$

10.1.2 Fundamental Cutoff ($\theta^*$)

1. Use the strict monotonicity of $A(\theta)$ in $\theta$ to find a unique $\theta^* = A(\theta^*)$

   (a) We note that $A(\theta)$ is falling strictly monotonically in $\theta$.
   (b) Hence $A(\theta) - \theta$ is falling strictly monotonically in $\theta$
   (c) We also note that $A(0) - 0 \geq 0$ and $A(1) - 1 \leq 1$
   (d) Hence there must be a unique $\theta^* \in [0, 1]$ such that $\theta^* = A(\theta^*)$, that is

$$\theta^* = \Phi \left( \sqrt{\lambda_\varepsilon} (x^* - \theta^*) \right)$$

10.2 b) Indifference Condition of Marginal Trader

10.2.1 Indifference Condition in Probability Terms

1. Write indifference condition for marginal individual at $x_i = x^*$

   Individual $i$ who gets the signal $x_i = x^*$ must be indifferent between attacking and not, so

$$\mathbb{E} \left[ u \left( 1, \{I(x_j(\theta))\}, \theta|x_i = x^*, \bar{\theta} \right) \right] = \mathbb{E} \left[ u \left( 0, \{I(x_j(\theta))\}, \theta|x_i = x^*, \bar{\theta} \right) \right]$$

2. Re-write in terms of probabilities and outcomes

   (a) Note that $u(0, \cdot, \cdot) = 0$

$$1 - c \Pr (A(\theta) > \theta|x, \bar{\theta}) - c \Pr (A(\theta) \leq \theta|x, \bar{\theta}) = 0$$

$$1 \cdot P(A(\theta) > \theta|x, \bar{\theta}) - c = 0$$

$$P(\theta < \theta^*|x, \bar{\theta}) - c = 0$$
10.2.2 Using Conditional Normal to Further Characterize Indifference Condition

1. Write the conditional distribution of $\theta|x, \bar{\theta}$, using the formulas given

(a) We are told that the Bayesian posterior expectation of $\theta$, given two normally distributed signals $a$ (with precision $\lambda_a$) and $b$ (with precision $\lambda_b$) is given by:

$$E(\theta|a, b) = a \frac{\lambda_a}{\lambda_a + \lambda_b} + b \frac{\lambda_b}{\lambda_a + \lambda_b}$$

and precision $\lambda_a + \lambda_b$

(b) This means that we are given that the formula for the distribution of an RV $\theta$ with

i. two normally distributed signals $a, b$

ii. precision of $a = \lambda_a$, precision of $b = \lambda_b$

$$\theta|a, b \sim N\left(a \frac{\lambda_a}{\lambda_a + \lambda_b} + b \frac{\lambda_b}{\lambda_a + \lambda_b}, \frac{1}{\lambda_a + \lambda_b}\right)$$

since precision $= \frac{1}{\text{variance}}$

(c) Hence have that the posterior distribution of $\theta$ given $x, \bar{\theta}$ is:

$$\theta|x, \bar{\theta} \sim N \left(x^* \frac{\lambda_x}{\lambda_x + \lambda_{\theta}} + \bar{\theta} \frac{\lambda_{\theta}}{\lambda_x + \lambda_{\theta}}, \frac{1}{\lambda_x + \lambda_{\theta}}\right)$$

2. Use the above to re-write ("update") the probability that $P(\theta < \theta^*|x, \bar{\theta})$

(a) (Since again, $\Phi\left(\frac{Z - \mu(\theta|x, \bar{\theta})}{\sigma(\theta|x, \bar{\theta})}\right)$ is probability $\theta < Z$, where $\sigma(\theta|x, \bar{\theta})$ is $\sqrt{\text{Var}}$ of $\theta|x, \bar{\theta}$)

$$P(\theta < \theta^*|x, \bar{\theta}) =$$

$$\Phi\left(\frac{\theta^* - \left(x^* \frac{\lambda_x}{\lambda_x + \lambda_{\theta}} + \bar{\theta} \frac{\lambda_{\theta}}{\lambda_x + \lambda_{\theta}}\right)}{\sqrt{\lambda_x + \lambda_{\theta}}}\right)$$

