Intergenerational Risksharing and Equilibrium Asset Prices: Appendix

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1 Derivation of risk exposures under laissez faire

It is easy to derive closed form relationships relating the share of physical capital holdings, consumption and wealth of an agent of generation t in all future periods to dividend realizations since the time of birth and parameters of the model. The share of physical capital holdings purchased in period $t+s$, conditional on survival, is given by the following recursive formula:

$$\begin{aligned}
\widehat{\theta}^t_{t+s} &= \frac{\beta\chi\widehat{W}^t_{t+s}}{p_{t+s}} \\
&= \frac{\beta\chi\widehat{\theta}^t_{t+s-1}(p_{t+s}+d_{t+s}+A_{t+s})}{p_{t+s}} \\
&= \frac{\beta\chi\widehat{\theta}^t_{t+s-1}\frac{1}{\chi}(p_{t+s}+d_{t+s})}{p_{t+s}} \\
&= \frac{1}{\chi}\frac{\beta\chi h_{t+s}+d_{t+s}}{h_{t+s}+d_{t+s}}\widehat{\theta}^t_{t+s-1},
\end{aligned}$$

or, by iterated substitution,

$$\widehat{\theta}^t_{t+s} = \frac{1-\beta\chi}{1-\chi}\frac{h_t}{h_t+d_t}\frac{1}{\chi^s}\prod_{r=1}^{s}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}.$$

Expressions for wealth and consumption of an agent of generation t in period $t+s$ are given by:

$$\widehat{W}^t_{t+s} = \frac{\widehat{\theta}^t_{t+s}p_{t+s}}{\beta\chi} = \frac{1}{1-\chi}\frac{h_t}{h_t+d_t}\frac{1}{\chi^s}\left[\prod_{r=1}^{s-1}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}\right](\beta\chi h_{t+s}+d_{t+s}),$$

and

$$\widehat{C}^t_{t+s} = (1-\beta\chi)\widehat{W}^t_{t+s} = \frac{1-\beta\chi}{1-\chi}\frac{h_t}{h_t+d_t}\frac{1}{\chi^s}\left[\prod_{r=1}^{s-1}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}\right](\beta\chi h_{t+s}+d_{t+s}),$$

or in terms of period t consumption:

$$\widehat{C}^t_{t+s} = \frac{\widehat{C}^t_t}{h_t+d_t}\frac{1}{\chi^s}\left[\prod_{r=1}^{s-1}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}\right](\beta\chi h_{t+s}+d_{t+s}),$$

Expected utility of period $t+s$ consumption is related to period t consumption by the following relation:

$$E_t\log(\widehat{C}^t_{t+s}) = \log(\widehat{C}^t_t) - \log(h_t+d_t) + E_t\log\left\{\frac{1}{\chi^s}\left[\prod_{r=1}^{s-1}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}\right](\beta\chi h_{t+s}+d_{t+s})\right\},$$

In fact, for agents of any generation $r \leq t$, and for any $s > 0$:

$$\begin{aligned}
E_t\log(\widehat{C}^r_{t+s}) &= \log(\widehat{C}^r_t) - \log(h_t+d_t) + E_t\log\left\{\frac{1}{\chi^s}\left[\prod_{r=1}^{s-1}\frac{\beta\chi h_{t+r}+d_{t+r}}{h_{t+r}+d_{t+r}}\right](\beta\chi h_{t+s}+d_{t+s})\right\} \\
&= \log(\widehat{C}^r_t) - \log(h_t+d_t) + \varphi_{st},
\end{aligned}$$

where φ_{st} is a common constant for all generations. From this it follows that expected lifetime utility, as of the beginning of period t, of agents of any generation $r \leq t$ is given by:

$$
\begin{aligned}
\widehat{U}_t^r &\equiv E_t\left\{\sum_{s=0}^{\infty}(\beta\chi)^s \log(\widehat{C}_{t+s}^r)\right\} \\
&= \log(\widehat{C}_t^r) + \sum_{s=1}^{\infty}(\beta\chi)^s E_t \log(\widehat{C}_{t+s}^r) \\
&= \log(\widehat{C}_t^r) + \sum_{s=1}^{\infty}(\beta\chi)^s \left[\log(\widehat{C}_t^r) - \log(h_t + d_t) + \varphi_{st}\right] \\
\widehat{U}_t^r &= \frac{1}{1-\beta\chi}\log(\widehat{C}_t^r) - \frac{\beta\chi}{1-\beta\chi}\log(h_t + d_t) + \varphi_t,
\end{aligned}
$$

where $\varphi_t = \sum_{s=1}^{\infty}(\beta\chi)^s\varphi_{st}$ is a common constant for all generations.

