Explicit Solutions and Stability Properties of Homogeneous Polynomial Dynamical Systems

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Abstract—In this paper, we provide a system theoretic treatment of certain continuous-time homogeneous polynomial dynamical systems (HPDS) via tensor algebra. In particular, if a system of homogeneous polynomial differential equations can be represented by an orthogonally decomposable (odeco) tensor, we can construct its explicit solution by exploiting tensor Z-eigenvalues and Z-eigenvectors. We refer to such HPDS as odeco HPDS. By utilizing the form of the explicit solution, we are able to discuss the stability properties of the odeco HPDS. We illustrate that the Z-eigenvalues of the corresponding dynamic tensor can be utilized to establish necessary and sufficient stability conditions, similar to these from linear systems theory. In addition, we are able to obtain the complete solution to the odeco HPDS with constant control. Finally, we establish results which enable one to determine if a general HPDS can be transformed to an odeco HPDS, where the previous results can be applied. We demonstrate our framework with numerical examples.

Index Terms—Homogeneous polynomial dynamical systems, explicit solutions, stability, tensor algebra, orthogonal decomposition, Z-eigenvalues, Z-eigenvectors.

I. INTRODUCTION

Tensor algebra has been applied to model and simulate nonlinear dynamics [1]–[5]. The essence is to represent nonlinear dynamics using tensor products where one can exploit tensor decomposition techniques such that CANDECOMP/PARAFAC (CP) decomposition, higher-order singular value decomposition, Tucker decomposition, and tensor train decomposition [6]–[9]. Kruppa [4], [5] represented a multilinear polynomial dynamical system by a contracted product between a parameter tensor and a monomial tensor, and utilized CP decomposition and Tucker decomposition for efficiently simulating the evolution of the dynamics. In addition, Chen et al. [2] proposed a new tensor-based multilinear dynamical system for characterizing the dynamics of hypergraphs, a generalization of graphs in which edges can contain more than one nodes. The multilinear dynamical system evolution is described by the action of tensor vector multiplications between a dynamic tensor and the static vector. In fact, the multilinear dynamical system belongs to the family of homogeneous polynomial dynamical systems (HPDS) if one expands the tensor vector multiplications.

The explicit solution and stability properties of a linear dynamical system can be readily obtained from the eigenvalue decomposition of the dynamic matrix. However, the results can hardly be extended to HPDS due to its nonlinear nature [10]–[14]. In terms of stability, many methods such as generalized characteristic value problems [13] and optimization-based Lyapunov functions [10] have been proposed to establish stability of certain HPDS. In this paper, we will exploit tensor orthogonal decomposition with Z-eigenvalues and Z-eigenvectors to summarize the explicit solutions and the stability properties of certain HPDS that can be represented by orthogonally decomposable (odeco) tensors. We refer to such HPDS as odeco HPDS. We will also provide criteria to determine if a general HPDS can be transformed to an odeco HPDS, where the results of explicit solutions and stability can be applied.

Tensor eigenvalue problems of real supersymmetric tensors were first explored by Qi [15], [16] and Lim [17] independently in 2005. There are many different notions of tensor eigenvalues such as H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [15], [16], [18], which have different applications in network theory, machine learning, elasticity theory, and dynamical systems. Surana et al. [19] compared the H-eigenvalue spectrum between the two Laplacian tensors for measuring hypergraph distance. Chen et al. [20] showed that the Z-eigenvector associated with the second smallest Z-eigenvalue of a normalized Laplacian tensor can be used for hypergraph partition. Moreover, Huang and Qi [21] used M-eigenvalues to prove the strong ellipticity of elasticity tensors in solid mechanics. Furthermore, Chen et al. [18], [22] utilized U-eigenvalues to determine the stability of multilinear time-invariant systems, which can be unfolded to linear dynamical systems via tensor unfolding, an operation that transforms a tensor into a matrix. Of particular interest of this paper are Z-eigenvalues.

Recently, Chen [1] investigated the explicit solutions and stability properties of the discrete-time odeco HPDS (also called multilinear dynamical systems in [1]) via tensor orthogonal decomposition. In particular, the author showed that Z-eigenvalues play a significant role in the stability analysis, offering necessary and sufficient conditions [1]. This paper will focus on continuous-time HPDS. Continuous-time HPDS are a popular tool in modeling various robotic systems [23]–[25]. The key contributions of the paper are:

1) We investigate the explicit solutions of the continuous-time odeco HPDS. We derive an explicit solution formula by using the Z-eigenvalues and Z-eigenvectors of the corresponding dynamic tensors.