(b) Simplifying

$$= \Phi\left(\sqrt{\lambda_x + \lambda_{\theta}} \left(\theta^* - \left(x^* \frac{\lambda_x}{\lambda_x + \lambda_{\theta}} + \bar{\theta} \frac{\lambda_{\theta}}{\lambda_x + \lambda_{\theta}}\right)\right)\right)$$

3. Rearrange to form requested

(a) Rearranging above

$$= 1 - \Phi\left(\sqrt{\lambda_x + \lambda_{\theta}} \left(x^* \frac{\lambda_x}{\lambda_x + \lambda_{\theta}} + \bar{\theta} \frac{\lambda_{\theta}}{\lambda_x + \lambda_{\theta}} - \theta^*\right)\right)$$

(b) Plug into indifference condition:

$$P(\theta < \theta^*|x, \theta) - c = 0$$

$$1 - \Phi\left(\sqrt{\lambda_x + \lambda_{\theta}} \left(x^* \frac{\lambda_x}{\lambda_x + \lambda_{\theta}} + \bar{\theta} \frac{\lambda_{\theta}}{\lambda_x + \lambda_{\theta}} - \theta^*\right)\right) - c = 0$$
\[ 1 - c = \Phi \left( \sqrt{\lambda_x + \lambda_\theta} \left( x^* \frac{\lambda_x}{\lambda_x + \lambda_\theta} + \bar{\theta} \frac{\lambda_\theta}{\lambda_x + \lambda_\theta} - \theta^* \right) \right) \]

as desired.

10.3 c) Solve expression for \( \theta^* \) and \( x^* \); Characterize Equilibrium

10.3.1 Solve for \( x^* \), then substitute for expression of \( \theta^* \):

1. Solve for \( x^* \) with answer from a)

   \[ \theta^* = \Phi \left( \sqrt{\lambda_x} (x^* - \theta^*) \right) \]

   \[ \Phi^{-1}(\theta^*) = \sqrt{\lambda_x} (x^* - \theta^*) \]

   \[ \frac{\Phi^{-1}(\theta^*)}{\sqrt{\lambda_x}} + \theta^* = x^* \]

2. Find expression for \( \theta^* \) only:

   (a) Manipulate expression from part b):

   \[ 1 - c = \Phi \left( \sqrt{\lambda_x + \lambda_\theta} \left( x^* \frac{\lambda_x}{\lambda_x + \lambda_\theta} + \bar{\theta} \frac{\lambda_\theta}{\lambda_x + \lambda_\theta} - \theta^* \right) \right) \]

   \[ \Phi^{-1} (1 - c) = \sqrt{\lambda_x + \lambda_\theta} \left( x^* \frac{\lambda_x}{\lambda_x + \lambda_\theta} + \bar{\theta} \frac{\lambda_\theta}{\lambda_x + \lambda_\theta} - \theta^* \right) \]

   \[ \Phi^{-1} (1 - c) = \frac{\sqrt{\lambda_x + \lambda_\theta}}{\lambda_x + \lambda_\theta} (x^* \lambda_x + \bar{\theta} \lambda_\theta - \theta^* (\lambda_x + \lambda_\theta)) \]

