Appendix to John Y. Campbell and John H. Cochrane, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior"

1. Pricing a dividend claim

Substituting the definition of D in equation (19), we obtain

$$\frac{P_t}{D_t}(s_t) = GE_t\left[M_{t+1}e^{w_{t+1}}\left(1+\frac{P_{t+1}}{D_{t+1}}(s_{t+1})\right)\right].$$

To avoid integrating over two random variables, v and w, we use the Law of Iterated Expectations to derive

$$E\left[g(v)e^{w}\right] = E\left[E(g(v)e^{w}|v)\right] = E\left[g(v)E(e^{w}|v)\right].$$

Then, the dividend claim satisfies

$$\frac{P_t}{D_t}(s_t) = Ge^{\frac{1}{2}(1-\rho^2)\sigma_w^2}E_t\left[M_{t+1}e^{\rho\frac{\sigma_w}{\sigma}v_{t+1}}\left(1+\frac{P_{t+1}}{D_{t+1}}(s_{t+1})\right)\right]. \quad (27)$$

We plug in the definition of M in terms of the shock v_{t+1} and solve this equation numerically on a grid, using a numerical integrator to evaluate the conditional expectation over the normally distributed v_{t+1}. .

Comparing this equation with the price-dividend ratio of the consumption claim, (17), the difference lies in the term $e^{\frac{1}{2}(1-\rho^2)\sigma_w^2}$ in front and the term $e^{\rho\frac{\sigma_w}{\sigma}v_{t+1}}$ rather than $e^{v_{t+1}}$ inside. We will use parameters $\sigma_w \approx 0.10$ at an annual frequency and $\rho = 0.2$, so the first term is roughly $e^{0.005}$ which is very close to one. The term $\rho\frac{\sigma_w}{\sigma}$ is the regression coefficient of dividend growth on consumption growth. Although dividend growth and consumption growth are poorly correlated, dividend growth is much more volatile than consumption growth, so this regression coefficient is in fact about one. This explains our finding that the price-dividend ratio of the dividend claim is similar to the price-dividend ratio of the consumption claim.

2. Density of s in continuous time.

The continuous time version of the surplus consumption ratio transition equation is

$$ds_t = (1-\phi)(\bar{s}-s_t)\,dt + \lambda(s_t)\sigma dB_t. \quad (28)$$

The forward equation implies that the stationary density $q(s)$ of a diffusion $ds = \mu(s)dt + \sigma(s)dB$, if it exists, can be expressed as

$$q(s) = \frac{z(s)}{\int z(s)ds}$$

where

$$z(s) = \frac{e^{2\int^s dv \frac{\mu(v)}{\sigma^2(v)}}}{\sigma^2(s)}.$$

Evaluating the integral with

$$\mu(s) = (1-\phi)(\bar{s} - s)$$

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}},$$

$$\lambda(s) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2\left(s - \bar{s}\right)} - 1, & s \leq s_{\max} \\ 0 & s \geq s_{\max} \end{cases}$$

we have

$$\begin{aligned} \ln z(s) &= 2\int^s dv \frac{\mu(v)}{\sigma^2(v)} - 2\ln\sigma(s) \\ &= 2\int^s dv \frac{\mu(v)}{\sigma^2\lambda^2(v)} - 2\ln\lambda(s) - 2\ln\sigma \end{aligned}$$

Express $\mu(v)\ dv$ in terms of $\lambda(v)$.

$$\lambda(v) = \frac{1}{\bar{S}}\sqrt{1 - 2\left(v - \bar{s}\right)} - 1$$

$$\mu(v) = (1-\phi)\left(\bar{s} - v\right) = (1-\phi)\frac{\bar{S}^2\left[1 + \lambda(v)\right]^2 - 1}{2}$$

$$d\lambda = \frac{-dv}{\bar{S}\sqrt{1 - 2\left(v - \bar{s}\right)}} = \frac{-dv}{\bar{S}^2\left[\lambda(v) + 1\right]}$$

$$-\bar{S}^2\left[1 + \lambda(v)\right]d\lambda = dv$$

now the integral,

$$\begin{aligned} \ln z(s) &= 2\int^s dv \frac{\mu(v)}{\sigma^2(v)} - 2\ln\sigma(s) \\ &= -2\int^s \bar{S}^2\left[1+\lambda\right](1-\phi)\frac{\bar{S}^2\left[1+\lambda\right]^2 - 1}{2\sigma^2\lambda^2}d\lambda - 2\ln\lambda(s) - 2\ln\sigma \\ &= -\frac{(1-\phi)\bar{S}^2}{\sigma^2}\int^s \left[1+\lambda\right]\frac{\bar{S}^2\left[1 + 2\lambda + \lambda^2\right] - 1}{\lambda^2}d\lambda - 2\ln\lambda(s) - 2\ln\sigma \\ &= -\frac{(1-\phi)\bar{S}^2}{\sigma^2}\int^s \left[1+\lambda\right]\frac{\bar{S}^2 - 1 + 2\lambda\bar{S}^2 + \lambda^2\bar{S}^2}{\lambda^2}d\lambda - 2\ln\lambda(s) - 2\ln\sigma \end{aligned}$$