For an agent of generation t:

$$\widehat{C}_t^t = (1-\beta\chi)\frac{h_t}{1-\chi},$$

so that expected lifetime utility at the beginning of period t is given:

$$\widehat{U}_t^t = \frac{1}{1-\beta\chi}\log\frac{1-\beta\chi}{1-\chi} + \frac{1}{1-\beta\chi}\log(h_t) - \frac{\beta\chi}{1-\beta\chi}\log(h_t + d_t) + \varphi_t$$

For any generation $r < t$:

$$\widehat{C}_t^r = (1-\beta\chi)\widehat{\theta}_{t-1}^r\frac{1}{\chi}(d_t + p_t) = \widehat{\theta}_{t-1}^r\frac{1}{\chi}(\beta\chi h_t + d_t),$$

so that expected lifetime utility at the beginning of period t is given by:

$$\widehat{U}_t^r = \frac{1}{1-\beta\chi}\log\widehat{\theta}_{t-1}^r + \frac{1}{1-\beta\chi}\log\frac{1}{\chi}(\beta\chi h_t + d_t) - \frac{\beta\chi}{1-\beta\chi}\log(h_t + d_t) + \varphi_t.$$

2 Derivation of the price of physical capital with social security

We can use the SDF in the presence of social security to price physical capital:

$$p_t^s = E_t[M_{t+1}^s(d_{t+1} + p_{t+1})].$$

Substituting in the expression for the SDF:

$$p_t^s = E_t\left[\frac{h_t + d_t}{h_{t+1} + \frac{\theta\mu_{dt+1}}{\beta\chi} + \varepsilon_{t+1}}(d_{t+1} + p_{t+1})\right].$$

Iterating forward:

$$p_t^s = (h_t + d_t)E_t \left\{ \begin{array}{c} \frac{d_{t+1}}{h_{t+1}+\frac{\theta\mu_{dt+1}}{\beta\chi}+\varepsilon_{t+1}} \\ +\frac{1}{h_{t+1}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+1}} E_{t+1}\left[\frac{h_{t+1}+d_{t+1}}{h_{t+2}+\frac{\theta\mu_{dt+2}}{\beta\chi}+\varepsilon_{t+2}}(d_{t+2}+E_{t+2}[....])\right] \end{array} \right\}.$$

By the law of iterated expectations:

$$p_t^s = (h_t + d_t) \left\{ \begin{array}{c} E_t\left[\frac{d_{t+1}}{h_{t+1}+\frac{\theta\mu_{dt+1}}{\beta\chi}+\varepsilon_{t+1}}\right] \\ +E_t\left[\frac{h_{t+1}+d_{t+1}}{h_{t+1}+\frac{\theta\mu_{dt+1}}{\beta\chi}+\varepsilon_{t+1}}\frac{d_{t+2}}{h_{t+2}+\frac{\theta\mu_{dt+2}}{\beta\chi}+\varepsilon_{t+2}}\right] \\ +E_t\left[\frac{h_{t+1}+d_{t+1}}{h_{t+1}+\frac{\theta\mu_{dt+1}}{\beta\chi}+\varepsilon_{t+1}}\frac{h_{t+2}+d_{t+2}}{h_{t+2}+\frac{\theta\mu_{dt+2}}{\beta\chi}+\varepsilon_{t+2}}\frac{d_{t+3}}{h_{t+3}+\frac{\theta\mu_{dt+3}}{\beta\chi}+\varepsilon_{t+3}}\right]+.... \end{array} \right\},$$

or

$$p_t^s = (h_t + d_t)E_t\left\{\sum_{s=1}^{\infty}\left[\prod_{r=1}^{s-1}\frac{h_{t+r}+d_{t+r}}{h_{t+r}+\frac{\theta\mu_{dt+r}}{\beta\chi}+\varepsilon_{t+r}}\right]\frac{d_{t+s}}{h_{t+s}+\frac{\theta\mu_{dt+s}}{\beta\chi}+\varepsilon_{t+s}}\right\}.$$

This is a closed form solution for the price of physical capital in terms of parameters of the model, current realizations of h_t and d_t, and expected future realizations of h_t and d_t.

The assumption of i.i.d. dividends leads to a particularly simple form since it implies that, for any $r \geq 1$:

$$E_t\left[\frac{h_{t+r}+d_{t+r}}{h_{t+r}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+r}}\right] = E_t\left[\frac{h_{t+1}+d_{t+1}}{h_{t+1}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+1}}\right] \equiv E_1$$

$$E_t\left[\frac{d_{t+r}}{h_{t+r}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+r}}\right] = E_t\left[\frac{d_{t+1}}{h_{t+1}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+1}}\right] \equiv E_2.$$

Rewriting the expression for E_1 as:

$$\begin{aligned} E_1 &= E_t\left[\frac{h_{t+1}+d_{t+1}}{h_{t+1}+\frac{\theta\mu_d}{\beta\chi}+\varepsilon_{t+1}}\right] \\ &= E_t\left[\frac{h_{t+1}+d_{t+1}}{h_{t+1}+d_{t+1}+(\frac{\theta}{\beta\chi}-1)\mu_d}\right] \\ &= 1-E_t\left[\frac{(\frac{\theta}{\beta\chi}-1)\mu_d}{h_{t+1}+d_{t+1}+(\frac{\theta}{\beta\chi}-1)\mu_d}\right], \end{aligned}$$

shows that $E_1 < 1$[1] iff $\theta > \beta\chi$[2].

[1] We only consider strictly positive dividend processes.

[2] The limit case $\theta = \beta\chi$ corresponds to the first best consumption allocation. The price of physical capital goes to infinity in this case.

