Formal Moduli Problems and Partition Lie Algebras

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Overview. If k is a field of characteristic zero, a theorem of Lurie–Pridham asserts an equivalence between formal moduli problems and d.g. Lie algebras over k. We generalise this equivalence to arbitrary fields by using partition Lie algebras. These new gadgets are intimately related to the equivariant topology of the partition complex, which allows us to access the operations acting on their homotopy groups.

An introduction to formal moduli problems. In order to study a given kind of algebro–geometric object over a ground field k in families (e.g. elliptic curves or GL_n-bundles), it is desirable to construct a representing geometric object X satisfying the following informal identity for all k-algebras R:

$$\mathrm{Map}(\mathrm{Spec}(R), X) \simeq \{ \ \mathrm{Spec}(R)\text{-families of objects of the given kind } \}.$$

Usually, we cannot find a variety or scheme with this property due to the presence of automorphisms. This obstacle can be circumvented by passing to *stacks*, i.e. functors $X : \{\text{Commutative } k\text{-algebras}\} \to \{\text{Groupoids}\}$ satisfying suitable geometricity conditions. By definition, we have $X(R) = \mathrm{Map}(\mathrm{Spec}(R), X)$.

Recent advances in distinct branches of mathematics (e.g. [3],[6],[8]) have highlighted the importance of "homotopical enhancements" of algebraic geometry. Following Toën–Vezzosi [10] and Lurie [5][7], one can proceed in two ways: derived algebraic geometry replaces commutative k-algebras with *simplicial* commutative k-algebras, whereas spectral algebraic geometry is based on connective $\mathbb{E}_\infty$-k-algebras. The former theory seems more suitable for algebro–geometric applications, whereas the latter applies in homotopical contexts. If $\mathrm{char}(k) = 0$, the two theories agree. We shall focus on the derived case, but will comment on how our results can be modified to apply in the spectral setting. Families of derived algebro–geometric objects of a given kind can often be represented by *derived stacks*, i.e. functors $X : \mathrm{SCR}_k \to \mathcal{S}$, from the ∞-category of simplicial commutative k-algebras to the ∞-category $\mathcal{S}$ of spaces, satisfying suitable geometricity conditions.

The formal neighbourhood of a k-valued point $x \in X(k)$ in a derived stack X is then described by the functor $\mathrm{SCR}_k^{\mathrm{art}} \to \mathcal{S}$ given by $R \mapsto X(R) \times^h_{X(k)} \{x\}$. Here, $\mathrm{SCR}_k^{\mathrm{art}}$ denotes the ∞-category of all $A \in \mathrm{SCR}_k$ such that $\pi_0(A)$ is local Artinian with residue field k and $\dim_k(\pi_*(A)) < \infty$. If X represents some family of derived algebro–geometric objects, then a point $x \in X(k)$ corresponds to a specific object defined over $\mathrm{Spec}(k)$, and $X^\wedge_x$ is the space of its infinitesimal deformations. In sufficiently geometric situations, the functor $X^\wedge_x$ satisfies the following conditions:

Definition 1. A *formal moduli problem* is a functor $X : \mathrm{SCR}_k^{\mathrm{art}} \to \mathcal{S}$ such that $X(k) \simeq *$ and whenever $A \simeq B \times^h_D C$ is a pullback in $\mathrm{SCR}_k^{\mathrm{art}}$ with $\pi_0(B) \twoheadrightarrow \pi_0(D)$, $\pi_0(C) \twoheadrightarrow \pi_0(D)$ surjective, applying X gives a pullback $X(A) \simeq X(B) \times^h_{X(D)} X(C)$. Write $\mathrm{Moduli}_k \subset \mathrm{Fun}(\mathrm{SCR}_k^{\mathrm{art}}, \mathcal{S})$ for the ∞-category of formal moduli problems.

