Appendix to
Strategic Asset Allocation:
Portfolio Choice for Long-Term Investors

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Contents

Chapter 1

Appendix: Introduction

This Appendix contains mathematical derivations of some selected results presented in John Campbell and Luis M. Viceira's book "Strategic Asset Allocation: Portfolio Choice for Long-Term Investors." To avoid confusion between equations in the main text of the book and equations in this Appendix, we number equations in the Appendix as (A1), (A2), etc.

Chapter 2

Appendix: Mathematical Derivations

2.1 Derivation of selected mathematical results in Chapter 3

2.1.1 Derivation of the approximation to the log portfolio return

In the case where there are two assets, one risky and one riskless, we have from (2.1) that

$$\frac{1+R_{p,t+1}}{1+R_{f,t+1}} = 1+\alpha_t\left(\frac{1+R_{t+1}}{1+R_{f,t+1}}-1\right).$$

Taking logs, this can be rewritten as

$$r_{p,t+1}-r_{f,t+1} = \log\left[1+\alpha_t\left(\exp(r_{t+1}-r_{f,t+1})-1\right)\right].$$

This equation gives a nonlinear relation between the log excess return on the single risky asset, $r_{t+1}-r_{f,t+1}$, and the log excess return on the portfolio, $r_{p,t+1}-r_{f,t+1}$. This relation can be approximated using a second-order Taylor expansion around the point $r_{t+1}-r_{f,t+1}=0$. The function $f_t(r_{t+1}-r_{f,t+1}) = \log\left[1+\alpha_t\left(\exp(r_{t+1}-r_{f,t+1})-1\right)\right]$ is approximated as

$$f_t(r_{t+1}-r_{f,t+1}) \approx f_t(0)+f_t'(0)(r_{t+1}-r_{f,t+1})+\frac{1}{2}f_t''(0)(r_{t+1}-r_{f,t+1})^2.$$

The derivatives of the function f_t, evaluated at $r_{t+1}-r_{f,t+1}=0$, are $f_t'(0)=\alpha_t$ and $f_t''(0)=\alpha_t(1-\alpha_t)$. Also, we replace $(r_{t+1}-r_{f,t+1})^2$ by its conditional expectation σ_t^2. Thus the Taylor approximation is

$$r_{p,t+1} - r_{f,t+1} = \alpha_t(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2.$$

The log excess portfolio return takes the same form as the simple excess portfolio return, with an adjustment factor in the variance of the risky asset return. The adjustment factor is zero if the portfolio weight in the risky asset is zero (for then the log portfolio return is just the log riskless return), and it is also zero if the weight in the risky asset is one (for then the log portfolio return is just the log risky return). The approximation in (2.21) can be justified rigorously by considering shorter and shorter time intervals. As the time interval shrinks, the higher-order terms that are dropped from (**??**) become negligible relative to those that are included, and the deviation of the realized squared excess return $(r_{t+1} - r_{f,t+1})^2$ from its expectation σ_t^2 also become negligible.

In the limit of continuous time, the approximation is exact and can be derived using Ito's Lemma. For completeness we present the derivation in the most general case where there are multiple risky assets and no riskless asset. The log return on the portfolio $r_{p,t+1}$ is a discrete-time approximation to its continuous-time counterpart. We assume that there are $(n+1)$ risky assets, one of which we use as a benchmark. Without loss of generality, we assume that the benchmark asset is a risky short-term instrument whose price we denote by B_t. We begin by specifying the return processes for the short-term instrument B_t and all other risky assets $\mathbf{P}_t$ in continuous time:

$$\begin{aligned} \frac{dB_t}{B_t} &= \mu_{b,t}dt + \boldsymbol{\sigma}_b d\mathbf{W}_t, \\ \frac{d\mathbf{P}_t}{\mathbf{P}_t} &= \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{W}_t, \end{aligned}$$

where $\mu_{b,t}$ and $\boldsymbol{\mu}_t$ are the drifts, $\boldsymbol{\sigma}_b$ and $\boldsymbol{\sigma}$ are the diffusion, and $\mathbf{W}_t$ is a m-dimensional standard Brownian motion. The dimensions of $\mu_b, \boldsymbol{\mu}, \boldsymbol{\sigma}_b, \boldsymbol{\sigma}$ are $1\times 1, n\times 1, 1\times m, n\times m$, respectively. We allow the drifts to depend on other state variables, but for notational simplicity, we suppress this dependency and simply use the time subscript. Moreover, note that the same $\mathbf{W}_t$ appears in these two equations.

Since we are working with log returns, we apply Ito's Lemma to each asset:

$$\begin{aligned} d\log B_t &= \left(\frac{dB_t}{B_t}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt, \\ d\log P_{i,t} &= \left(\frac{dP_{i,t}}{P_{i,t}}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right)dt, \end{aligned}$$

where $\boldsymbol{\sigma}_i$ is the ith row of the diffusion matrix $\boldsymbol{\sigma}$, and $i = 1, .., n$.

Let V_t be the value of the portfolio at time t. We will use $d\log V_t$ to approximate $r_{p,t+1}$. By Ito's Lemma,

$$d\log V_t = \left(\frac{dV_t}{V_t}\right) - \frac{1}{2}\left(\frac{dV_t}{V_t}\right)^2.$$

We will now derive these 2 terms in order:

$$\begin{aligned}
\frac{dV_t}{V_t} &= \boldsymbol{\alpha}_t'\left(\frac{d\mathbf{P}_t}{\mathbf{P}_t}\right) + \left(1 - \boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\frac{dB_t}{B_t} \\
&= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t + \frac{1}{2}\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right]dt\right) + \left(1 - \boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\left(d\log B_t + \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt\right) \\
&= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right) + d\log B_t \\
&\quad + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] - \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\cdot\boldsymbol{\iota}\right)dt + \frac{1}{2}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'dt,
\end{aligned}$$

where $\boldsymbol{\iota}$ is a $n \times 1$ vector of ones and the bracket $[\cdot]$ denotes a vector with $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'$ the ith entry. Next,

$$\begin{aligned}
\left(\frac{dV_t}{V_t}\right)^2 &= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)'\boldsymbol{\alpha}_t + \left(d\log B_t\right)^2 \\
&\quad + 2\boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log B_t\right) + o\left(dt\right),
\end{aligned}$$

where the $o(dt)$ terms vanish because they involve either $(dt)^2$ or $(dt)\,(d\mathbf{W}_t)$.