Under this condition the price of physical capital is finite, given by the following expression:

$$\begin{aligned} p_t^s &= (h_t + d_t)\left\{E_2 + E_2 E_1 + E_2 (E_1)^2 ...\right\} \\ &= (h_t + d_t)\frac{E_2}{1 - E_1}. \end{aligned}$$

The price of physical capital in the presence of social security is thus equal to a constant fraction of the price under laissez faire:

$$p_t^s = F p_t^{lf},$$

where

$$F \equiv \frac{E_2}{1 - E_1}\frac{1 - \beta\chi}{\beta\chi}.$$

3 Interpretation of equation (29)

An important point to note is that, from the perspective of the old generations, the current-period transfer, denoted by S_t, is only one of three components of the change in total wealth achieved by the social security system when compared to the laissez faire outcome. The other two components are the present discounted value of all future social security payments, equal to $\beta\chi z_t$, and the change in the price of physical capital compared to the laissez faire outcome, $p_t^s - p_t^{lf}$.

For standard parameter values (including the ones we use in our benchmark calibration), the current-period transfer to the old S_t is negatively correlated with the physical capital dividend. This is the intuitive case where the current-period transfer provides insurance to the old generations. We refer to this as the "normal" regime in what follows. However, it turns out that there is a second theoretical possibility. For some (empirically implausible) parameter values, the *current-period* transfer to the old S_t, defined by expression (29) in the main text, can be *positively* correlated with the shock to the physical capital dividend. We thank the referee for pointing this possibility out to us. We will refer to this as the "counterintuitive" regime.

However, even in the counterintuitive case where S_t is positively correlated with the shock to the physical capital dividend under pure risksharing social security, the *total wealth* transfer to the old through social security

$$\Delta W_t^{r<t} = S_t + \beta\chi z_t + p_t^s - p_t^{lf}$$

is still negatively correlated with the shock to the physical capital dividend. It is this change in total wealth that matters for the consumption decision. This means that, by design, a pure risksharing social security system of the form that we propose leads, from the perspective of the old generations, to smoother total wealth, and hence smoother consumption, than would be the case under laissez faire. This means that the denominator of the SDF, which is just the aggregate consumption of all old generations, is less variable under pure risksharing social security than under laissez faire. By Jensen's inequality, this implies a rise in the riskless rate of interest as claimed in Proposition 1, even in the case where the dividends to human and physical capital are perfectly correlated.

We now provide more formal arguments supporting these statements. First, we look at the total wealth transfers achieved by social security. Under laissez faire, the consumption allocations are given by equations (10) and (11) in the main text:

$$C_t^t = (1 - \beta\chi)h_t$$

$$C_t^{r<t} = \beta\chi h_t + d_t.$$

Under pure risksharing social security ($\theta = 1$) the following expression for the consumption allocations can be obtained by slightly rewriting equations (25) and (26) in the main text:

$$C_t^t = (1 - \beta\chi)h_t + (1 - \beta\chi)\varepsilon_t$$

$$C_t^{r<t} = \beta\chi h_t + d_t - (1 - \beta\chi)\varepsilon_t.$$

Hence pure risksharing social security is designed to change the consumption of the young by $(1 - \beta\chi)\varepsilon_t$, and the consumption of the old by $-(1 - \beta\chi)\varepsilon_t$, compared to the laissez faire equilibrium. Given the assumption of log utility, this means that social security must change the total wealth of the two groups by

$$\Delta W_t^t = \varepsilon_t$$

$$\Delta W_t^{r<t} = -\varepsilon_t,$$

where for each of the two groups ΔW denotes the difference in total wealth between the pure risksharing social security equilibrium and the laissez faire equilibrium.

In bad states of the world (a low realization of the dividend, which implies a low return on the savings of the old), ε_t is negative and the total wealth transfer from the young to the old is positive. This means that in bad states, the old consume more under social security than under laissez faire. In good states of the world, the opposite holds: ε_t is positive and the total wealth transfer goes from the old to the young. This is why pure risksharing social security is a source of risk reduction to the old.

While this argument shows that the total wealth transfer to the old through the pure risksharing social security system is always negatively correlated with the shock to the physical capital dividend, the same is not necessarily true for the current-period social security payment S_t. From the expressions for total wealth, the change in total wealth for the two groups produced by the social security system can be broken down into the following components

$$\Delta W_t^t = \varepsilon_t = -S_t + (1 - \beta\chi)z_t$$

$$\Delta W_t^{r<t} = -\varepsilon_t = S_t + \beta\chi z_t + p_t^s - p_t^{lf}.$$

Under the normal regime, the current-period transfer S_t is negatively correlated with ε_t. Figure A illustrates the effects of pure risksharing social security on the different components of the total wealth of the old generations ($\Delta W_t^{r<t}$) as a function of the underlying shock to the physical capital dividend ε_t for our benchmark set of parameters. As shown earlier, the total change in wealth compared to the laissez faire outcome is equal to $-\varepsilon_t$. Figure A illustrates how S_t is decreasing in ε_t under the normal regime: the worse the shock to the dividend, the higher the transfer. The present value of future social security transfers $\beta\chi z_t$ is positive and increasing in ε_t. It is positive because the future insurance is valuable. The fact that it is increasing in ε_t comes from the effect of the dividend shock on the discount rate. In a bad state of the world where ε_t is negative, equilibrium consumption is low. With i.i.d. dividends, this implies low expected marginal utility growth and therefore a high discount rate. The third effect on the wealth of the old through the change in the price of physical capital. Since the price of physical capital falls in the presence of social security, this effect is negative.