If $\mathrm{char}(k) = 0$, then formal moduli problems are controlled by d.g. Lie algebras. More precisely, let $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{DGLA}_k$ be the right adjoint to the Chevalley-

Eilenberg cochains functor from the ∞-category of d.g. Lie algebras to the ∞-category of augmented simplicial commutative k-algebras. The underlying chain complex of $\mathfrak{D}(R)$ is the linear dual of the cotangent fibre $\cot(R)$, which can be computed explicitly as $\cot(R) = |\operatorname{Bar}_\bullet(1, \operatorname{Sym}^*, \mathfrak{m}_R)|$ for $\mathfrak{m}_R$ the augmentation ideal of R and Sym^* the monad parametrising nonunital simplicial commutative k-algebras. The following theorem of Lurie [7] and Pridham [9] clarifies previous seminal work of Deligne, Drinfel'd, Feigin, Hinich, Kontsevich-Soibelman, Manetti, and others:

Theorem 2. (Lurie , Pridham) If k is a field of characteristic zero, the functor $\mathrm{DGLA}_k \to \mathrm{Moduli}_k$ given by $\mathfrak{g} \mapsto \big(R \mapsto \mathrm{Map}_{\mathrm{DGLA}_k}(\mathfrak{D}(R), \mathfrak{g})\big)$ is an equivalence.

Partition Lie Algebras. We generalise Theorem 2 to arbitrary fields k and thus give a Lie–algebraic description of the infinitesimal structure of moduli stacks.

To construct our equivalence, we want to define a functor $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \Lambda$ to some ∞-category Λ of generalised Lie algebras in a way that makes the induced functor $\Lambda \to \mathrm{Moduli}_k$ given by $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_\Lambda(\mathfrak{D}(R), \mathfrak{g}))$ an equivalence.

In a first attempt to define $\mathfrak{D}$ and Λ, we observe that the tangent fibre functor $\cot^\vee : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Mod}_k$ admits a left adjoint. Writing L for the monad associated with this adjunction, we obtain a functor $(\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$. Unfortunately, this very natural functor *does not* allow us to establish an equivalence between $\mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$ and Moduli_k. Roughly speaking, the monad L fails to preserve sifted colimits because it involves a double dualisation, which in turn prohibits us from using Lurie's ∞-categorical version of the Barr-Beck theorem. Even though $\mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$ is therefore the wrong target category, the assignment $(A \mapsto \cot(A)^\vee)$ is still the correct functor whenever $A \in \mathrm{SCR}_k^{\mathrm{art}}$ is Artinian. We therefore want to replace L with a sifted-colimit-preserving monad L^π that agrees with L on some full subcategory of Mod_k containing $\cot(A)^\vee$ for all Artinian A.

Indeed, let $\mathrm{Coh}_k^{\leq 0}$ be the full subcategory of Mod_k spanned by all coconnective k-module spectra with finite-dimensional homotopy groups in all degrees. Any Artinian $A \in \mathrm{SCR}_k^{\mathrm{art}}$ has $\cot(A)^\vee \in \mathrm{Coh}_k^{\leq 0}$. In fact, the monad L from above preserves $\mathrm{Coh}_k^{\leq 0}$ (cf. [5, Proposition 3.2.14.]) and is well-behaved on this subcategory: if $X_\bullet$ is a simplicial diagram in $\mathrm{Coh}_k^{\leq 0}$ with $|X_\bullet| \in \mathrm{Coh}_k^{\leq 0}$, then $|\mathrm{L}(X_\bullet)| \simeq \mathrm{L}(|X_\bullet|)$. From this, we can show that $\mathrm{L}\,|_{\mathrm{Coh}_k^{\leq 0}}$ lies in the image of the fully faithful monoidal restriction functor $\mathrm{End}_\Sigma^{\mathrm{Coh}_k^{\leq 0}}(\mathrm{Mod}_k) \to \mathrm{End}(\mathrm{Coh}_k^{\leq 0})$. Here, $\mathrm{End}_\Sigma^{\mathrm{Coh}_k^{\leq 0}}(\mathrm{Mod}_k)$ is the ∞-category of sifted-colimit-preserving endofunctors of Mod_k which preserve $\mathrm{Coh}_k^{\leq 0}$. Let L^π be the unique monad lifting $\mathrm{L}\,|_{\mathrm{Coh}_k^{\leq 0}}$ under the above restriction functor.