$$
\begin{aligned}
&= -\frac{(1-\phi)\bar{S}^2}{\sigma^2}\int^s \frac{\bar{S}^2-1+2\lambda\bar{S}^2+\lambda^2\bar{S}^2+\lambda\bar{S}^2-\lambda+2\lambda^2\bar{S}^2+\lambda^3\bar{S}^2}{\lambda^2}d\lambda - 2\ln\lambda(s) - 2\ln\sigma \\
&= -\frac{(1-\phi)\bar{S}^2}{\sigma^2}\int^s \left[\frac{\bar{S}^2-1}{\lambda^2}+\frac{\left(3\bar{S}^2-1\right)}{\lambda}+3\bar{S}^2+\bar{S}^2\lambda\right]d\lambda - 2\ln\lambda(s) - 2\ln\sigma \\
&= -\frac{(1-\phi)\bar{S}^2}{\sigma^2}\left[\frac{1-\bar{S}^2}{\lambda(s)}+\left(3\bar{S}^2-1\right)\ln\lambda(s)+3\bar{S}^2\lambda(s)+\frac{\bar{S}^2\lambda(s)^2}{2}\right] - 2\ln\lambda(s) - 2\ln\sigma
\end{aligned}
$$

using $\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}}$,

$$
\ln z(s) = -\gamma\bar{S}^2\left[\frac{\bar{S}^{-2}-1}{\lambda(s)}+3\lambda(s)+\frac{\lambda(s)^2}{2}\right] - \left[\gamma\left(3\bar{S}^2-1\right)+2\right]\ln\lambda(s) - 2\ln\sigma.
$$

As $s \to -\infty$, the λ^2 and λ terms dominate, so

$$
\ln z(s) \approx -\gamma\,|s| - 2\gamma\bar{S}\sqrt{2\,|s|}.
$$

With the more general model that allows interest rate variation, we have the more general version,

$$
\ln z(s) = \frac{\gamma^4\sigma^2\,(\phi-1)}{(\gamma\,(1-\phi)-B)^2}\left[\frac{\lambda(s)^2}{2}+3\lambda(s)+\frac{\bar{S}^{-2}-1}{\lambda(s)}\right] + \left[\frac{\left(3-\bar{S}^{-2}\right)\gamma^4\sigma^2\,(\phi-1)}{(\gamma\,(1-\phi)-B)^2}-2\right]\ln\lambda(s).
$$

3. Aggregation

Aggregating heterogeneous groups

Suppose each individual i in group j has the same endowment, which is a constant fraction of average consumption

$$
C_t^{ij} = C_t^j = \alpha^j C_t^a
$$

where C^j is group j average consumption, $C_t^j \equiv \frac{1}{N^j}\sum_{i\in j} C_t^{ij}$, N^j is the number of people in group j, C^a is economywide average consumption, $C_t^a \equiv \frac{1}{N}\sum_{ij} C_t^{ij}$, N is the number of people, and α^j is a constant. We use the symbol C for the endowment, but we have to verify that the consumer does not trade away from this endowment. Once this is done, we can interpret C as the post-trade allocation from a different endowment stream. Everyone's log consumption growth is the same,

$$
c_t^{ij} - c_{t-1}^{ij} = c_t^a - c_{t-1}^a.
$$

However, some groups have lower *levels* of consumption than other groups.

Habit for group j evolves from average consumption for group j, As before, we describe habit formation via the evolution of the surplus consumption ratio for group j, $S_t^j \equiv (C_t^j - X_t^j)/C_t^j$ by

$$s_{t+1}^j = (1-\phi)\bar{s} + \phi s_t^j + \lambda\left(s_t^j\right)\left(c_{t+1}^j - c_t^j - g\right).$$

Since consumption growth is the same for everyone, and starting from an initial state with equal surplus consumption ratios, all surplus consumption ratios are the same,

$$s_t^j = s_t^a \;\; \forall i, j$$

where s_t^a denotes the common value of the surplus consumption ratio.

Individuals in group j use group j habit. Therefore, marginal utility for individual i in group j is

$$MU_t^j = \left(C_t^j - X_t^j\right)^{-\gamma} = \left(C_t^j S_t^j\right)^{-\gamma} = (C_t^j S_t^a)^{-\gamma}$$

Plugging in the endowment rule, we find that all marginal utilities move in lockstep,

$$MU_t^j = (\alpha^j)^{-\gamma}(C_t^a S_t^a)^{-\gamma}$$

Ratios of marginal utilities MU_t^j/MU_{t-1}^j are the same for all agents, and they all agree on asset prices. They have no reason to trade away from the endowments.