The corresponding effect of pure risksharing social security on the wealth of the young (ΔW_t^t) is shown in Figure B. Since the young do not hold physical capital initially, the change in the price of physical capital leaves them unaffected. The current-period transfer from the young to the old S_t has the opposite effect on the wealth of the young than on that of the old. The present value of future social security transfers $(1-\beta\chi)z_t$ is positive and increasing in ε_t, just as it is for the old generation. This is possible because all generations currently alive will benefit from the future insurance provided by those generations that are still unborn.

The relationships we have just described hold for the normal regime. It turns out that, for extreme parameter values, the features of pure risksharing social security can look different in equilibrium. In particular, it is possible that the current-period transfer S_t is positively correlated with ε_t, while at the same time, the change in the wealth of the old $\Delta W_t^{r<t}$ is negatively correlated with ε_t. This somewhat counterintuitive result can perhaps best be illustrated for the case of perfectly correlated dividends to human and physical capital, as suggested by the referee. In fact, to make it even more transparent, assume that the two dividends are equal, up to a constant:

$$h_t = d_t + \mu_h - \mu_d.$$

In this case, the expression for the pure risksharing ($\theta = 1$) social security transfer in equation (29) reduces to:

$$S_t = \text{constant} + \left[2\beta\chi - 1 - 2(1-\beta\chi)\frac{E_2}{1-E_1}\right] d_t,$$

where we used the expression for the price of physical capital under social security derived in the previous section.

Hence

$$Cov(S_t, d_t) < 0 \ \text{ iff } 2\beta\chi - 1 - 2(1-\beta\chi)\frac{E_2}{1-E_1} < 0.$$

Using the expressions for E_1 and E_2 derived in the previous section, some tedious but straightforward calculations show that this last condition is equivalent to

$$E_t\left[\frac{1}{1+\frac{1}{\beta\chi}\frac{\mu_d}{\mu_h}+\frac{2}{\mu_h}(d_{t+1}-\mu_d)}\right]<1. \tag{A.1}$$

By Jensen's inequality,

$$\frac{1}{1+\frac{1}{\beta\chi}\frac{\mu_d}{\mu_h}}<E_t\left[\frac{1}{1+\frac{1}{\beta\chi}\frac{\mu_d}{\mu_h}+\frac{2}{\mu_h}(d_{t+1}-\mu_d)}\right]$$

and the expectation on the right-hand-side of this last inequality is increasing in the volatility of the shock to the physical capital dividend σ_d.

Thus the inequality in equation (A.1) will hold for small values of σ_d, but it can fail as σ_d becomes large. All else equal it is also more likely to fail for larger values of $\beta\chi$. It is therefore possible that $Cov(S_t, d_t) > 0$, for large values of σ_d and $\beta\chi$.

Figures C and D provide an example of the counterintuitive regime. For the purpose of this example we make the extreme assumption that $\beta\chi = 0.999$ and $\sigma_d = 2$, on a 20 year basis. Figure C1 plots the effects of pure risksharing social security on the different components of the wealth of the old. Note the scale on the y axis. The asset prices p_t and z_t are extreme in this case, when compared to the magnitude of the underlying dividends. In Figure C2 we plot the current-period transfer from the young to the old separately on a rescaled axis, together with the total wealth change for the old. As pointed out earlier, the schedule for S_t is now *upward* sloping, meaning that, in contrast to the normal regime, the old get a *smaller* transfer in bad states of the world where the dividend is low. Figure D plots the corresponding components of the change in the total wealth of the young.

To conclude, we would like to emphasize again that the normal regime is the one that prevails for any plausible values of the parameters, including the ones used in our benchmark calibration. The counterintuitive regime can only occur for extremely high values of the effective discount factor $\beta\chi$ and the dividend volatility σ_d. In the numerical example reported in Figures C and D, we assume that $\beta\chi = 0.999$ and $\sigma_d = 2$, on a 20 year basis. In contrast, for the numerical example in Figures A and B we assume that $\beta\chi = (0.96^{20}) * (2/3) = 0.295$ and $\sigma_d = 0.2$, which corresponds to the benchmark calibration in Section 4 of the main text.

For the benchmark value of $\beta\chi = 0.295$, the normal regime prevails, even for extremely high values of σ_d. When we assume an implausibly high effective discount factor of $\beta\chi = 0.999$ on a 20 year basis, the threshold value where the equilibrium switches from the normal to the counterintuitive regime is still very high: around $\sigma_d = 0.8$. Such a high dividend volatility implies implausibly high annual return volatilities in equilibrium.

4 Proof of Proposition 1

Noting that,

$$\frac{1}{1+R_{f,t+1}}=E_t[M_{t+1}],$$

$$R^s_{f,t+1} > R^{lf}_{f,t+1} \text{ iff } E_t[M^s_{t+1}] < E_t[M^{lf}_{t+1}].$$

By Jensen's inequality, this condition is equivalent to:

$$Var_t[h_{t+1} + \frac{\mu_{dt+1}}{\beta\chi} + \varepsilon_{t+1}] < Var_t[h_{t+1} + \frac{\mu_{dt+1}}{\beta\chi} + \frac{\varepsilon_{t+1}}{\beta\chi}],$$

which says that the denominator of the SDF in the decentralized equilibrium is a mean-preserving spread of the denominator of the SDF under pure risksharing social security. This last condition is clearly satisfied in the case of deterministic dividends to human capital since $\beta\chi < 1$.