Now, from equation (**??**)–(**??**) and ignoring dt terms,

$$d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota} = \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)d\mathbf{W}_t.$$

Thus,

$$\begin{aligned}
\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)' &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)', \\
\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log B_t\right) &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\cdot\boldsymbol{\sigma}_b'.
\end{aligned}$$

Collecting these results and using our notation for excess returns and the return on the benchmark risky asset ($\mathbf{x}_{t+1} = d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}$, and

$r_{0,t+1} = d\log(B_t))$ and setting $dt = 1$, we have:

$$\begin{aligned}
& r_{p,t+1} \\
= \; & d\log V_t \\
= \; & \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{0,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] - \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota}\right) \\
& -\frac{1}{2}\left[\boldsymbol{\alpha}_t'\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)'\boldsymbol{\alpha}_t + 2\boldsymbol{\alpha}_t'\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\boldsymbol{\sigma}_b'\right] \\
= \; & \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{0,t+1} - \frac{1}{2}\boldsymbol{\alpha}_t'\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)'\boldsymbol{\alpha}_t \\
& +\frac{1}{2}\boldsymbol{\alpha}_t'\left(\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] + \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota} - \mathbf{2}\boldsymbol{\sigma}\boldsymbol{\sigma}_b'\right).
\end{aligned}$$

Similarly, using the notation in book for variances and covariances, we have

$$\begin{aligned}
\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)' & \equiv \Sigma_t, \\
\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b' & \equiv \sigma_{0t}^2, \\
\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] + \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota} - \mathbf{2}\boldsymbol{\sigma}\boldsymbol{\sigma}_b' & = \boldsymbol{\sigma}_t^2.
\end{aligned}$$

With these terms, the return on the portfolio is

$$r_{p,t+1} = \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{0,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t'\boldsymbol{\sigma}_t^2 - \frac{1}{2}\boldsymbol{\alpha}_t'\Sigma_t\boldsymbol{\alpha}_t,$$

which is equation (2.23) in text.

2.2 Derivation of selected mathematical results in Chapter 3

2.2.1 Derivation of the approximation to the log intertemporal budget constraint

Taking logs on both sides of the intertemporal budget constraint (3.2) we obtain equation (3.3) in text:

$$\Delta w_{t+1} = r_{p,t+1} + \log(1 - \exp(c_t - w_t)). \tag{A1}$$

The second-term on the right-hand side of (A1) is a non-linear function of the log consumption-wealth ratio. A first-order approximation of this function around the mean of the log consumption-wealth ratio gives:

$$\begin{aligned}\log(1-\exp(c_t-w_t)) &\approx \log\left(1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)\right)\\ &\quad -\frac{\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}{1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}\left(\left(c_t-w_t\right)-\mathrm{E}\left[c_t-w_t\right]\right)\\ &= \log\left(1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)\right)\\ &\quad +\frac{\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}{1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}\mathrm{E}\left[c_t-w_t\right]\\ &\quad -\frac{\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}{1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}\left(c_t-w_t\right). \qquad \text{(A2)}\end{aligned}$$

Defining

$$\rho \equiv 1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right), \qquad \text{(A3)}$$

we can rewrite (A2) as

$$\log(1-\exp(c_t-w_t)) \approx k+\left(1-\frac{1}{\rho}\right)\left(c_t-w_t\right), \qquad \text{(A4)}$$

where

$$\begin{aligned}k &\equiv \log\left(1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)\right)+\frac{\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}{1-\exp\left(\mathrm{E}\left[c_t-w_t\right]\right)}\left(\mathrm{E}\left[c_t-w_t\right]\right)\\ &= \log\left(\rho\right)+\frac{1-\rho}{\rho}\log\left(1-\rho\right).\end{aligned}$$

Note that this approximation is exact when the optimal consumption-wealth ratio is constant. so that $c_t - w_t = \mathrm{E}_t[c_t - w_t]$.

Direct substitution of (A4) into (A1) gives equation (3.4) for the log intertemporal budget constraint in text.

2.2.2 Solution to model with constant variances and risk premia when there are multiple risky assets

Chapter 2 shows (see equation [2.51]) that under Epstein-Zin utility with multiple risky assets, the premium on each risky asset over the risky benchmark asset is given by

$$\begin{aligned}&\mathrm{E}_t\left(r_{i,t+1}-r_{0,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{i,t+1}-r_{0,t+1}\right)\\ =\ &\frac{\theta}{\psi}\mathrm{Cov}_t\left(\Delta c_{t+1},r_{i,t+1}-r_{0,t+1}\right)\\ &+\left(1-\theta\right)\mathrm{Cov}_t\left(r_{p,t+1},r_{i,t+1}-r_{0,t+1}\right)\\ &-\mathrm{Cov}_t\left(r_{i,t+1}-r_{0,t+1},r_{0,t+1}\right), \qquad \text{(A5)}\end{aligned}$$

where $\theta = (1-\gamma)/(1-\psi^{-1})$.