Asset prices and aggregate quantities can be represented from the preferences of a fictitious representative agent with marginal utility

$$MU_t = (C_t^a S_t^a)^{-\gamma}.$$

Since all consumption growths and surplus consumption ratios are the same, the common surplus consumption ratio evolves as it should,

$$s_{t+1}^a = (1-\phi)\bar{s} + \phi s_t^a + \lambda\left(s_t\right)\left(c_{t+1}^a - c_t^a - g\right).$$

C^a is average consumption and S^a is the average surplus consumption ratio. However, the "average habit" in the representative agent surplus consumption ratio is a weighted average of individual group habits. Define X_t^a so that $S_t^a = (C_t^a - X_t^a)/C_t^a$. Solving for X_t^a,

$$X_t^a = C_t^a(1 - S_t^a) = C_t^a \frac{X_t^j}{C_t^j} = \frac{X_t^j}{\alpha^j}$$

Therefore, aggregate habit is a weighted average of individual habits.

$$X_t^a = \frac{1}{N}\sum_{i,j}\frac{X_t^j}{\alpha^j}$$

Aggregating heterogenous individuals

Each consumer i receives an endowment C_t^i such that

$$\xi_i \left(C_t^i - X_t\right)^{-\gamma} = \xi(C_t^a - X_t)^{-\gamma}.$$

Therefore,

$$\begin{aligned}\xi_i^{-1/\gamma}\left(C_t^i - X_t\right) &= \xi^{-1/\gamma}(C_t^a - X_t)\\ \xi_i^{-1/\gamma}S_t^iC_t^i &= \xi^{-1/\gamma}S_t^aC_t^a,\end{aligned} \tag{29}$$

where

$$S^i \equiv \frac{C^i - X}{C^i}.$$

S^i has the property that

$$\begin{aligned}C^i(1-S^i) &= C^a(1-S^a)\\ C^iS^i &= C^i - C^a + C^aS^a\end{aligned}$$

Substituting in (29),

$$\begin{aligned}\xi_i^{-1/\gamma}\left[C_t^i - C_t^a + C_t^aS_t^a\right] &= \xi^{-1/\gamma}S_t^aC_t^a\\ (C_t^i - C_t^a) &= \frac{\left[\xi^{-1/\gamma} - \xi_i^{-1/\gamma}\right]}{\xi_i^{-1/\gamma}}S_t^aC_t^a\end{aligned}$$

Summing over i and if we choose

$$\xi^{-1/\gamma} = \frac{1}{N}\sum_i \xi_i^{-1/\gamma},$$

we find that C^a is in fact the average of the C^i.

4. Risk aversion

To calculate risk aversion in our economy, consider an individual consumer. His problem is to choose contingent consumption claims at each date and state, given initial wealth. Using a superscript a to distinguish individual and aggregate quantities, and focusing on period 0 for notational simplicity, the consumer's problem is

$$V(W_0, W_0^a, S_0^a) \equiv \max_{\{C_t\}} E_0\sum_t \delta^t u(C_t, X_t) \;\; s.t. \; E_0\sum_t \delta^t\frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_t)}C_t \le W_0. \tag{30}$$

The $u_c(C^a, X)$ terms give asset prices. In equilibrium for a representative agent, wealth is the value of a share of aggregate consumption,

$$W_0 \equiv E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u(C_0^a, X_t)} C_t^a.$$

The extra state variables W^a and S^a enter the value function in order to describe asset prices and the level of external habit. Asset prices depend on the state variable S_t^a. Then one of W_t^a, C_t^a or X_t can describe the level of external habit X_{t+j} that appears in individual utility.

Risk aversion is defined as

$$rra \equiv -\frac{W V_{WW}}{V_W} = -\frac{\partial \ln V_W(W_0, W_0^a, S_0^a)}{\partial \ln V_W}$$

Since it is defined from the value function, risk aversion is a property of preferences, technology, and market structure, not a property of preferences alone. Risk aversion measures aversion to purely idiosyncratic bets on wealth. Crucially, the increase in wealth ∂W_t occurs for the individual, holding all aggregate variables constant.

The envelope condition $u_c = V_w$ means that the risk aversion coefficient can be written

$$rra = -\frac{\partial \ln V_W(W_0, W_0^a, S_0^a)}{\partial \ln V_W} = -\frac{\partial \ln u_c(C_0, X)}{\partial \ln C_0} \times \frac{\partial \ln C_0}{\partial \ln W_0} = \eta_0 \frac{\partial \ln C_0}{\partial \ln W_0}, \qquad (31)$$

where we denote curvature by η_t,

$$\eta_t \equiv -\frac{C_t u_{cc}(C_t, X_t)}{u_c(C_t, X_t)} = -\frac{\partial \ln u_c(C_t, X_t)}{\partial \ln C_t}.$$

In our model, $\eta_t = \gamma / S_t$. Finding risk aversion is therefore reduced to finding out how much consumption at date 0 reacts to an idiosyncratic wealth change. To answer this question, we have to find out how much consumption at all dates and states responds to the wealth change and impose the budget constraint.