2) According to the formula of the explicit solutions, we are able to discuss the stability properties of the odeco HPDS. We find that the Z-eigenvalues of the corresponding dynamic tensors can offer necessary and sufficient stability conditions. Furthermore, we apply an upper bound of the largest Z-eigenvalue to determine the asymptotic stability efficiently.

3) We explore the complete solutions of the odeco HPDS with constant control. We discover that the complete solutions can be solved implicitly by exploiting Gauss hypergeometric functions.

4) We establish results which enable one to determine if a general HPDS can be transformed to an odeco HPDS, in which all the previous results can be applied.

The paper is organized into seven sections. In Section II, we review tensor preliminaries including tensor vector multiplications, tensor eigenvalues, CP decomposition, and orthogonal decomposition. We derive an explicit solution formula for the continuous-time odeco HPDS and discuss the stability properties of such HPDS based on the form of the explicit solutions in Section III. In Section IV, we explore the complete solutions of the odeco HPDS with constant control. We provide criteria to determine if a general HPDS can be transformed to an odeco HPDS in Section V. Four numerical...
examples are presented in Section VI. Finally, we conclude in Section VII with future research directions.

II. TENSOR PRELIMINARIES

A tensor is a multidimensional array [6]–[8], [18], [26], [27]. The order of a tensor is the number of its dimensions, and each dimension is called a mode. A kth order tensor usually is denoted by \( T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \). It is therefore reasonable to consider scalars \( x \in \mathbb{R} \) as zero-order tensors, vectors \( v \in \mathbb{R}^n \) as first-order tensors, and matrices \( M \in \mathbb{R}^{n \times n} \) as second-order tensors. A tensor is called cubical if every mode is the same size, i.e., \( \mathbb{R} \) matrices as zero-order tensors, vectors \( v \in \mathbb{R}^n \) as first-order tensors, and matrices \( M \in \mathbb{R}^{n \times n} \) as second-order tensors. A tensor is called cubical if every mode is the same size, i.e., \( T \in \mathbb{R}^{n \times n \times \cdots \times n} \). A cubical tensor \( T \) is called supersymmetric if \( T_{j_1 j_2 \cdots j_k} \) is invariant under any permutation of the indices.

A. Tensor Vector Multiplication

The tensor vector multiplication \( T \times_p v \) along mode \( p \) for a vector \( v \in \mathbb{R}^{np} \) is defined by

\[
(T \times_p v)_{j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_k} = \sum_{j_p=1}^{n_p} T_{j_1 j_2 \cdots j_p \cdots j_k} v_{j_p},
\]

which can be extended to

\[
T \times_1 v_1 \times_2 v_2 \times_3 \cdots \times_k v_k = T v_1 v_2 \cdots v_k \in \mathbb{R}
\]

for \( v_p \in \mathbb{R}^{np} \). If \( T \) is supersymmetric and \( v_p = v \) for all \( p = 1, 2, \ldots, k \), the product (2) is also known as the homogeneous polynomial associated with \( T \), and we write it as \( T v^k \) for simplicity.

B. Tensor Eigenvalues

The tensor eigenvalues of real supersymmetric tensors were first explored by Qi [15], [16] and Lim [17] independently. There are many different notions of tensor eigenvalues including H-eigenvalues, Z-eigenvalues, M-eigenvalues, and U-eigenvalues [15], [16], [18]. Of particular interest of this paper are Z-eigenvalues. Given a kth order supersymmetric tensor \( T \in \mathbb{R}^{n \times n \times \cdots \times n} \), the E-eigenvalues \( \lambda \in \mathbb{C} \) and E-eigenvectors \( v \in \mathbb{C}^n \) of \( T \) are defined as

\[
\begin{aligned}
T v_k & = \lambda v \\
v^\top v & = 1
\end{aligned}
\]

The E-eigenvalues \( \lambda \) could be complex. If \( \lambda \) are real, we call them Z-eigenvalues. Computing the E-eigenvalues and the Z-eigenvalues of a tensor is NP-hard [28]. However, many numerical algorithms such that homotopy continuation approaches [29], [30] and adaptive shifted power methods [31] are proposed in order to compute the E-eigenvalues or Z-eigenvalues of a tensor.