Using the log budget constraint (3.4) and the trivial identity $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$, we have

$$\begin{aligned}
& \operatorname{Cov}_t\left(\Delta c_{t+1}, r_{i,t+1} - r_{0,t+1}\right) \\
= \; & \operatorname{Cov}_t\left((c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}, r_{i,t+1} - r_{0,t+1}\right) \\
= \; & \operatorname{Cov}_t\left(c_{t+1} - w_{t+1}, r_{i,t+1} - r_{0,t+1}\right) + \operatorname{Cov}_t\left(\Delta w_{t+1}, r_{i,t+1} - r_{0,t+1}\right) \\
= \; & \operatorname{Cov}_t\left(c_{t+1} - w_{t+1}, r_{i,t+1} - r_{0,t+1}\right) + \operatorname{Cov}_t\left(r_{pt+1}, r_{i,t+1} - r_{0,t+1}\right). \qquad \text{(A6)}
\end{aligned}$$

Substitution of (A6) into (A5) gives

$$\begin{aligned}
& \mathrm{E}_t\left(r_{i,t+1} - r_{0,t+1}\right) + \frac{1}{2}\operatorname{Var}_t\left(r_{i,t+1} - r_{0,t+1}\right) \\
= \; & \frac{\theta}{\psi}\operatorname{Cov}_t\left(c_{t+1} - w_{t+1}, r_{i,t+1} - r_{0,t+1}\right) \\
& + \left(1 - \theta + \frac{\theta}{\psi}\right)\operatorname{Cov}_t\left(r_{p,t+1}, r_{i,t+1} - r_{0,t+1}\right) \\
& - \operatorname{Cov}_t\left(r_{i,t+1} - r_{0,t+1}, r_{0,t+1}\right),
\end{aligned}$$

or, in vector notation,

$$\begin{aligned}
& \mathrm{E}_t\left(\mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_t^2 \\
= \; & -\left(\frac{1-\gamma}{1-\psi}\right)\operatorname{Cov}_t\left(c_{t+1} - w_{t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\
& + \gamma\operatorname{Cov}_t\left(r_{p,t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\
& - \operatorname{Cov}_t\left(\mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}, r_{0,t+1}\right), \qquad \text{(A7)}
\end{aligned}$$

where we have substituted $(1-\gamma)/(1-\psi^{-1})$ for θ. $\boldsymbol{\sigma}_t^2$ denotes a column vector with the variance of the excess return on each asset over the return on the benchmark risky asset:

$$\boldsymbol{\sigma}_t^2 \equiv \left(\operatorname{Var}_t\left(r_{1,t+1} - r_{0,t+1}\right), ..., \operatorname{Var}_t\left(r_{n,t+1} - r_{0,t+1}\right)\right)'.$$

The equation for log portfolio return (2.23) implies that the second covariance term in (A7) is

$$\begin{aligned}
& \operatorname{Cov}_t\left(r_{p,t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\
= \; & \operatorname{Cov}_t\left((\mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}) + r_{0,t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\
= \; & \boldsymbol{\alpha}_t'\operatorname{Var}_t\left(\mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) + \operatorname{Cov}_t\left(r_{0,t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right),
\end{aligned}$$

so that (A7) becomes

$$\begin{aligned}
& \mathrm{E}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right)+\frac{1}{2}\boldsymbol{\sigma}_t^2 \\
= & -\left(\frac{1-\gamma}{1-\psi}\right)\mathrm{Cov}_t\left(c_{t+1}-w_{t+1},\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right) \\
& +\gamma\boldsymbol{\alpha}_t'\,\mathrm{Var}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right) \\
& -(1-\gamma)\,\mathrm{Cov}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota},r_{0,t+1}\right), \qquad \text{(A8)}
\end{aligned}$$

from which we obtain immediately an expression for $\boldsymbol{\alpha}_t$.

Defining

$$\boldsymbol{\sigma}_{ht}\equiv\frac{\mathrm{Cov}_t\left(-\left(c_{t+1}-w_{t+1}\right),\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right)}{1-\psi},$$

$$\boldsymbol{\Sigma}_t\equiv\mathrm{Var}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right),$$

and

$$\boldsymbol{\sigma}_{0t}\equiv\mathrm{Cov}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota},r_{0,t+1}\right),$$

the expression for $\boldsymbol{\alpha}_t$ resulting from (A8) becomes

$$\boldsymbol{\alpha}_t=\frac{1}{\gamma}\boldsymbol{\Sigma}_t^{-1}\left(\mathrm{E}_t\,\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}+\boldsymbol{\sigma}_t^2/2\right)+\left(1-\frac{1}{\gamma}\right)\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{\sigma}_{ht}-\boldsymbol{\sigma}_{0t}\right),$$

which is (3.21) in text. Equation (3.20) obtains when the benchmark asset is riskless one-period ahead, so that $\boldsymbol{\sigma}_{0t}=0$.

Note that section 3.1.3 shows that

$$\frac{\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\left(c_{t+1}-w_{t+1}\right)}{1-\psi}=\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j}. \qquad \text{(A9)}$$

This section also shows that, when time-variation in interest rates is the only source of variation in investment opportunities, the right-hand-side of (A9) is equal to

$$\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j}=\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j},$$

so that

$$\begin{aligned}
\boldsymbol{\sigma}_{ht} &\equiv \mathrm{Cov}_t\left(-\frac{c_{t+1}-w_{t+1}}{1-\psi},\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right) \\
&= \mathrm{Cov}_t\left(-\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j},\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right), \qquad \text{(A10)}
\end{aligned}$$

as stated in (3.19).

From equation (3.18) in text, it is easy to see that, if $\rho = \rho_c$, then equation (A10) becomes

$$\begin{aligned}\boldsymbol{\sigma}_{ht} &\equiv \mathrm{Cov}_t\left(-(\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j r_{f,t+1+j}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\ &= \mathrm{Cov}_t\left(r_{c,t+1}, \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota}\right) \\ &\equiv \boldsymbol{\sigma}_{ct},\end{aligned}$$

so that $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\sigma}_{ht} = \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\sigma}_{ct}$ is the vector of population regression coefficients from a multiple regression of an inflation-indexed consol return onto the set of risky asset returns, as stated in text.

2.2.3 Recursive expression for A_n

The recursive equation for the coefficient A_n in the indexed zero-coupon bond pricing equation (3.27) is given by

$$A_n - A_{n-1} = (1 - \phi_x)\,\mu_x B_{n-1} - \frac{1}{2}\left[(\beta_{mx} + B_{n-1})^2\,\sigma_x^2 + \sigma_m^2\right].$$

with $A_0 = B_0 = 0$. See Campbell, Lo and Mackinlay (1997) for a derivation of the pricing equation (3.27).