In the case of stochastic dividends to human capital, the condition can be rewritten as

$$\frac{\sigma_d}{\sigma_h} > -\rho_{hd}\frac{2\beta\chi}{1+\beta\chi}.$$

Hence a sufficient condition in the case of stochastic dividends to human capital is that $\rho_{hd} \geq 0$.

5 Proof of Proposition 2

With a purely deterministic social security transfer, the SDF is given by:

$$M^s_{t+1} = \frac{h_t + d_t}{h_{t+1} + \frac{d_{t+1}}{\beta\chi} + \frac{\tau}{\beta\chi}}.$$

Comparing this expression to the SDF under laissez-faire,

$$M^{lf}_{t+1} = \frac{h_t + d_t}{h_{t+1} + \frac{d_{t+1}}{\beta\chi}},$$

shows that:

$$E_t[M^s_{t+1}] < E_t[M^{lf}_{t+1}] \text{ iff } \tau > 0.$$

6 Proof of Proposition 3

In the case where $\theta = 1$, the expression for F derived in Section 2 of the Appendix may be rewritten:

$$F = \frac{E_t\left[\frac{d_{t+1}}{h_{t+1}+d_{t+1}+\frac{1-\beta\chi}{\beta\chi}\mu_d}\right]}{E_t\left[\frac{\mu_d}{h_{t+1}+d_{t+1}+\frac{1-\beta\chi}{\beta\chi}\mu_d}\right]},$$

or,

$$F = 1 + \frac{Cov_t\left[d_{t+1}, \frac{1}{h_{t+1}+d_{t+1}+\frac{1-\beta\chi}{\beta\chi}\mu_d}\right]}{E_t\left[\frac{\mu_d}{h_{t+1}+d_{t+1}+\frac{1-\beta\chi}{\beta\chi}\mu_d}\right]}.$$

Note that $\lim_{\sigma_d^2 \to 0} F = 1$. More generally, the expectation term in the denominator is positive since dividends are positive by assumption. Hence:

$$F < 1 \text{ iff } Cov_t\left[d_{t+1}, \frac{1}{h_{t+1} + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d}\right] < 0.$$

This condition holds unambiguously in the special case when the human capital dividend is deterministic ($h_{t+1} = h$). Indeed, in this case:

$$\begin{aligned}
Cov_t\left[d_{t+1}, \frac{1}{h + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d}\right] &= Cov_t\left[h + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d, \frac{1}{h + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d}\right] \\
&= 1 - E_t\left[h + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d\right] E_t\left[\frac{1}{h + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d}\right] \\
&< 0,
\end{aligned}$$

by Jensen's inequality.

For stochastic human capital dividends on the other hand, we need to impose that:

$$Cov_t\left[d_{t+1}, \frac{1}{h_{t+1} + d_{t+1} + \frac{1-\beta\chi}{\beta\chi}\mu_d}\right] < 0$$

for the result to hold.

7 Proof of Proposition 4

With a purely deterministic social security transfer, the expressions for E_1 and E_2 are modified as follows:

$$\begin{aligned}
E_1 &= E_t\left[\frac{h_{t+1} + d_{t+1}}{h_{t+1} + \frac{d_{t+1}}{\beta\chi} + \frac{\tau}{\beta\chi}}\right] \\
E_2 &= E_t\left[\frac{d_{t+1}}{h_{t+1} + \frac{d_{t+1}}{\beta\chi} + \frac{\tau}{\beta\chi}}\right].
\end{aligned}$$

In this case, the expression for F can be rewritten as:

$$\frac{1}{F} = 1 + \frac{\tau}{1-\beta\chi} \frac{E_t\left[\frac{1}{h_{t+1} + \frac{d_{t+1}}{\beta\chi} + \frac{\tau}{\beta\chi}}\right]}{E_t\left[\frac{d_{t+1}}{h_{t+1} + \frac{d_{t+1}}{\beta\chi} + \frac{\tau}{\beta\chi}}\right]}$$

The two expectations in this expressions are positive since dividends are positive by assumption and the expression in the denominator of the expectations is a scalar multiple of the consumption of the old age group, which is positive as long as $-\beta\chi\underline{h} < \tau < (1-\beta\chi)\underline{h}$ (the restriction on transfers discussed in the main text).

Hence $F < 1$ iff $\tau > 0$, which means that the transfer is from the young to the old.

8 Proof of Proposition 5

Note that the return on physical capital in the presence of social security may be rewritten:

$$1 + R^s_{t+1} = \frac{1}{h_t + d_t}\left[h_{t+1} + \left(1 + \frac{1-\beta\chi}{F\beta\chi}\right)d_{t+1}\right].$$

Hence $\frac{\partial E_t(R^s_{t+1})}{\partial F} < 0$ and $\frac{\partial Var_t(R^s_{t+1})}{\partial F} < 0$. Noting that $\lim_{F\to 1} R^s_{t+1} = R^{lf}_{t+1}$ completes the proof.