C. CANDECOMP/PARAFAC Decomposition

Like rank-one matrices, a tensor \( T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \) is rank-one if it can be written as the outer product of \( k \) vectors, i.e., \( T = v^{(1)} \circ v^{(2)} \circ \cdots \circ v^{(k)} \). The CANDECOMP/PARAFAC (CP) decomposition decomposes a tensor \( T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k} \) into a sum of rank-one tensors as form of outer products. It is often useful to normalize all the vectors and have weights \( \lambda_r \) in descending order in front:

\[
T = \sum_{r=1}^{R} \lambda_r v^{(1)}_r \circ v^{(2)}_r \circ \cdots \circ v^{(k)}_r,
\]

where \( v^{(p)}_r \in \mathbb{R}^{np} \) have unit length, and \( R \) is called the CP rank of \( T \) if it is the minimum integer that achieves (4).

Tensor orthogonal decomposition is a special case of CP decomposition. A kth order supersymmetric tensor \( T \in \mathbb{R}^{n \times n \times \cdots \times n} \) is called orthogonally decomposable (odeco) if it can be written as a sum of vector outer products

\[
T = \sum_{r=1}^{n} \lambda_r v_r \circ v_r \circ \cdots \circ v_r,
\]

where \( \lambda_r \in \mathbb{R} \) in the descending order, and \( v_r \in \mathbb{R}^n \) are orthonormal [32].

It is easy to show that \( \lambda_r \) are the Z-eigenvalues of \( T \) with the corresponding Z-eigenvectors \( v_r \). Note that \( \lambda_r \) do not include all the Z-eigenvalues of \( T \), which means that \( \lambda_1 \) may not be the largest Z-eigenvalue of \( T \). Reobeva [32] speculated that odeco tensors satisfy a set of polynomial equations that vanish on the odeco variety, which is the Zariski closure of the set of odeco tensors inside the space of kth order n-dimensional complex supersymmetric tensors. Although the author only proved for the case when \( n = 2 \), she provided strong evidence for its overall correctness [32]. Furthermore, a tensor power method was proposed in [32] in order to obtain the orthogonal decomposition of an odeco tensor.

III. MAIN RESULTS

In this paper, we are interested in finding the explicit solution to a continuous-time homogeneous polynomial dynamical system (HPDS) of degree \( k - 1 \) that can be represented by

\[
x(t) = A x(t) = A x(t)(k-1),
\]

where \( A \in \mathbb{R}^{n \times n \times \cdots \times n} \) is a kth order n-dimensional odeco tensor, and \( x(t) \in \mathbb{R}^n \) is the state variable. We refer to such HPDS as odeco HPDS.

A. Explicit solutions

We find that the explicit solutions of an odeco HPDS can be solved in a simple fashion by exploiting tensor orthogonal decomposition with Z-eigenvalues and Z-eigenvectors.

**Proposition 1:** Suppose that \( k \geq 3 \) and \( A \in \mathbb{R}^{n \times n \times \cdots \times n} \) is odeco. Let the initial condition \( x_0 = \sum_{r=1}^{1} \alpha_r v_r \). Then the explicit solution to the odeco HPDS (6), given initial condition \( x_0 \), is given by

\[
x(t) = \sum_{r=1}^{n} \left( 1 - (k-2) \lambda_r \alpha_r^{k-2} t \right)^{-\frac{1}{k-2}} \alpha_r v_r,
\]

where \( \lambda_r \) are the Z-eigenvalues with the corresponding Z-eigenvectors \( v_r \) in the orthogonal decomposition of \( A \). Moreover, if \( \lambda_r \alpha_r^{k-2} > 0 \) for some \( r \), the solution (7) is only defined over the interval

\[
t \in \left[ 0, \frac{1}{k-2} \left( \frac{S}{(k-2) \lambda_r \alpha_r^{k-2}} - 1 \right) \right],
\]

where \( S = \{ r = 1, 2, \ldots, n | \lambda_r \alpha_r^{k-2} > 0 \} \).