2.2.4 Pricing nominal bonds

The pricing of default-free nominal bonds follows the same steps as the pricing of indexed bonds. The relevant stochastic discount factor to price nominal bonds is the nominal SDF $M_{t+1}^{\$}$, whose log is given

$$m_{t+1}^{\$} = m_{t+1} - \pi_{t+1}. \tag{A11}$$

Since both M_{t+1} and Π_{t+1} are jointly lognormal and homoskedastic, $M_{t+1}^{\$}$ is also lognormal. The log nominal return on a one-period nominal bond is $r_{1,t+1}^{\$} = -\log \mathrm{E}_t[M_{t+1}]$, or

$$\begin{aligned}r_{1,t+1}^{\$} &= -\mathrm{E}_t\left[m_{t+1}^{\$}\right] - \frac{1}{2}\mathrm{Var}_t\left[m_{t+1}^{\$}\right] \\ &= x_t + z_t - \frac{1}{2}\left[(\beta_{mx} + \beta_{\pi x})\,2\sigma_x^2 + \beta_{\pi z}^2\sigma_z^2 + (1 + \beta_\pi)^2\,\sigma_m^2 + \sigma_\pi^2\right],\end{aligned}$$

a linear combination of the expected log real SDF and expected inflation.

The risk premium on a 1-period nominal bond over a 1-period real bond can be written as

$$\mathrm{E}_t\left[r^{\$}_{1,t+1}-\pi_{t+1}-r_{1,t+1}\right]+\frac{1}{2}\,\mathrm{Var}_t\left[\pi_{t+1}\right]=-\beta_{mx}\beta_{\pi x}\sigma_x^2-\beta_{\pi m}\sigma_m^2,$$

which has the same form as equation (3.38) for equities.

The log price of an n-period nominal bond, $p^{\$}_{n,t}$, also has an affine structure. It is a linear combination of x_t and z_t whose coefficients are time-invariant, though they vary with the maturity of the bond. As shown in equation (3.36), $-p^{\$}_{n,t}=A^{\$}_n+B^{\$}_{1,n}x_t+B^{\$}_{2,n}z_t$,where

$$\begin{aligned}
B^{\$}_{1,n} &= 1+\phi_x B^{\$}_{1,n-1}=\frac{1-\phi_x^n}{1-\phi_x}\\
B^{\$}_{2,n} &= 1+\phi_z B^{\$}_{2,n-1}=\frac{1-\phi_z^n}{1-\phi_z}\\
A^{\$}_n-A^{\$}_{n-1} &= \left(1-\phi_x\right)\mu_x B^{\$}_{1,n-1}+\left(1-\phi_z\right)\mu_z B^{\$}_{2,n-1}\\
&\quad -\frac{1}{2}\left(\beta_{mx}+\beta_{\pi_x}+B^{\$}_{1,n-1}+\beta_{zx}B^{\$}_{2,n-1}\right)^2\sigma_x^2\\
&\quad -\frac{1}{2}\left(\beta_{\pi z}+B^{\$}_{2,n-1}\right)^2\sigma_z^2-\frac{1}{2}\left(1+\beta_{\pi m}+\beta_{zm}\right)^2\sigma_m^2\\
&\quad -\frac{1}{2}\sigma_\pi^2,
\end{aligned}$$

and $A^{\$}_0=B^{\$}_{1,0}=B^{\$}_{2,0}=0$.

The excess return on a n-period bond over the one-period log nominal interest rate is

$$\begin{aligned}
r^{\$}_{n,t+1}-r^{\$}_{1,t+1} &= p^{\$}_{n-1,t+1}-p^{\$}_{n,t}+p^{\$}_{1,t}\\
&= -\left(B^{\$}_{1,n-1}+B^{\$}_{2,n-1}\beta_{zx}\right)\left(\beta_{mx}+\beta_{\pi x}\right)\sigma_x^2-B^{\$}_{2,n-1}\beta_\pi\sigma_z^2\\
&\quad -\left(1+\beta_{\pi m}\right)\beta_{zm}B^{\$}_{2,n-1}\sigma_m^2-\frac{1}{2}\left(B^{\$}_{1,n-1}+B^{\$}_{2,n-1}\beta_{zx}\right)^2\sigma_x^2\\
&\quad -\frac{1}{2}\left(B^{\$}_{2,n-1}\right)^2\sigma_z^2-\frac{1}{2}\beta_{zm}^2\left(B^{\$}_{2,n-1}\right)^2\sigma_m^2\\
&\quad -\left(B^{\$}_{1,n-1}+B^{\$}_{2,n-1}\beta_{zx}\right)\varepsilon_{x,t+1}-B^{\$}_{2,n-1}\beta_{zm}\varepsilon_{m,t+1}\\
&\quad -B^{\$}_{2,n-1}\varepsilon_{z,t+1}.
\end{aligned}$$

The terms in $B^{\$}_{2,n-1}\beta_{zx}$ and $B^{\$}_{2,n-1}\beta_{zm}$ arise because shocks to expected inflation are correlated with shocks to the expected and unexpected log real SDF. Thus risk premia in the nominal term structure are different from risk premia in the real term structure because they include compensation for inflation risk. Like real risk premia, however, nominal risk premia are constant over time.

2.3 Derivation of selected mathematical results in Chapter 5

2.3.1 Coefficients of the value function in the model with time-varying interest rates and power utility

Substitution of (5.22) into (5.20) leads to

$$\begin{aligned} 0 &= \frac{\gamma h_0}{1-\gamma} - \frac{\gamma h_1}{1-\gamma}\left(C_0 + C_1 r\right) + \frac{\lambda^2}{2\gamma\sigma^2} - \frac{\beta}{1-\gamma} + r \\ &\quad + \left(\frac{\gamma\kappa}{1-\gamma}\left(\theta - r\right) - \lambda\right) C_1 + \frac{\gamma\sigma^2}{2\left(1-\gamma\right)} C_1^2, \end{aligned} \tag{A12}$$

where we must determine C_0 and C_1 so that the equation holds for all values of the instantaneous interest rate. Simple inspection of the terms in the equation shows that the right hand side of the equation is a linear combination of the instantaneous interest rate. Thus C_0 and C_1 must be such that both the intercept and the slope of the linear equation are zero simultaneously:

$$0 = -\frac{\gamma\left(h_1 + \kappa\right)}{1-\gamma} C_1 + 1, \tag{A13}$$

$$\begin{aligned} 0 &= -\frac{\gamma h_1}{1-\gamma} C_0 + \frac{\gamma h_0}{1-\gamma} + \frac{\lambda^2}{2\gamma\sigma^2} - \frac{\beta}{1-\gamma} \\ &\quad \left(\frac{\gamma\kappa\theta}{1-\gamma} - \lambda\right) C_1 + \frac{\gamma\sigma^2}{2\left(1-\gamma\right)} C_1^2. \end{aligned} \tag{A14}$$