Figure A: Effects of social security on the wealth of the old (normal regime)

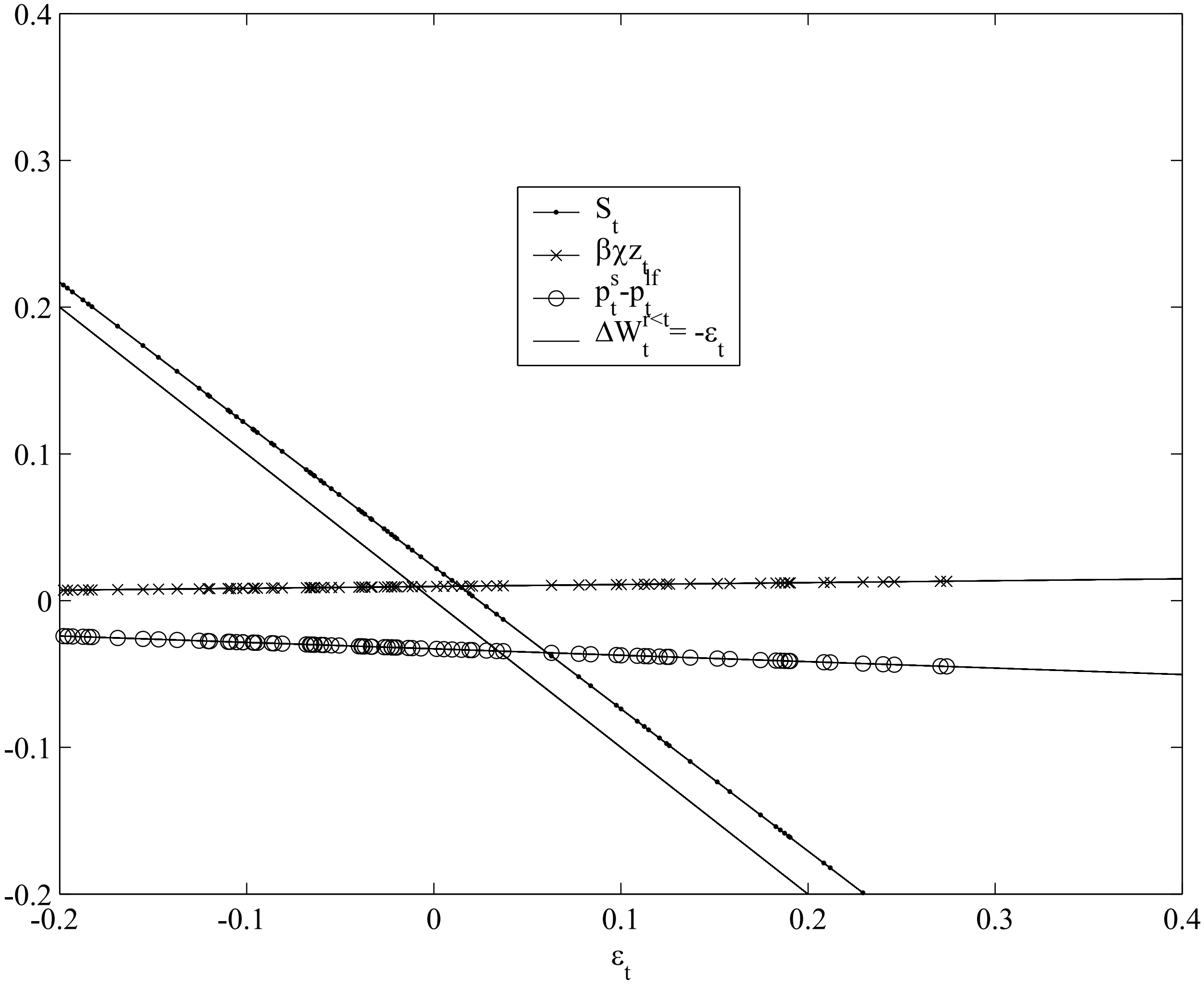

Figure B: Effects of social security on the wealth of the young (normal regime)