**Proof:** Since \( v_r \) are orthonormal, suppose that

\[
x(t) = \sum_{r=1}^{n} c_r(t) v_r = V c(t),
\]

where

\[
V = [v_1 \ v_2 \ \ldots \ v_n],
\]

\[
c(t) = [c_1(t) \ c_2(t) \ \ldots \ c_n(t)]^\top.
\]
Clearly, \( c_r(0) = \alpha_r \) for all \( r = 1, 2, \ldots, n \). Based on the property of tensor vector multiplications, it can be shown that
\[
\dot{x}(t) = \left( \sum_{r=1}^{n} \lambda_r v_r \odot v_r \odot \cdots \odot v_r \right) \times_1 x(t) \times_2 \cdots \times_{k-1} x(t)
\]
\[
= \left( \sum_{r=1}^{n} \lambda_r v_r \odot v_r \odot \cdots \odot v_r \right) \times_1 \left( \sum_{i=1}^{n} c_i(t) v_i \right) \times_2 \cdots \times_{k-1} \left( \sum_{i=1}^{n} c_i(t) v_i \right) v_r
\]
\[
= \sum_{r=1}^{n} \lambda_r c_r(t) v_r.
\]
Thus, we have
\[
\dot{c}_r(t) = \lambda_r c_r(t) - \frac{1}{\lambda_r} c_r(0),
\]
and the result follows immediately. Moreover, if \( \lambda_r \alpha_r^{k-2} > 0 \) for some \( r \), the corresponding coefficient functions \( c_r(t) \) will have singularities at \( t = \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}} \). Thus, the domains of \( c_r(t) \) are given by \( t \in \left( 0, \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}} \right) \). The other branches of \( c_r(t) \) over \( t \in \left( \frac{1}{(k-2)\lambda_r \alpha_r^{k-2}}, \infty \right) \) do not satisfy the initial conditions, so they are not included in the solutions of \( c_r(t) \). Therefore, the domain of the solution (7) will be
\[
D = \left\{ t \in [0, \min S : (k-2)\lambda_r \alpha_r^{k-2} > 0, \forall r \right\},
\]
where \( S = \{ r = 1, 2, \ldots, n | \lambda_r \alpha_r^{k-2} > 0 \} \). Note that if \( \lambda_r \alpha_r^{k-2} \leq 0 \) for all \( r \), the domain of the solution (7) will be \( D = [0, \infty) \).

The coefficients \( \alpha_r \) can be found from the inner product between \( x_0 \) and \( v_r \). When \( k = 2 \), the result reduces to the famous linear systems’ solutions, i.e.,
\[
\lim_{k \to 2} x(t) = \lim_{k \to 2} \sum_{r=1}^{n} (1 \to 2) \lambda_r \alpha_r^{k-2} - \frac{1}{\lambda_r} \alpha_r v_r
\]
\[
= \lim_{p \to \infty} \sum_{r=1}^{n} \left( 1 + \frac{1}{p} \right)^p \alpha_r v_r = \sum_{r=1}^{n} \exp(\lambda_r t) \alpha_r v_r,
\]
where \( \lambda_r \) become the eigenvalues of the dynamic matrix with the corresponding eigenvectors \( v_r \). Furthermore, based on the form of the explicit solution, we can discuss the stability properties of the odeco HPDS (6).

### B. Stability

In linear control theory, it is conventional to investigate so-called (internal) stability [33]. The stability of a linear dynamical system only relies on the locations of the eigenvalues of the dynamic matrix. Similarly, the equilibrium point \( x = 0 \) of a HPDS is called stable if \( \|x(t)\| \leq \gamma \|x_0\| \) for some initial condition \( x_0 \) and \( \gamma > 0 \), asymptotically stable if \( \|x(t)\| \to 0 \) as \( t \to \infty \), and unstable if \( \|x(t)\| \to \infty \) as \( t \to \infty \). Here "\( \| \cdot \| \)" denotes the Frobenius norm. We discover that the stability properties of the odeco HPDS (6) are similar to those of linear systems, but depend on both the Z-eigenvalues of \( A \) and initial conditions.

**Proposition 2:** Suppose that \( k \geq 3 \). Let the initial condition \( x_0 = \sum_{r=1}^{n} \alpha_r v_r \). For the odeco HPDS (6), the equilibrium point \( x = 0 \) is:

1. stable if and only if \( \lambda_r \alpha_r^{k-2} \leq 0 \) for all \( r = 1, 2, \ldots, n \);
2. asymptotically stable if and only if \( \lambda_r \alpha_r^{k-2} < 0 \) for all \( r = 1, 2, \ldots, n \);
3. unstable if and only if \( \lambda_r \alpha_r^{k-2} > 0 \) for some \( r = 1, 2, \ldots, n \), where \( \lambda_r \) are the Z-eigenvalues in the orthogonal decomposition of \( A \).