Equation (A13) is a linear equation whose only unknown is C_1. The solution to this equation is given in (5.23). Equation (A14) depends on both C_1 and C_0, but it is linear in C_0 given C_1. Substituting the expression for C_1 that obtains from (A13) into (A14), we can solve for C_0 immediately.

2.3.2 Coefficients of the value function in the model with time-varying interest rates and recursive utility (unit elasticity of intertemporal substitution case)

The solution procedure is analogous to the solution procedure shown in the previous section. Substitution of (5.72) into (5.71) leads to

$$\begin{aligned} 0 &= -\frac{\beta}{1-\gamma}\left(C_0+C_1 r\right)+\left(\beta\log\beta-\beta+\frac{\lambda^2}{2\gamma\sigma^2}+r\right) \\ &\quad +\frac{\sigma^2}{2\gamma}C_1^2+\left(\frac{\kappa}{1-\gamma}\left(\theta-r\right)-\frac{\lambda}{\gamma}\right)C_1+\frac{\sigma^2}{2\left(1-\gamma\right)}C_1^2, \end{aligned} \tag{A15}$$

where we must determine C_0 and C_1 so that the equation holds for all values of the instantaneous interest rate. Once again, simple inspection of the terms in the equation shows that the right hand side of the equation is a linear combination of the instantaneous interest rate. Thus C_0 and C_1 must be such that both the intercept and the slope of the linear equation are zero simultaneously. This leads to two algebraic equations for C_0 and C_1.

The first equation obtains from collecting terms in r in (A15) and setting them to zero:

$$0=-\frac{\gamma\left(\beta+\kappa\right)}{1-\gamma}C_1+1. \tag{A16}$$

This equation is identical to (A13) with $h_1=\beta$. The second equation is again linear in C_0 given C_1, and obtains by collecting all other terms in (A15).

2.3.3 Coefficients of the value function in the model with stochastic volatility and recursive utility (unit elasticity of intertemporal substitution case)

Substitution of guess $I=\exp\{C_0+C_1 q_t\}$ into (5.85) leads to

$$\begin{aligned} 0 &= \beta\log\beta-\beta-\frac{\beta\gamma}{1-\gamma}\log\left(1-\gamma\right)+\frac{\left(\mu_P-r\right)^2}{2\gamma}q+r \\ &\quad -\frac{1}{1-\gamma}\beta\left(C_0+C_1 q\right)+\left(\frac{\rho_{Pq}\sigma_q\left(\mu_P-r\right)}{\gamma}y+\frac{\kappa_q}{1-\gamma}\left(\theta_q-q\right)\right)C_1 \\ &\quad +\frac{\rho_{Pq}^2\sigma_q^2}{2\gamma}qC_1^2+\frac{\sigma_q^2}{2\left(1-\gamma\right)}qC_1^2. \end{aligned} \tag{A17}$$

where we must determine C_0 and C_1 so that the equation holds for all values of precision q_t. Once again, simple inspection of the terms in the equation shows that the right hand side of the equation is a linear combination of q_t. Thus C_0 and C_1 must be such that both the intercept and the slope of the linear equation are zero simultaneously. This leads to the following two algebraic equations for C_0 and C_1:

$$0 = aC_1^2 + bC_1 + c, \tag{A18}$$
$$0 = (1-\gamma)(\beta \log \beta + r - \beta) - \beta\gamma \log(1-\gamma) - \beta C_0 + \kappa_q \theta_q C_1, \tag{A19}$$

where

$$a = \frac{\sigma_q^2}{2\gamma(1-\gamma)} \left[\gamma\left(1-\rho_{Pq}^2\right) + \rho_{Pq}^2\right], \tag{A20}$$
$$b = \frac{\rho_{Pq}\sigma_q(\mu_P - r)}{\gamma} - \frac{\beta + \kappa_q}{1-\gamma}, \tag{A21}$$
$$c = \frac{(\mu_P - r)^2}{2\gamma}. \tag{A22}$$

Equation(A18) is a quadratic equation in C_1, and equation (A19) is linear in C_0 given C_1. For general parameter values the equation for C_1 has two roots. These roots are always real provided that $\gamma > 1$. From standard theory on quadratic equations, the product of the roots is equal to c/a. When $\gamma > 1$, this ratio is always negative so that the roots have opposite signs. It is easy to check that only the negative root maximizes the value function for all values of q_t.[1] This root is obtained by selecting the positive root of the discriminant of the quadratic equation. Therefore, $C_1 < 0$ when $\gamma > 1$.

When $\gamma < 1$, the roots are real—and a solution to the problem exists—if and only if

$$\left(\frac{1-\gamma}{\gamma}\frac{\sigma_q(\mu_P - r)}{\beta + \kappa_q}\right)\left(2\rho_{Pq} + \frac{\sigma_q(\mu_P - r)}{\beta + \kappa_q}\right) < 1.$$

This condition implies that both roots of the quadratic equation are positive. In this case the largest root—again, the root associated with the positive root of the discriminant—maximizes the value function. Therefore, $C_1 > 0$ when $\gamma < 1$. Putting together the results for $\gamma > 1$ and $\gamma < 1$, we have that $\widetilde{C}_1 = C_1/(1-\gamma) > 0$.

[1] Note that the equation for B implies that $\partial C_0/\partial C_1 > 0$.