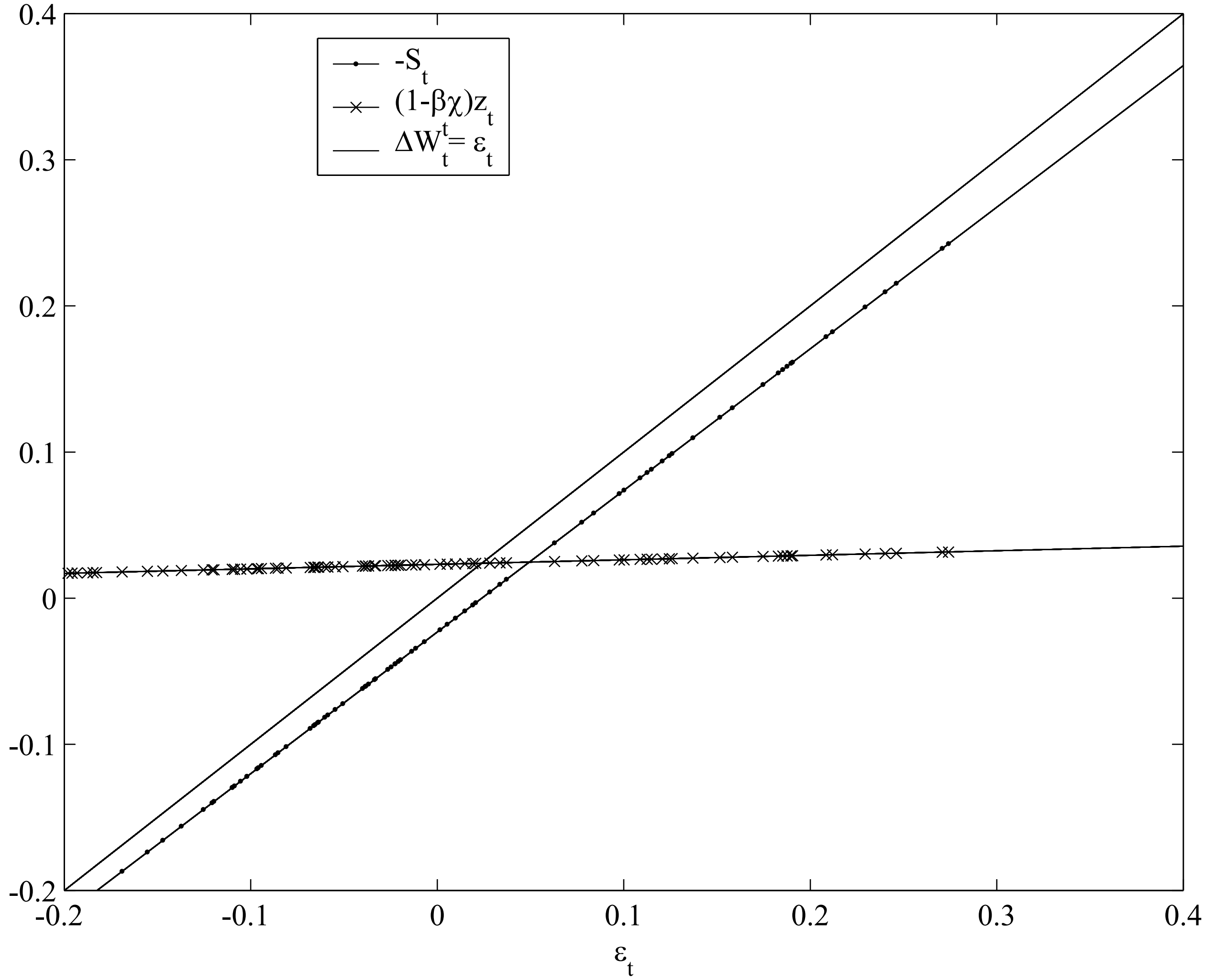

Figure C1: Effects of social security on the wealth of the old (unintuitive regime)