**Proof:** By the triangle inequality, it can be shown that
\[
\|x(t)\| = \| \sum_{r=1}^{n} c_r(t) v_r \| \leq \sum_{r=1}^{n} |c_r(t)| \|v_r\| = \sum_{r=1}^{n} |c_r(t)|.
\]
Since \( \lambda_r \alpha_r^{k-2} \leq 0 \) for all \( r = 1, 2, \ldots, n \), the coefficient functions \( |c_r(t)| \) are bounded by \( |\alpha_r| \) over \( t \in [0, \infty) \). Then we have
\[
\|x(t)\| \leq \sum_{r=1}^{n} |\alpha_r| = \|x_0\| \leq \sqrt{n} \|x_0\|.
\]
Therefore, the equilibrium point \( x = 0 \) is stable. On the other hand, since \( v_r \) are orthonormal, \( \|x(t)\| = \| V c(t) \| = \| c(t) \| \) where \( V \) and \( c(t) \) are the same as defined in Proposition 1. If \( \|x(t)\| = \| c(t) \| \leq \gamma \|x_0\| \), all the coefficient functions \( c_r(t) \) must be bounded for \( t \geq 0 \). Thus, \( \alpha_r \alpha_r^{k-2} \) must lie in the closed left-half plane for all \( r = 1, 2, \ldots, n \). The other two cases can be shown similarly.

When \( k = 2 \), the above conditions reduce to the famous linearity stability conditions. The inequalities obtained from the asymptotic stability condition can provide us with the region of attraction of the odeco HPDS (6), i.e.,
\[
R = \{ x : \lambda_r \alpha_r^{k-2} < 0 \text{ for some } r = 1, 2, \ldots, n \} = \sum_{r=1}^{n} \alpha_r v_r,
\]
where \( v_r \) are the Z-eigenvectors in the orthogonal decomposition of \( A \) corresponding to the Z-eigenvalues \( \lambda_r \). Furthermore, when \( k \) is even, \( \alpha_r^{k-2} \) will be always greater than or equal to zero. Thus, the stability conditions can be simplified for the odeco HPDS (6) of odd degree.

**Corollary I:** Suppose that \( k \geq 4 \) is even. For the odeco HPDS (6), the equilibrium point \( x = 0 \) is:

1. stable if and only if \( \lambda_r \leq 0 \) for all \( r = 1, 2, \ldots, n \);
2. asymptotically stable if and only if \( \lambda_r < 0 \) for all \( r = 1, 2, \ldots, n \);
3. unstable if and only if \( \lambda_r > 0 \) for some \( r = 1, 2, \ldots, n \), where \( \lambda_r \) are the Z-eigenvalues in the orthogonal decomposition of \( A \).

**Proof:** The results follow immediately from Proposition 2 when \( k \) is even.

When \( k \) is even, the stability conditions are exactly same as those of linear systems, i.e., the odeco HPDS (6) of odd degree is globally stable if and only if all the Z-eigenvalues \( \lambda_r \) from the orthogonal decomposition of \( A \) lie in the left-half plane. On the other hand, computing the orthogonal decomposition or Z-eigenvalues of a supersymmetric tensor is NP-hard [28], [32]. If we know an upper bound of the largest Z-eigenvalue of a supersymmetric tensor, it will save a great amount of computations to determine the asymptotic
stability of the odeco HPDS (6). Chen [1] found that the largest Z-eigenvalue of an even-order supersymmetric tensor is upper bounded by the largest eigenvalue of one of its unfolded matrices.

**Lemma 1:** Let $\mathbf{A} \in \mathbb{R}^{n \times n \times \ldots \times n}$ be an even-order supersymmetric tensor. Then the largest Z-eigenvalue $\lambda_{\max}$ of $\mathbf{A}$ is upper bounded by $\mu_{\max}$ where $\mu_{\max}$ is the largest eigenvalue of $\psi(\mathbf{A})$ defined by:

$$\mathbf{A} = \psi(\mathbf{A}) \text{ such that } \mathbf{a}_{j_1i_1\ldots j_ik} \to \mathbf{a}_{j_i},$$

(10)

with $j = j_1 + \sum_{p=2}^{k}(j_p - 1)n^{p-1}$ and $i = i_1 + \sum_{p=2}^{k}(i_p - 1)n^{p-1}$.