   (b) Plugging in expression for \( x^* \):

   i. From (1) we have that: \( \frac{\Phi^{-1}(\theta^*)}{\sqrt{\lambda_x}} + \theta^* = x^* \)

   \[ \Phi^{-1} (1 - c) = \frac{\sqrt{\lambda_x + \lambda_\theta}}{\lambda_x + \lambda_\theta} \left( \left( \frac{\Phi^{-1}(\theta^*)}{\sqrt{\lambda_x}} + \theta^* \right) \lambda_x + \bar{\theta} \lambda_\theta - \theta^* (\lambda_x + \lambda_\theta) \right) \]

   ii. Simplifying

   \[ \Phi^{-1} (1 - c) = \frac{1}{\sqrt{\lambda_x + \lambda_\theta}} \left( \Phi^{-1}(\theta^*) \sqrt{\lambda_x} + \theta^* \lambda_x + \bar{\theta} \lambda_\theta - \theta^* (\lambda_x + \lambda_\theta) \right) \]

   \[ \Phi^{-1} (1 - c) = \frac{1}{\sqrt{\lambda_x + \lambda_\theta}} \left( \Phi^{-1}(\theta^*) \sqrt{\lambda_x} - \theta^* \lambda_\theta \right) \]
(c) Solving for $\theta^*$ on RHS, with constants on LHS:
(This allows us to characterize functional form of $\theta^*$)

$$
\Phi^{-1} \left( 1 - c \right) \sqrt{\lambda_c + \lambda_\theta - \theta \lambda_\theta} = \Phi^{-1} \left( \theta^* \right) \sqrt{\lambda_c - \theta^* \lambda_\theta}
$$

$$
\frac{\Phi^{-1} \left( 1 - c \right) \sqrt{\lambda_c + \lambda_\theta - \theta \lambda_\theta}}{\sqrt{\lambda_c}} = \Phi^{-1} \left( \theta^* \right) - \theta^* \frac{\lambda_\theta}{\sqrt{\lambda_c}}
$$

i. We see that the LHS is constant, so we can re-write with $\psi = \frac{\Phi^{-1} \left( 1 - c \right) \sqrt{\lambda_c + \lambda_\theta - \theta \lambda_\theta}}{\sqrt{\lambda_c}}$:

$$
\psi = \Phi^{-1} \left( \theta^* \right) - \theta^* \frac{\lambda_\theta}{\sqrt{\lambda_c}}
$$

10.3.2 Interpreting Equilibrium

1. Given that $\Phi^{-1} \left( \theta^* \right)$ is the inverse of the normal cdf, and $-\theta^* \frac{\lambda_\theta}{\sqrt{\lambda_c}}$ is a decreasing linear function in $\theta^*$, $\theta^*$ will have several solutions for a given $\psi$ for sufficiently large $\lambda_\theta$.

2. As we take $\lambda_c \to \infty$, we see that the function will go to $\psi \to \Phi^{-1} \left( \theta^* \right)$. This will have a unique solution for any $\theta^*$. Hence we see that as the precision of private information increases, we get uniqueness of the equilibrium. We will get the same result if we increase the noise in the prior on $\theta$ (≈ public signal), taking $\lambda_\theta \to 0$.

Appendix: Math Reminders

Covariance properties:

- $ck \cdot \text{Cov} \left( X, Y \right) = \text{Cov} \left( cX, kY \right)$
  - Multiplying a random variable by a constant multiplies the covariance by that constant

- $\text{Cov} \left( X, Y \right) = \text{Cov} \left( X + c, Y + k \right)$
  - Adding a constant to either or both random variables does not change their covariances

Limits

Leibniz’s Rule: Derivative of a Definite Integral:

$$
\frac{d}{dx} \left[ \int_{a(x)}^{b(x)} f \left( x \right) \right] = f \left( b \right) \frac{da}{dx} - f \left( a \right) \frac{db}{dx} + \int_{a}^{b} f' \left( x \right) dx
$$

L'Hôpital’s Rule

1. If

$$
\lim_{x \to c} f \left( x \right) = \lim_{x \to c} g \left( x \right) = 0 \text{ or } \pm \infty
$$

(a) [they have to be the same] and $\lim_{x \to c} \frac{f' \left( x \right)}{g' \left( x \right)}$ exists, and $g' \left( x \right) \neq 0 \forall x$ in $I$ with $x \neq c$
2. Then

\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \]

**Infinite Sum**

If \( |r| < 1 \), then:

\[ \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r} \]