The consumer's first order conditions for choice of C_t are

$$u_c(C_t, X_t) = -\xi(W_0, W_0^a, S_0^a)\; u_c(C_t^a, X_t)$$

where ξ is a constant of proportionality. Differentiating to find the effect of a wealth change,

$$u_{cc}(C_t, X_t) \frac{\partial C_t}{\partial W_0} = -\frac{\partial \xi}{\partial W_0} u_c(C_t^a, X_t)$$

Simplifying and evaluating at $C = C^a$,

$$\left[-\frac{C_t u_{cc}(C_t, X_t)}{u_c(C_t, X_t)} \right] \frac{W_0 \partial C_t}{C_t \partial W_0} = \frac{W_0 \partial \xi}{\partial W_0}$$

Therefore, the elasticity of date (state-) t consumption with respect to initial wealth is inversely proportional to curvature,

$$\frac{\partial \ln C_t}{\partial \ln W_0} = \frac{const}{\eta_t}.$$

To raise marginal utility by the same proportion in each state, consumption rises by a larger proportion in good states with low curvature. Evaluating this equation at time 0, and comparing to the definition of risk aversion, we find that the constant is time-0 risk aversion,

$$\frac{\partial \ln C_t}{\partial \ln W_0} = \frac{rra_0}{\eta_t}. \tag{32}$$

To evaluate the constant, and therefore initial period risk aversion we differentiate the budget constraint,

$$E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_0)} \frac{\partial C_t}{\partial W_0} = 1 \tag{33}$$

$$E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_t)} \frac{C_t}{W_0} \frac{\partial \ln C_t}{\partial \ln W_0} = 1 \tag{34}$$

$$E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_t)} C_t \frac{rra_0}{\eta_t} = W_0$$

$$rra_0 = \frac{E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_t)} C_t}{E_0 \sum_t \delta^t \frac{u_c(C_t^a, X_t)}{u_c(C_0^a, X_t)} C_t \eta_t^{-1}}$$

Thus the risk aversion coefficient is the price of the consumption stream divided by the price of a security that pays the consumption stream times the inverse of the local curvature of the utility function.

In our model, marginal utility is $u_c(t) = (C_t S_t)^{-\gamma}$, and utility curvature is $\eta_t = \gamma / S_t$. Hence, the risk aversion coefficient is a function only of the surplus consumption ratio S and it can be expressed as the price of a claim to C_t divided by the price of a claim to $C_t S_t$.

$$rra_t = rra(S_t) \ = \gamma \times \frac{E_t \sum_{j=0}^{\infty} \delta^j (S_{t+j} C_{t+j})^{-\gamma} C_{t+j}}{E_t \sum_{j=0}^{\infty} \delta^j (S_{t+j} C_{t+j})^{-\gamma} S_{t+j} C_{t+j}}. \tag{35}$$

We calculate the price of the claim to C_tS_t on a grid like the other assets and find, as explained in the text, that the risk aversion coefficient is greater than the utility curvature coefficient. This behavior is illustrated in the figure below.

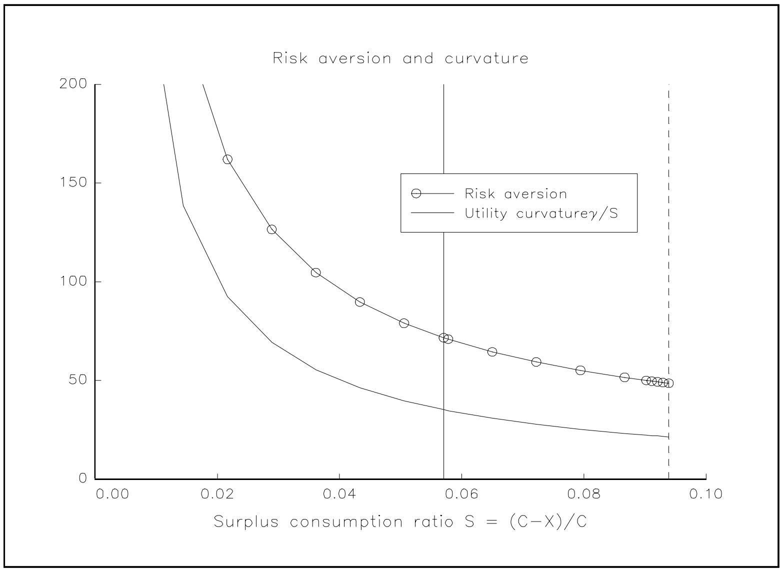

Figure A1. Risk aversion and utility curvature .