2.4 Derivation of selected mathematical results in Chapter 6

2.4.1 Optimal consumption and portfolio choice for retired investors

Optimal portfolio rule

A retired investor does not have labor income. Thus he faces the intertemporal budget constraint

$$W^r_{t+1} = (W_t - C^r_t)\left(1 + R^r_{p,t+1}\right),$$

whose loglinear approximation is given in equation (3.4) in text. To facilitate comparisons with the labor income case, it is convenient to rewrite (3.4) as follows:

$$w^r_{t+1} - w_t = k^r - \rho^r_c\left(c^r_t - w_t\right) + r^r_{p,t+1}. \tag{A23}$$

where $\rho^r_c \equiv -(1 - 1/\rho) = \exp\{\mathrm{E}[c^r - w_t]\}/(1 - \exp\{\mathrm{E}[c^r - w_t]\})$, and $k^r = -(1+\rho^r_c)\log(1+\rho^r_c) + \rho^r_c \log(\rho^r_c)$. Note that (A23) holds exactly when the consumption-wealth ratio is constant—as it is in this case.

We have also shown in this appendix that we can approximate the log portfolio return with the following expression:

$$r_{p,t+1} = r_f + \alpha_t(r_{t+1} - r_f) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma^2_u. \tag{A24}$$

This is equation (2.21) when investment opportunities are constant.

We have shown in section 2.2.3 that the Euler equation for an investor with power utility of consumption and no labor income implies the following expression for the risk-premium on the risky asset (see equation [2.43]):

$$\mathrm{E}_t\, r_{t+1} - r_f + \frac{1}{2}\mathrm{Var}_t\left(r_{t+1}\right) = \gamma\,\mathrm{Cov}_t\left(c^r_{t+1} - c^r_t, r_{t+1}\right). \tag{A25}$$

or, given our assumptions about the investment opportunity set,

$$\mu + \frac{1}{2}\sigma^2_u = \gamma\,\mathrm{Cov}_t\left(c^r_{t+1} - c^r_t, r_{t+1}\right). \tag{A26}$$

We can compute the covariance term in the right-hand side of equation (A26) by noting that (6.42) implies

$$c^r_{t+1} - c^r_t = b^r_1(w^r_{t+1} - w_t), \tag{A27}$$

so that

$$\begin{aligned}\mathrm{Cov}_t\left(c^r_{t+1}-c^r_t, r_{t+1}\right) &= b^r_1\,\mathrm{Cov}_t\left(w^r_{t+1}-w_t, r_{t+1}\right)\\ &= b^r_1\,\mathrm{Cov}_t\left(r_{p,t+1}, r_{t+1}\right)\\ &= b^r_1\alpha^r_t\sigma^2_u,\end{aligned}$$

where the second equality obtains from (A23), and the third equality obtains from (A24).

Therefore,

$$\mu+\frac{1}{2}\sigma^2_u=\gamma b^r_1\alpha^r_t\sigma^2_u,$$

from which (6.43) in text follows.

Optimal consumption rule

To derive the optimal consumption rule, note that the log of the Euler equation (6.41) with $i=p$ yields the following equation for expected log consumption growth:

$$\mathrm{E}_t\left[c^r_{t+1}-c^r_t\right]=\frac{1}{\gamma}\left[\mathrm{E}\left[r^r_{p,t+1}\right]+\log\delta^r+\frac{1}{2}\mathrm{Var}\left[r^r_{p,t+1}-\gamma\left(c^r_{t+1}-c^r_t\right)\right]\right], \tag{A28}$$

where $\delta^r\equiv(1-\pi^d)\delta$, and I have suppressed the subscript t from the conditional moments on the right-hand-side because $\alpha^r_t\equiv\alpha^r$ implies that they are constant. Moreover, equation (A27) implies $\mathrm{Var}_t[r^r_{p,t+1}-\gamma(c^r_{t+1}-c^r_t)]=(1-b^r_1\gamma)^2\,\mathrm{Var}(r^r_{p,t+1})$.

On the other hand, from equation (A27) and the log budget constraint (A23) we have

$$\begin{aligned}\mathrm{E}_t\left[c^r_{t+1}-c^r_t\right] &= b^r_1\,\mathrm{E}_t\left[w^r_{t+1}-w_t\right]\\ &= b^r_1\,\mathrm{E}\left[r^r_{p,t+1}\right]-b^r_1\rho^r_c b^r_0+b^r_1k^r+b^r_1\rho^r_c\left(1-b^r_1\right)w_t.\end{aligned} \tag{A29}$$

Equalizing the right-hand side of equations (A28) and (A29), and identifying coefficients, we obtain two equations. The first one implies

$$b^r_1=1,$$

while the second one implies

$$\begin{aligned}b^r_0 = &-\left(\frac{1}{b^r_1\rho^r_c}\right)\left[\left(\frac{1}{\gamma}-b^r_1\right)\mathrm{E}\left[r^r_{p,t+1}\right]+\frac{1}{\gamma}\log\delta^r\right.\\ &\left.+\frac{1}{2\gamma}\left(1-b^r_1\gamma\right)^2\mathrm{Var}\left(r^r_{p,t+1}\right)-b^r_1k^r\right].\end{aligned} \tag{A30}$$

Since $b_1^r = 1$, we have $b_0^r \equiv \mathrm{E}[c_t^r - w_t] = \log(\rho_c^r) - \log(1 + \rho_c^r)$. We can easily substitute the log-linearization constants ρ_c^r and k^r out from equation (A30) and obtain

$$b_0^r = \log\left(1 - \exp\left\{\left[\left(\frac{1}{\gamma} - b_1^r\right)\mathrm{E}\left[r_{p,t+1}^r\right] + \frac{1}{\gamma}\log\delta^r + \frac{1}{2\gamma}\left(1 - b_1^r\gamma\right)^2 \mathrm{Var}\left(r_{p,t+1}^r\right)\right]\right\}\right). \tag{A31}$$

2.4.2 Optimal consumption and portfolio choice for employed investors

To derive the optimal portfolio rule in the employment state we first need to derive loglinear expressions for the intertemporal budget constraint (6.39) and the Euler equation (6.40) for an employed investor.