Definition 3. A *partition Lie algebra* is an algebra over the monad L^π on Mod_k. If $M \in \mathrm{Coh}_k^{\leq 0}$ is represented by a cosimplicial k-module, $\mathrm{L}^\pi(M)$ is given by

$$\mathrm{L}^\pi(M) = \textstyle\bigoplus_{n \geq 1} (\tilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes M^{\otimes n})^{\Sigma_n}.$$

Here, $\Sigma|\Pi_n|^\diamond$ denotes the reduced-unreduced suspension of the n^{th} partition complex $|\Pi_n|$, i.e. the realisation of the poset of proper nontrivial partitions of $\{1, \ldots, n\}$. The functor $\tilde{C}^\bullet(-, k)$ sends a space X to the cosimplicial set of reduced k-valued singular cochains on X, and the functor $(-)^{\Sigma_n}$ takes strict Σ_n-fixed points.

Since L^π and L agree on $\mathrm{Coh}_k^{\leq 0}$, we have a functor $\cot(-)^\vee$ from finitely presented simplicial commutative k-algebras to partition Lie algebras. Extending in a filtered-limit-preserving way, we obtain a functor $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k)$. This assignment is not fully faithful; for example, $k[x] \to k[[x]]$ gives an equivalence after applying $\mathfrak{D}$. However, it becomes fully faithful on a suitable subcategory:

Definition 4. An augmented simplicial commutative k-algebra $A \in \mathrm{SCR}_k^{\mathrm{aug}}$ is *topologically almost finitely generated* if $\pi_0(A)$ is Noetherian and complete with respect to $\mathfrak{m} = \ker(\pi_0(A) \to k)$ and $\pi_i(A)$ is finitely generated over $\pi_0(A)$ for all i.

Theorem 5. The functor $\mathfrak{D}$ restricts to an equivalence between the ∞-category $\mathrm{SCR}_k^{\mathrm{taf}}$ of topologically almost finitely generated $A \in \mathrm{SCR}_k^{\mathrm{aug}}$ and the ∞-category $\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Coh}_k^{\leq 0})$ of partition Lie algebras whose underlying module lies in $\mathrm{Coh}_k^{\leq 0}$.

Using this equivalence and an analysis of the functor $\mathfrak{D}$ on "π_0-surjective pullbacks", we can prove that $\mathfrak{D}$ defines a deformation theory in the sense of Lurie (cf. [7, Definition 12.3.3.2.]), which in turn implies our generalisation of Theorem 2:

Theorem 6. For any field k, the functor $\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k) \to \mathrm{Moduli}_k$ given by $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_{\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k)}(\mathfrak{D}(R), \mathfrak{g}))$ establishes an equivalence between the ∞-category of partition Lie algebras and the ∞-category of formal moduli problems.

Remark 7. There is a parallel equivalence between formal moduli problems based on connective $\mathbb{E}_\infty$-k-algebras and an ∞-category of *spectral partition Lie algebras*.

Remark 8. Relying on the connection between partition complexes and partition Lie algebras (or spectral partition Lie algebras), we compute the homotopy groups of free objects (thus extending computations in [1],[2],[4]). These groups parametrise operations acting on the homotopy groups of partition Lie algebras.

Remark 9. Our equivalence generalises to mixed characteristic contexts, where we can describe infinitesimal deformations of $\mathrm{Spec}(A)$-valued families for A Noetherian with a map to a field k such that $\pi_0(A)$ is local with residue field k and complete with respect to the augmentation ideal.

References

[1] G. Arone, L. Brantner, *The action of Young subgroups on the partition complex*, arXiv preprint arXiv:1801.01491 (2018).

[2] G. Arone, M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Inventiones mathematicae, **135(3)** (1999), 743–788.

[3] S. Galatius, A. Venkatesh, *Derived Galois deformation rings*, Advances in Mathematics **327** (2018), 470-623.

[4] P. Goerss, *On the André-Quillen cohomology of commutative $\mathbb{F}_2$-algebras: Astérisque 186*, Société mathématique de France (1990).

[5] J. Lurie, *Derived Algebraic Geometry*, Ph.D. thesis, available on the author's website.

[6] J. Lurie, *Elliptic Cohomology II: Orientations*, preprint available from the author's website.

[7] J. Lurie, *Spectral Algebraic Geometry*, preprint available on the author's website.

[8] T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, *Shifted symplectic structures*, Publications mathématiques de l'IHÉS, **117(1)** (2013), 271-328.

[9] J. Pridham, *Unifying derived deformation theories*, Advances in Mathematics, **224(3)** (2010), 772-826.

[10] B. Toën, G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, Memoirs of the AMS (2008).