Equation (35) gives one explanation for this behavior. In a nonstochastic steady state, this formula for risk aversion reduces to curvature, $rra = \gamma/S$. When there are shocks, the price of CS is lower than S times the price of C, for the simple reason that S is procyclical (low in high marginal utility states) and hence risky.

The fact that we find risk aversion higher than curvature, together with equation (31) means that consumption responds more than proportionally to wealth in every state. This result might seem to be inconsistent with the budget constraint (33) that "on average" consumption must move proportionally with wealth. However, the budget constraint is not violated. First, "on average" is weighted by contingent claims prices and the C/W or dividend/price ratio. Second, one should not confuse $\partial \ln C_t/\partial \ln W_t$ in a different initial state S_t with $\partial C_{t+j}/\partial W_t$ across potential future states S_{t+j}. In response to a change in W_t, equation (32) shows that consumption in future dates (and states) increases in inverse proportion to local curvature,

$$\frac{1}{\eta_{t+j}}\frac{\partial \ln C_{t+j}}{\partial \ln W_t} = \frac{1}{\eta_t}\frac{\partial \ln C_t}{\partial \ln W_t}.$$

Thus, although initial consumption responds more than proportionally to wealth in every initial state, $\partial \ln C_t/\partial \ln W_t > 1$ at every value of S_t, future consumption responds less ($\partial \ln C_{t+j}/\partial \ln W_t < \partial \ln C_t/\partial \ln W_t$) in future states with higher curvature η_{t+j} (lower surplus consumption ratios) than the initial state. States with high curvature also have higher contingent claims prices and higher C/W (price/dividend) ratios, and so count more in the budget constraint. There are enough such states to satisfy the budget constraint.

5. Marginal utility with internal habit

Marginal utility for the internal-habit version of our model, with random walk consumption, is given by the following procedure.

1) Find a function $Z(S)$ from the recursive relation

$$Z(S_t) = S_t^{1-\gamma} + \delta G^{1-\gamma} \phi E_t \left\{ e^{(1-\gamma)v_{t+1}} \left[1 + \lambda'(s_t) v_{t+1}\right] Z(S_{t+1}) \right\}.$$

2) Marginal utility is then given by

$$\begin{aligned} \frac{\partial U_t}{\partial C_t}(S_t, C_t) &= C_t^{-\gamma} f(S_t) \\ f(S_t) &= S_t^{1-\gamma}\left[1 + \lambda(s_t)\right] \\ &\quad -\lambda(s_t)\delta G^{1-\gamma} E\left\{ e^{(1-\gamma)v_{t+1}} Z(S_{t+1}) \left[1 - \phi - \phi\lambda'(s_t)v_{t+1}\right] | S_t \right\}. \end{aligned}$$

In the external case, marginal utility is given by $C^{-\gamma}S^{-\gamma}$. In the internal case, marginal utility is still given by a function separable between consumption and the surplus consumption ratio, $C^{-\gamma} f(S)$. Beyond this observation, the formula is ugly, and we are not able to provide much intuition.

The derivation consists of laboriously evaluating the derivatives $\partial X_{t+j}/\partial C_t$ in (26) from the surplus consumption ratio evolution equation. To simplify the notation, specialize to $t = 0$. The setup is

$$U_0 = E_0 \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma}$$

s.t.

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t)\left[c_{t+1} - c_t - g\right]; \quad s_t = \ln S_t = \ln\left(1 - \frac{X_t}{C_t}\right).$$

Marginal utility is

$$\frac{\partial U_0}{\partial C_0} = (C_0 - X_0)^{-\gamma} - E_t \left\{ \sum_{j=0}^{\infty} \delta^t \left(C_t - X_t\right)^{-\gamma} \frac{\partial X_t}{\partial C_0} \right\} \tag{36}$$

Examining (36), our task is simply to evaluate partial derivatives $\partial X_t/\partial C_0$. To do this directly, we have to iterate the state transition equation t times and then take the derivative. This is infeasible, but we can take the derivative and then iterate, since $\partial X_2/\partial C_0 = \partial X_2/\partial X_1 \times \partial X_1/\partial C_0$.