**Corollary 2:** Suppose that $k \geq 4$ is even. For the odeco HPDS (6), the equilibrium point $x = 0$ is:

1) stable if $\mu_{\max} \leq 0$;
2) asymptotically stable if $\mu_{\max} < 0$,

where $\mu_{\max}$ is the largest eigenvalue of $\psi(\mathbf{A})$ defined in (10).

**Proof:** Based on Lemma 1, we know that $\lambda_1 \leq \lambda_{\max} \leq \mu_{\max}$. Therefore, the result follows immediately from Corollary 1. $\blacksquare$

Note that $\lambda_1$ is the largest Z-eigenvalue in the orthogonal decomposition of $\mathbf{A}$, while $\lambda_{\max}$ is the largest Z-eigenvalue of $\mathbf{A}$. There are many other upper bounds for the largest Z-eigenvalue or Z-spectral radius of a supersymmetric tensor [34]-[37]. Given an odeco dynamic tensor, the better upper bound of the largest Z-eigenvalue, the more strong stability conditions we can obtain.

**IV. ODECO HPDS WITH CONSTANT CONTROL**

The odeco HPDS with constant control is given by

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)^{k-1} + \mathbf{b},$$

(11)

where $\mathbf{A} \in \mathbb{R}^{n \times n \times \ldots \times n}$ is a $k$th order $n$-dimensional odeco tensor, and $\mathbf{b} \in \mathbb{R}^n$ is a constant control vector. We find that the complete solution to this polynomial dynamical system (11) can be solved implicitly by using Gauss hypergeometric functions.

**Proposition 3:** Suppose that $k \geq 3$. Let $\mathbf{x}(t) = \sum_{r=1}^{n} c_r(t)\mathbf{v}_r$ with initial conditions $c_r(0) = \alpha_r$. For the odeco HPDS with constant control (11), the coefficient functions $c_r(t)$ can be solved implicitly by

$$t = \frac{g\left(\frac{k-2}{k-1} - \frac{\beta_r}{\lambda_r c_r(t)^{k-2}}\right)}{(k-2)\lambda_r c_r(t)^{k-2}} - \frac{g\left(\frac{k-2}{k-1} - \frac{\beta_r}{\lambda_r \alpha_r^{k-2}}\right)}{(k-2)\lambda_r \alpha_r^{k-2}},$$

(12)

where $\lambda_r$ are the Z-eigenvalues with the corresponding Z-eigenvectors $\mathbf{v}_r$ in the orthogonal decomposition of $\mathbf{A}$, $\beta_r$ is the $r$th entry of $\mathbf{v}_r^\top \mathbf{b}$, and $g(a, z)$ is the specified Gauss hypergeometric function [38] defined by

$$g(a, z) = 2F_1(1, a; a + 1; z) = \sum_{m=0}^{\infty} \frac{z^m}{(a + m) m!}.$$

**Proof:** Since $\mathbf{x}(t) = \sum_{r=1}^{n} c_r(t)\mathbf{v}_r$, we can rewrite the polynomial dynamical system (11) as follows:

$$\dot{\mathbf{v}}(t) = \mathbf{V}(\mathbf{A} + \mathbf{c}(t)^{k-1}) + \mathbf{V}^\top \mathbf{b},$$

$$\Rightarrow \dot{c}(t) = \mathbf{A} c(t)^{k-1} + \mathbf{b},$$

where $\mathbf{b}_r = \mathbf{V}^\top \mathbf{b}$. Therefore, for each coefficient function $c_r(t)$, we have

$$\dot{c}_r(t) = \lambda_r c_r(t)^{k-1} + \beta_r$$

(13)

for $r = 1, 2, \ldots, n$. The differential equation (13) is a particular form of the Chini’s equation [39], and can be solved implicitly by using Gauss hypergeometric functions. Based on the method of separation of variables, it can be shown that

$$\int \frac{1}{\lambda_r c_r(t)^{k-1} + \beta_r} dt = \int 1 dt$$

$$\Rightarrow \frac{g\left(\frac{k-2}{k-1} - \frac{\beta_r}{\lambda_r c_r(t)^{k-2}}\right)}{(k-2)\lambda_r c_r(t)^{k-2}} = t + w_r.$$