Loglinear intertemporal budget constraint

We can rewrite the intertemporal budget constraint (6.39) as

$$\frac{W_{t+1}^e}{L_{t+1}} = \left(1 + \frac{W_t}{L_t} - \frac{C_t^e}{L_t}\right)\left(\frac{L_t}{L_{t+1}}\right)R_{p,t+1}^e,$$

or, in logs,

$$w_{t+1}^e - l_{t+1} = \log\left(\exp\left\{w_t - l_t\right\} - \exp\left\{c_t^e - l_t\right\}\right) - \Delta l_{t+1} + r_{p,t+1}^e. \tag{A32}$$

We can now linearize equation (A32) by taking a first-order Taylor expansion around $(c_t^e - l_t) = \mathrm{E}[c_t^e - l_t]$ and $(w_t^e - l_t) = \mathrm{E}[w_t^e - l_t]$. This gives

$$w_{t+1}^e - l_{t+1} \approx k^e + \rho_w^e\left(w_t - l_t\right) - \rho_c^e\left(c_t^e - l_t\right) - \Delta l_{t+1} + r_{p,t+1}^e, \tag{A33}$$

where

$$\rho_w^e = \frac{\exp\left\{\mathrm{E}[w_t^e - l_t]\right\}}{1 + \exp\left\{\mathrm{E}[w_t^e - l_t]\right\} - \exp\left\{\mathrm{E}[c_t^e - l_t]\right\}}, \tag{A34}$$

$$\rho_c^e = \frac{\exp\left\{\mathrm{E}[c_t^e - l_t]\right\}}{1 + \exp\left\{\mathrm{E}[w_t^e - l_t]\right\} - \exp\left\{\mathrm{E}[c_t^e - l_t]\right\}}, \tag{A35}$$

and

$$k^e = -(1 - \rho_w^e + \rho_c^e)\log(1 - \rho_w^e + \rho_c^e) - \rho_w^e\log(\rho_w^e) + \rho_c^e\log(\rho_c^e). \tag{A36}$$

Note that $\rho_w^e, \rho_c^e > 0$ because $W_t + L_t - C_t^e > 0$ along the optimal path.

Loglinear Euler equation

We can write equation (6.40) as

$$\begin{aligned} 1 &= \pi^e \ \mathrm{E}_t \left[\exp\left\{\log\delta - \gamma\left(c^e_{t+1} - c^e_t\right) + r_{i,t+1}\right\}\right] \\ &\quad + (1-\pi^e)\,\mathrm{E}_t\left[\exp\left\{\log\delta^r - \gamma\left(c^r_{t+1} - c^e_t\right) + r_{i,t+1}\right\}\right] \\ &\equiv \pi^e \ \mathrm{E}_t\left[\exp\left\{x_{t+1}\right\}\right] + (1-\pi^e)\ \mathrm{E}_t\left[\exp\left\{y_{t+1}\right\}\right]\ , \qquad \text{(A37)} \end{aligned}$$

where the notational correspondence between the first and second line is obvious, and $\delta^r \equiv (1-\pi^d)\delta$. Taking a second order Taylor expansion of $\exp\{x_{t+1}\}$ and $\exp\{y_{t+1}\}$ around $\overline{x}_t = \mathrm{E}_t[x_{t+1}]$ and $\overline{y}_t = \mathrm{E}_t[y_{t+1}]$ we can write:

$$\begin{aligned} 1 &\approx \pi^e \ \mathrm{E}_t\left[\exp\{\overline{x}_t\}\left(1 + (x_{t+1} - \overline{x}_t) + \frac{1}{2}(x_{t+1} - \overline{x}_t)^2\right)\right] \\ &\quad + (1-\pi^e)\ \mathrm{E}_t\left[\exp\{\overline{y}_t\}\left(1 + (y_{t+1} - \overline{y}_t) + \frac{1}{2}(y_{t+1} - \overline{y}_t)^2\right)\right] \\ &= \pi^e \exp\{\overline{x}_t\}\left(1 + \frac{1}{2}\ \mathrm{Var}_t(x_{t+1})\right) + (1-\pi^e)\exp\{\overline{y}_t\}\left(1 + \frac{1}{2}\ \mathrm{Var}_t(y_{t+1})\right)\ . \end{aligned}$$

Finally, a first-order Taylor expansion around zero gives

$$1 \approx \pi^e\left(1 + \overline{x}_t + \frac{1}{2}\ \mathrm{Var}_t(x_{t+1})\right) + (1-\pi^e)\left(1 + \overline{y}_t + \frac{1}{2}\ \mathrm{Var}_t(y_{t+1})\right),$$

or

$$\begin{aligned} 0 &= \pi^e\left(\log\delta - \gamma\mathrm{E}_t\left[c^e_{t+1} - c^e_t\right] + \mathrm{E}_t\left[r_{i,t+1}\right]\right. \qquad \text{(A38)} \\ &\quad \left. + \frac{1}{2}\mathrm{Var}_t\left[r_{i,t+1} - \gamma\left(c^e_{t+1} - c^e_t\right)\right]\right) \\ &\quad + (1-\pi^e)\left(\log\delta^r - \gamma\mathrm{E}_t\left[c^r_{t+1} - c^e_t\right] + \mathrm{E}_t\left[r_{i,t+1}\right]\right. \\ &\quad \left. \frac{1}{2}\mathrm{Var}_t\left[r_{i,t+1} - \gamma\left(c^r_{t+1} - c^e_t\right)\right]\right). \end{aligned}$$

Optimal portfolio rule

We start guessing the functional form of the optimal policies in the employment state is:

$$\begin{aligned} c^e_t - l_t &= b^e_0 + b^e_1(w_t - l_t)\,, \qquad \text{(A39)} \\ \alpha^e_t &= \alpha^e. \end{aligned}$$

Note that we can also write the optimal consumption policy in the retirement state (6.42) in the same form as equation (**??**):

$$c_{t+1}^r - l_{t+1} = b_0^r + b_1^r \left(w_{t+1} - l_{t+1} \right), \tag{A40}$$

where $b_1^r = 1$.