Write the s transition equation as

$$\ln\left(1-\frac{X_{t+1}}{C_{t+1}}\right) = (1-\phi)\bar{s} + \phi \ln\left(1-\frac{X_t}{C_t}\right) + \lambda(\ln\left(1-\frac{X_t}{C_t}\right))\left[\ln C_{t+1} - \ln C_t - g\right].$$

This defines a function

$$X_{t+1} = X_{t+1}(X_t, C_{t+1}, C_t).$$

Take a total derivative to get

$$-\frac{1}{S_{t+1}C_{t+1}}dX_{t+1} + \frac{X_{t+1}}{S_{t+1}C_{t+1}^2}dC_{t+1} = -\frac{\phi}{S_tC_t}dX_t + \frac{\phi X_t}{S_tC_t^2}dC_t$$

$$+\lambda'(s_t)\left[-\frac{\phi}{S_tC_t}dX_t + \frac{\phi X_t}{S_tC_t^2}dC_t\right]\left[\ln C_{t+1} - \ln C_t - g\right] + \lambda(s_t)\left[\frac{1}{C_{t+1}}dC_{t+1} - \frac{1}{C_t}dC_t\right]$$

Isolating dX terms,

$$-\frac{1}{S_{t+1}C_{t+1}}dX_{t+1} + \left[\frac{X_{t+1}}{S_{t+1}C_{t+1}^2} - \frac{\lambda(s_t)}{C_{t+1}}\right]dC_{t+1} = -\frac{\phi}{S_tC_t}\left[1 + \lambda'(s_t)\left(\ln C_{t+1} - \ln C_t - g\right)\right]dX_t$$

$$+\frac{1}{C_t}\left[\frac{\phi X_t}{S_tC_t}\left[1 + \lambda'(s_t)\left(\ln C_{t+1} - \ln C_t - g\right)\right] - \lambda(s_t)\right]dC_t.$$

Therefore, the required partial derivatives are

$$\frac{\partial X_{t+1}}{\partial X_t} = \phi\frac{S_{t+1}C_{t+1}}{S_tC_t}\left[1 + \lambda'(s_t)\left(\ln C_{t+1} - \ln C_t - g\right)\right]$$

$$\frac{\partial X_{t+1}}{\partial C_{t+1}} = \frac{X_{t+1}}{C_{t+1}} - S_{t+1}\lambda(s_t)$$

$$\frac{\partial X_{t+1}}{\partial C_t} = -S_{t+1}\frac{C_{t+1}}{C_t}\left[\frac{\phi X_t}{S_tC_t}\left[1 + \lambda'(s_t)\left(\ln C_{t+1} - \ln C_t - g\right)\right] - \lambda(s_t)\right]$$

The $\lambda(s)$ terms are

$$\lambda(s) = \frac{1}{\bar{S}}\sqrt{1 - 2\left(s - \bar{s}\right)} - 1;\ \bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}}$$

so

$$\lambda'(s) = \frac{d\lambda(s)}{ds} = \frac{1}{2}\frac{1}{\bar{S}}\frac{-2}{\sqrt{1-2\left(s-\bar{s}\right)}} = -\frac{1}{\bar{S}\sqrt{1-2\left(s-\bar{s}\right)}}$$

Now we are ready to evaluate marginal utility. The first step is to express the derivatives of the quantity X_t, $\partial X_t / \partial C_0$, in terms of the derivatives of the function $X_{t+1}(X_t, C_{t+1}, C_t)$. (We could introduce a notation $f(X_t, C_{t+1}, C_t)$ to keep the two straight, but it's not worth the bother). X_1 is a function $X_1(X_0, C_1, C_0)$, so C_0 affects X_1 directly and via X_0. Thus, we get

$$\frac{\partial U_0}{\partial C_0} = (C_0 - X_0)^{-\gamma} - (C_0 - X_0)^{-\gamma}\frac{\partial X_0}{\partial C_0} - E_0\delta\left(C_1 - X_1\right)^{-\gamma}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]$$
$$-E_0\delta^2\left(C_2 - X_2\right)^{-\gamma}\frac{\partial X_2}{\partial X_1}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]$$
$$-E_0\delta^3\left(C_3 - X_3\right)^{-\gamma}\frac{\partial X_3}{\partial X_2}\frac{\partial X_2}{\partial X_1}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]....$$

Simplifying a bit,

$$C_0^{\gamma}\frac{\partial U_0}{\partial C_0} = S_0^{-\gamma} - S_0^{-\gamma}\frac{\partial X_0}{\partial C_0} - E_0\delta S_1^{-\gamma}\left(\frac{C_1}{C_0}\right)^{-\gamma}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]$$
$$-E_0\delta^2 S_2^{-\gamma}\left(\frac{C_2}{C_1}\frac{C_1}{C_0}\right)^{-\gamma}\frac{\partial X_2}{\partial X_1}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]$$
$$-E_0\delta^3 S_3^{-\gamma}\left(\frac{C_3}{C_2}\frac{C_2}{C_1}\frac{C_1}{C_0}\right)^{-\gamma}\frac{\partial X_3}{\partial X_2}\frac{\partial X_2}{\partial X_1}\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]....$$