Plugging the initial conditions yields

$$t = \frac{g\left(\frac{k-2}{k-1} - \frac{\beta_r}{\lambda_r c_r(0)^{k-2}}\right)}{(k-2)\lambda_r c_r(0)^{k-2}} + \frac{g\left(\frac{k-2}{k-1} - \frac{\beta_r}{\lambda_r \alpha_r^{k-2}}\right)}{(k-2)\lambda_r \alpha_r^{k-2}},$$

and the proof is complete. $\blacksquare$

The solutions of $c_r(t)$ can be further solved by any nonlinear solver given a specific time point $t$. We then can recover the complete solution of $\mathbf{x}(t)$ based on the values of $c_r(t)$. Moreover, although $g(a, z)$ is defined for $|z| < 1$, it can be analytically continued along any path in the complex plane that avoids the branch points one and infinity [40]. When $k = 3$, the differential equation (13) is also known as the Riccati equation, which can be converted to a second-order linear system. Furthermore, we can use the implicit solutions to determine the system properties of the dynamics of $c_r(t)$ at some particular points. Denote the implicit solution (12) by $t = f(c_r(t) + \beta_r)$. For example, if we consider $c_r$ approaches to infinity, it can be shown that

$$\lim_{c_r \to \pm \infty} f(c_r) + \beta_r = \beta.$$

Thus, if the second terms $\beta$ in (12) are positive for some $r$, the domains of the coefficient functions $c_r(t)$ will be $[0, \beta]$. We can therefore conclude that the dynamical systems of the coefficient functions are unstable, which can be used for inferring the system properties of the original polynomial dynamical system.

**V. EXTENSION TO GENERAL HPDS**

In this section, we extend the previous results to general HPDS. First, we introduce the notion of almost symmetric for cubical tensors.

**Definition 1:** A kth order $n$-dimensional tensor $\mathbf{A} \in \mathbb{R}^{n \times n \times \ldots \times n}$ is called almost symmetric if it is symmetric only to its first $k - 1$ mode.

**Proposition 4:** Every HPDS of degree $k - 1$ can be represented by

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)^{k-1},$$

(14)

where $\mathbf{A} \in \mathbb{R}^{n \times n \times \ldots \times n}$ is a $k$th order $n$-dimensional almost symmetric tensor.

**Proof:** Since $\mathbf{A}$ is almost symmetric, its $(k - 1)$th order sub-tensors $\mathbf{A}_{\ldots\ldots\ldots\ldots\ldots\ldots}$ are symmetric for $j = 1, 2, \ldots, n$. We know that every homogeneous polynomial of degree $k - 1$ can be represented by a $(k - 1)$th order symmetric tensor. Therefore, the result follows immediately. $\blacksquare$

The colon operator “:” in the proof acts as shorthand to include all subscripts in a particular array dimension as used in MATLAB. Since $\mathbf{A}$ is almost symmetric, the CP decomposition of $\mathbf{A}$ is given by

$$\mathbf{A} = \sum_{r=1}^{R} \mathbf{v}_r \circ \mathbf{v}_r \circ \ldots \circ \mathbf{v}_r^{(j)}.$$

(15)

Without loss of generality, we multiply the weights $\lambda_r$ into the vector $\mathbf{v}_r^{(j)}$ beforehand. Our goal is to construct a linear transformation $\mathbf{P} \in \mathbb{R}^{n \times n}$ with $\mathbf{x}(t) = \mathbf{Py}(t)$ such that the transformed system can be represented by an odeco tensor, i.e.,

$$\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t)^{k-1},$$

(16)

where $\mathbf{A} \in \mathbb{R}^{n \times n \times \ldots \times n}$ is a $k$th order $n$-dimensional odeco tensor.
Proposition 5: Suppose that $A \in \mathbb{R}^{n \times n \times \cdots \times n}$ has a CP decomposition (15) with $R = n$. If one can find an invertible linear transformation $P \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ that satisfy the conditions:

1) $P^T V = P^{-1} V^{(f)} \Lambda^{-1}$;
2) $P^T V$ is an orthogonal matrix, where $V \in \mathbb{R}^{n \times n}$ and $V^{(f)} \in \mathbb{R}^{n \times n}$ are the matrices that contain all the vectors $v_r$ and $v_r^{(f)}$, respectively, then the HPDS (14) can be transformed to the odeco HPDS (16).