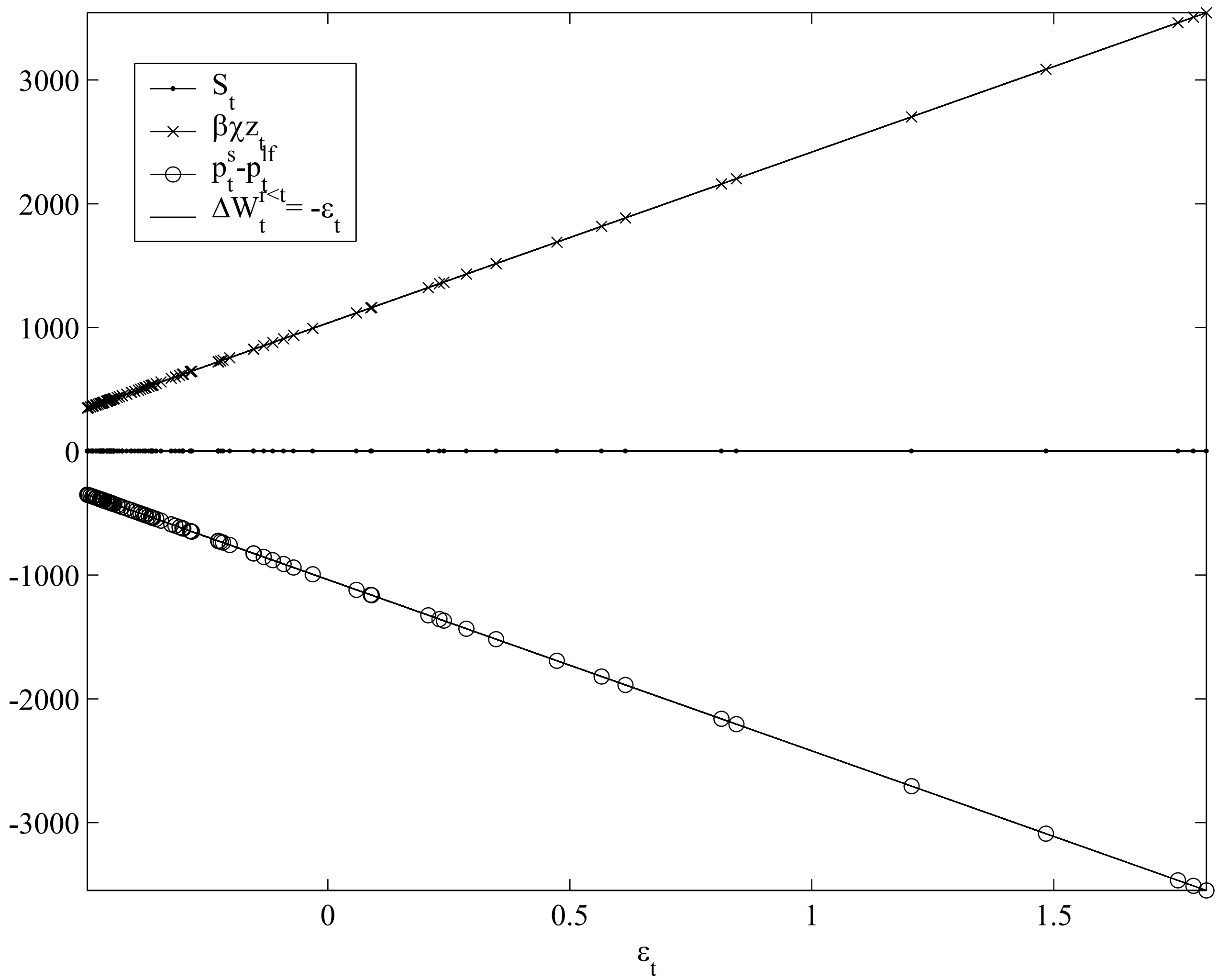

Figure C2: Effects of social security on the wealth of the old (unintuitive regime)

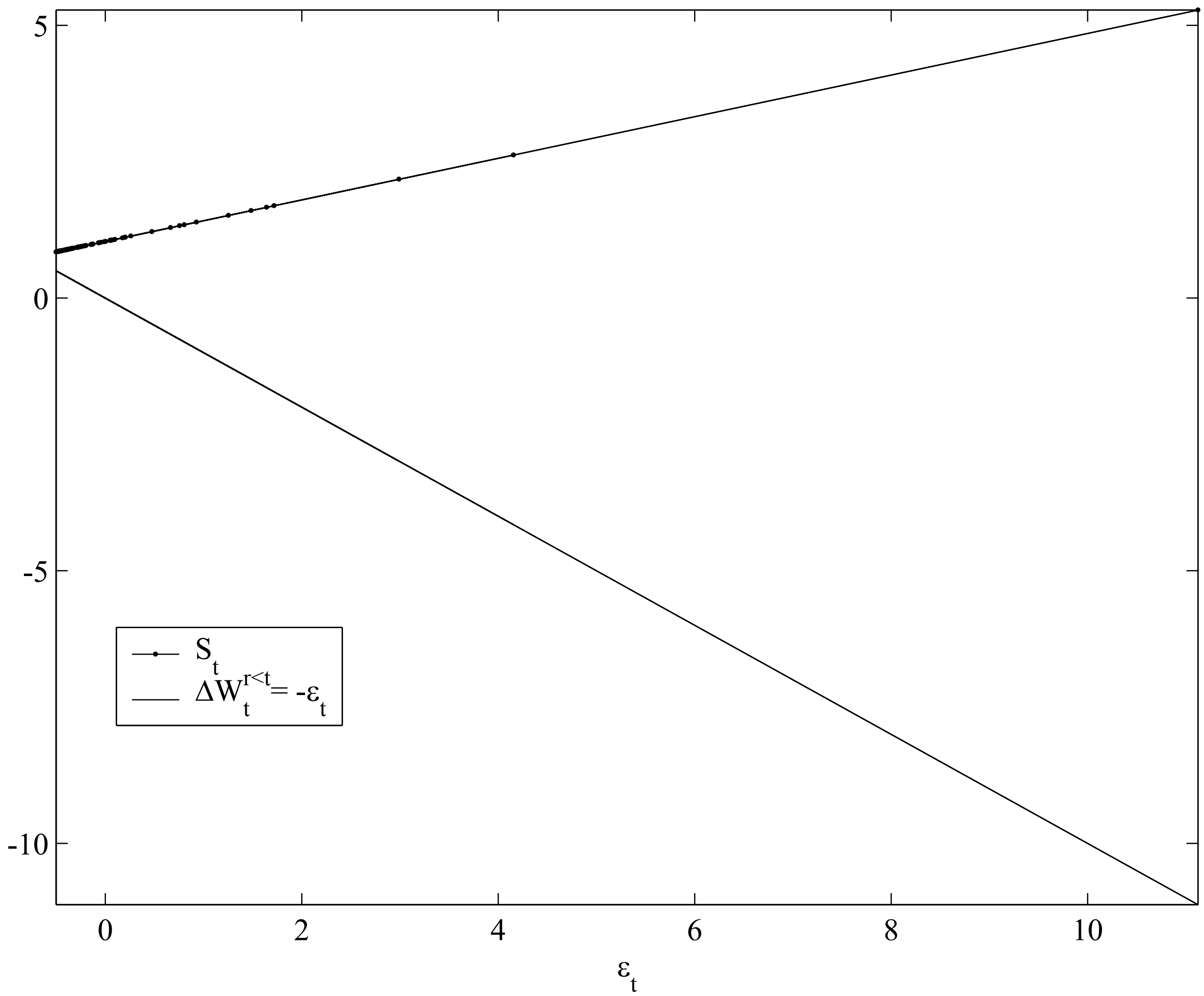

Figure D: Effects of social security on the wealth of the young (unintuitive regime)