Now, collect terms to write marginal utility as

$$C_0^{\gamma}\frac{\partial U_0}{\partial C_0} = S_0^{-\gamma}\left(1 - \frac{\partial X_0}{\partial C_0}\right) - E_0\left\{\delta\left(\frac{C_1}{C_0}\right)^{-\gamma} W_1\left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]\right\}$$

where

$$W_1 = S_1^{-\gamma} + \delta S_2^{-\gamma}\left(\frac{C_2}{C_1}\right)^{-\gamma}\frac{\partial X_2}{\partial X_1} + \delta^2 S_3^{-\gamma}\left(\frac{C_3}{C_2}\frac{C_2}{C_1}\right)^{-\gamma}\frac{\partial X_3}{\partial X_2}\frac{\partial X_2}{\partial X_1} + ...$$

W_t is recursive:

$$W_2 = S_2^{-\gamma} + \delta S_3^{-\gamma}\left(\frac{C_3}{C_2}\right)^{-\gamma}\frac{\partial X_3}{\partial X_2} + ...$$

$$W_1 = S_1^{-\gamma} + \delta \left(\frac{C_2}{C_1}\right)^{-\gamma} \frac{\partial X_2}{\partial X_1} W_2.$$

Now we have to handle the expectations. We want a recursive object that is a function of the state variable s. We can use the random walk rule for consumption, since will only evaluate marginal utility for the given consumption process, rather than for arbitrary consumption paths. We are *evaluating* a formula for the derivative, not *taking* a derivative at this stage.

By iterated expectations and since all the other terms in curly braces are functions of no more than time 1 information, we can replace W_1 with $E_1 W_1$ in the last expression for marginal utility,

$$C_0^{\gamma} \frac{\partial U_0}{\partial C_0} = S_0^{-\gamma} \left(1 - \frac{\partial X_0}{\partial C_0}\right) - E_0 \left\{\delta \left(\frac{C_1}{C_0}\right)^{-\gamma} E_1(W_1) \left[\frac{\partial X_1}{\partial C_0} + \frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial C_0}\right]\right\}$$

Substituting in the partial derivative formula $\partial X_2/\partial X_1$ from above into the W transition equation,

$$W_1 = S_1^{-\gamma} + \delta \left(\frac{C_2}{C_1}\right)^{-\gamma} \phi \frac{S_2 C_2}{S_1 C_1} \left[1 + \lambda'(s_1)\left(\ln C_2 - \ln C_1 - g\right)\right] W_2.$$

$$W_1 = S_1^{-\gamma} + \delta \phi G^{1-\gamma} e^{(1-\gamma)v_2} \frac{S_2}{S_1} \left[1 + \lambda'(s_1) v_2\right] W_2.$$

Therefore,

$$E_1(W_1) = S_1^{-\gamma} + \delta \phi G^{1-\gamma} E_1 \left\{ e^{(1-\gamma)v_2} \frac{S_2}{S_1} \left[1 + \lambda'(s_1) v_2\right] E_2(W_2)\right\}.$$

If $E_2(W_2)$ is a function of state S_2, then $E_1(W_1)$ is a function of state S_1. We use the notation $Z(S_{t+1}) \equiv S_{t+1} E_{t+1} W_{t+1}(S_{t+1})$.

Now we have

$$C_t^{\gamma} \frac{\partial U_t}{\partial C_t} = S_t^{-\gamma} \left(1 - \frac{\partial X_t}{\partial C_t}\right) - E_t \left\{\delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{Z(S_{t+1})}{S_{t+1}} \left[\frac{\partial X_{t+1}}{\partial C_t} + \frac{\partial X_{t+1}}{\partial X_t}\frac{\partial X_t}{\partial C_t}\right]\right\}$$

Note if we set all the X derivatives to zero we recover the external case $C_t^{-\gamma} S_t^{-\gamma}$.

¿From above, the partial derivatives are given by

$$\frac{\partial X_{t+1}}{\partial X_t} = \phi \frac{S_{t+1} C_{t+1}}{S_t C_t} \left[1 + \lambda'(s_t) v_{t+1}\right]$$

$$\frac{\partial X_t}{\partial C_t} = \frac{X_t}{C_t} - S_t \lambda(s_{t-1})$$

$$\frac{\partial X_{t+1}}{\partial C_t} = S_{t+1}\frac{C_{t+1}}{C_t}\left[\lambda(s_t) - \frac{\phi X_t}{S_t C_t}\left[1+\lambda'(s_t)v_{t+1}\right]\right]$$

Plug the formulas for partial derivatives into the formula for marginal utility and simplify.