Subtracting the log Euler equation (A38) for $r_{i.t+1} = r_f$ from the log Euler equation (A38) for $r_{i.t+1} = r_{t+1}$ yields:

$$\begin{aligned} \mathrm{E}_t\, r_{t+1} - r_f + \frac{1}{2} \mathrm{Var}_t \left(r_{t+1} \right) &= \gamma \pi^e \, \mathrm{Cov}_t \left(c_{t+1}^e - c_t^e, r_{t+1} \right) \\ &\quad + \gamma \left(1 - \pi^e \right) \mathrm{Cov}_t \left(c_{t+1}^r - c_t^e, r_{t+1} \right) \end{aligned} \tag{A41}$$

But equations (A39) and (A40), the log-linear intertemporal budget constraint (A33) and the trivial equality

$$c_{t+1}^s - c_t^e = (c_{t+1}^s - l_{t+1}) - (c_t^e - l_t) + (l_{t+1} - l_t), \tag{A42}$$

imply that

$$\begin{aligned} \mathrm{Cov}_t \left(c_{t+1}^s - c_t^e, r_{t+1} \right) &= \mathrm{Cov}_t \left(b_1^s r_{p,t+1}^e + \left(1 - b_1^s \right) \left(l_{t+1} - l_t \right), r_{t+1} \right) \\ &= b_1^s \alpha_t^e \sigma_u^2 + \left(1 - b_1^s \right) \sigma_{\xi u}, \end{aligned} \tag{A43}$$

for $s = e$, r. The second line follows from the assumptions on asset returns and labor income. Substituting back into equation (A41) and using equation (A26) we find

$$\mu + \frac{1}{2} \sigma_u^2 = \gamma \left[\left(\pi^e b_1^e + (1 - \pi^e) \right) \alpha_t^e \sigma_u^2 + \pi^e \left(1 - b_1^e \right) \sigma_{\xi u} \right], \tag{A44}$$

from which equation (6.45) in text obtains immediately.

Optimal consumption rule

The log-Euler equation (A38) for $i = p$ and the trivial equality (A42) imply

$$\pi^e \, \mathrm{E}_t \left[c_{t+1}^e - l_{t+1} \right] + \left(1 - \pi^e \right) \mathrm{E}_t \left[c_{t+1}^r - l_{t+1} \right] = \left(c_t^e - l_t \right) + \Upsilon_t^e, \tag{A45}$$

where

$$\Upsilon_t^e \equiv \Upsilon^e = \frac{1}{\gamma} \left(\mathrm{E} \left[r_{p,t+1}^e \right] + \frac{1}{2} \mathrm{V}^e + \pi^e \log \delta + \left(1 - \pi^e \right) \log \delta^r \right) - g, \tag{A46}$$

and

$$\begin{aligned} \mathrm{V}^e &= [\pi^e(1-b_1^e\gamma)^2 + (1-\pi^e)(1-b_1^r\gamma)^2]\,\mathrm{Var}[r_{p,t+1}^e] \\ &\quad +\pi^e\gamma(1-b_1^e)\,\mathrm{Var}[\Delta l_{t+1}] \\ &\quad -2\pi^e\gamma(1-\gamma b_1^e)(1-b_1^e)\,\mathrm{Cov}[r_{p,t+1}^e, \Delta l_{t+1}]. \end{aligned} \tag{A47}$$

If we substitute equations (A39) and (A40) into equation (A45) we obtain

$$\overline{b}_0 + \overline{b}_1\,\mathrm{E}_t\left[w_{t+1} - l_{t+1}\right] = \Upsilon^e + b_0^e + b_1^e\left(w_t - l_t\right), \tag{A48}$$

where $\overline{b}_0 = \pi^e b_0^e + (1-\pi^e)b_0^r$ and $\overline{b}_1 = \pi^e b_1^e + (1-\pi^e)b_1^r$. Further substitution of the log budget constraint in the employment state (A33) and guess (A39) in the left-hand side of equation (A48) yields

$$\begin{aligned} &\overline{b}_0 + \overline{b}_1\left(\rho_w^e - \rho_c^e b_1^e\right)(w_t - l_t) + \overline{b}_1\left(k^e - \rho_c^e b_0^e - g + \mathrm{E}\,r_{p,t+1}^e\right) \\ = \;&\Upsilon^e + b_0^e + b_1^e\left(w_t^e - l_t\right). \end{aligned}$$

Identifying coefficients on both sides of this equation we get the following two-equation system:

$$\begin{aligned} \overline{b}_1\left(\rho_w^e - \rho_c^e b_1^e\right) &= b_1^e, \\ \overline{b}_0 + \overline{b}_1\left(k^e - \rho_c^e b_0^e - g + \mathrm{E}\,r_{p,t+1}^e\right) &= \Upsilon^e + b_0^e. \end{aligned}$$

We can solve this system recursively, since the first equation depends only on b_1^e and the second on b_1^e and b_0^e.

Simple algebraic manipulation of the first equation gives the following quadratic equation for b_1^e:

$$0 = \pi^e\rho_c^e\left(b_1^e\right)^2 + \left[1 - \pi^e\rho_w^e + (1-\pi^e)\,\rho_c^e\right]b_1^e - (1-\pi^e)\,\rho_w^e. \tag{A49}$$

The expression for b_0^e is given by

$$\begin{aligned} b_0^e &= -\frac{1}{k_1}\Bigg[\left(\frac{1}{\gamma} - \overline{b}_1\right)\mathrm{E}\left[r_{p,t+1}^e\right] + \frac{1}{\gamma}\left(\pi^e\log\delta + (1-\pi^e)\log\delta^r\right) \\ &\quad + \frac{1}{2\gamma}\,\mathrm{V}^e - \pi^e\left(1-b_1^e\right)g - (1-\pi^e)\,b_0^r - k^e\Bigg], \end{aligned} \tag{A50}$$

with

$$k_1 = (1-\pi^e) + \rho_c^e\overline{b}_1 > 0. \tag{A51}$$