$$C_t^{\gamma}\frac{\partial U_t}{\partial C_t} = S_t^{1-\gamma}\left(1+\lambda(s_{t-1})\right) -$$

$$E_t\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{Z(S_{t+1})}{S_{t+1}}\left\{S_{t+1}\frac{C_{t+1}}{C_t}\left[\lambda(s_t) - \frac{\phi X_t}{S_t C_t}\left[1+\lambda'(s_t)v_{t+1}\right]\right]\right.\right.$$
$$\left.\left.+\phi\frac{S_{t+1}C_{t+1}}{S_t C_t}\left[1+\lambda'(s_t)v_{t+1}\right]\left[\frac{X_t}{C_t} - S_t\lambda(s_{t-1})\right]\right\}\right\}$$

or

$$C_t^{\gamma}\frac{\partial U_t}{\partial C_t} = S_t^{1-\gamma}\left(1+\lambda(s_{t-1})\right) -$$

$$E_t\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} Z(S_{t+1})\frac{C_{t+1}}{C_t}\times\right.$$

$$\left.\left[\lambda(s_t) - \frac{\phi X_t}{S_t C_t}\left[1+\lambda'(s_t)v_{t+1}\right] + \frac{\phi}{S_t}\left[1+\lambda'(s_t)v_{t+1}\right]\left[\frac{X_t}{C_t} - S_t\lambda(s_{t-1})\right]\right]\right\}$$

or

$$C_t^{\gamma}\frac{\partial U_t}{\partial C_t} = S_t^{1-\gamma}\left(1+\lambda(s_{t-1})\right) -$$

$$E_t\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} Z(S_{t+1})\left[\lambda(s_t) + \left\{\frac{\phi}{S_t}\left[\frac{X_t}{C_t} - S_t\lambda(s_{t-1})\right] - \frac{\phi X_t}{S_t C_t}\right\}\left[1+\lambda'(s_t)v_{t+1}\right]\right]\right\}$$

or, finally,

$$C_t^{\gamma}\frac{\partial U_t}{\partial C_t} = S_t^{1-\gamma}\left(1+\lambda(s_{t-1})\right) - E_t\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} Z(S_{t+1})\left[\lambda(s_t) - \phi\lambda(s_{t-1})\left(1+\lambda'(s_t)v_{t+1}\right)\right]\right\}.$$

The conditional expectations only require one-step ahead simulation and so we can find them easily by numerical integration. Marginal utility is a function of state S_t, S_{t-1}. The form is then reassuring: $U_c(C_t) = C_t^{-\gamma}f(S_t, S_{t-1})$. It's initially surprising that $\lambda(s_{t-1})$ enters. That feature comes from the fact that X_t changes as C_t changes, and $\lambda(s_{t-1})$ controls the sensitivity of X_t to C_t. In the continuous time limit, the distinction between $\lambda(s_{t+1})$ and $\lambda(s_t)$ vanishes. Therefore, we save a state variable and approximate marginal utility as

$$C_t^{\gamma}\frac{\partial U_t}{\partial C_t} = S_t^{1-\gamma}\left[1+\lambda(s_t)\right] - E_t\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} Z(S_{t+1})\left[\lambda(s_t) - \phi\lambda(s_t)\left(1+\lambda'(s_t)v_{t+1}\right)\right]\right\}$$

This approximation is exact when $s_t = s_{t-1}$.

6. An example in which internal and external habits are indistinguishable

Suppose habit accumulation is linear, and there is a constant riskfree rate or linear technology equal to the discount rate, $R^f = 1/\delta$. The consumer's problem is then

$$\max \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \quad s.t. \sum_t \delta^t C_t = \sum_t \delta^t e_t + W_0; \quad X_t = \theta \sum_{j=1}^{\infty} \phi^j C_{t-j}$$

where e_t is a stochastic endowment. The first order conditions are

$$MU_t = E_t \left[MU_{t+1} \right]$$

where MU denotes marginal utility. In the external case, marginal utility is simply

$$MU_t = (C_t - X_t)^{-\gamma}. \tag{37}$$

In the internal case, marginal utility is

$$MU_t = (C_t - X_t)^{-\gamma} - \theta \sum_{j=1}^{\infty} \delta^j \phi^j E_t \left(C_{t+j} - X_{t+j} \right)^{-\gamma} \tag{38}$$

The sum measures the habit-forming effect of consumption. Now, guess the same solution as for the external case,

$$(C_t - X_t)^{-\gamma} = E_t \left[(C_{t+1} - X_{t+1})^{-\gamma} \right]. \tag{39}$$

and plug in to (38). We find that the internal marginal utility is simply proportional to marginal utility (37) in the external case,

$$MU_t = \left(1 - \frac{\theta \delta \phi}{1 - \delta \phi} \right) (C_t - X_t)^{-\gamma}. \tag{40}$$

Since this expression satisfies the first order condition $MU_t = E_t MU_{t+1}$, we confirm the guess (39). Ratios of marginal utility are the same, so allocations and asset prices are completely unaffected by internal vs. external habit in this example.