Proof: Since $y(t) = P^{-1} x(t)$, we can write

$$
\dot{y}(t) = P^{-1} \dot{x}(t) = P^{-1} \left( A x(t)^{k-1} \right)
$$

$$
= P^{-1} \left[ \left( \sum_{r=1}^{n} v_r \circ v_r \circ \cdots \circ v_r^{(f)} \right) (Py(t))^{k-1} \right]
$$

$$
= \left( \sum_{r=1}^{n} P^T v_r \circ P^T v_r \circ \cdots \circ P^{-1} v_r^{(f)} \right) y(t)^{k-1}.
$$

If $P^T V = P^{-1} V^{(f)} \Lambda^{-1}$, then

$$
\left( \sum_{r=1}^{n} P^T v_r \circ P^T v_r \circ \cdots \circ P^{-1} v_r^{(f)} \right) y(t)^{k-1}
$$

$$
= \left( \sum_{r=1}^{n} \lambda_r P^T v_r \circ P^T v_r \circ \cdots \circ P^T v_r \right) y(t)^{k-1},
$$

where $\lambda_r$ are the $r$th diagonal of $\Lambda$. Moreover, if $P^T V$ is an orthogonal matrix, then the transformed dynamic tensor

$$
\tilde{A} = \sum_{r=1}^{n} \lambda_r P^T v_r \circ P^T v_r \circ \cdots \circ P^T v_r,
$$

which is odeco. Thus, the result follows immediately.

The above two conditions can be further simplified.

Corollary 3: Suppose that $A \in \mathbb{R}^{n \times n \times \cdots \times n}$ has CP decomposition (15) with $R = n$. Let $V \in \mathbb{R}^{n \times n}$ and $V^{(f)} \in \mathbb{R}^{n \times n}$ be the matrices that contain all the vectors $v_r$ and $v_r^{(f)}$, respectively. Let $W = (V^{-1})^T$. If there exist $\lambda_r \in \mathbb{R}$ such that $w_r = \lambda_r^{-1} v_r^{(f)}$ for all $r$ ($w_r$ are the column vectors of $W$), then the HPDS (14) can be transformed to the odeco HPDS (16).

Proof: The result follows immediately by combining the two conditions from Proposition 5, i.e., $PP^T V = V^{(f)} \Lambda^{-1} \Rightarrow PP^{-1} W = V^{(f)} \Lambda^{-1} \Rightarrow W = V^{(f)} \Lambda^{-1}$.

After obtaining $\Lambda$, one can compute the linear transformation $P$ for an arbitrary orthonormal basis, i.e., $P^T V = U$, where $U$ is an arbitrary orthogonal matrix. Since the transformed state $y(t)$ can be solved explicitly, the solution of the original HPDS is then given by $x(t) = Py(t)$. The results of stability also follow.

For the cases when $R < n$, one can simply add zero Z-eigenvalues to make $R = n$. For the cases when $R > n$, one may split the CP decomposition, i.e.,

$$
A = \sum_{r=1}^{n} v_r \circ v_r \circ \cdots \circ v_r^{(f)} + \sum_{r=n+1}^{R} v_r \circ v_r \circ \cdots \circ v_r^{(f)},
$$

and try to transform the system to

$$
\dot{y}(t) = (A + E) y(t)^{k-1},
$$

where $\tilde{A} \in \mathbb{R}^{n \times n \times \cdots \times n}$ is a $k$th order $n$-dimensional odeco tensor, and $E$ is an error tensor. More works need to be done in this case.

VI. NUMERICAL EXAMPLES

All the numerical examples presented were performed on a Macintosh machine with 16 GB RAM and a 2 GHz Quad-Core Intel Core i5 processor in MATLAB R2020b.

![Fig. 1. Trajectories of the odeco HPDS with the initial condition $x_0 = [0.6516 -1.3239 0.9070]^T$ using the MATLAB ODE45 solver.](image_url)

A. Explicit Solutions

In this example, we compute the explicit solution of an odeco HPDS, and compare it to the trajectory using the MATLAB ODE45 solver. Given a following 3-dimensional odeco HPDS of degree two

$$
\begin{align*}
\dot{x}_1 &= 0.0962 x_1^2 + 0.0291 x_2^3 + 0.0957 x_2^2 \\
&\quad -0.0170 x_1 x_2 - 0.0048 x_1 x_3 - 0.0322 x_2 x_3 \\
\dot{x}_2 &= -0.0085 x_1^2 + 0.1840 x_2^2 + 0.0992 x_2^2 \\
&\quad +0.0582 x_1 x_2 - 0.0322 x_1 x_3 + 0.0474 x_2 x_3 \\
\dot{x}_3 &= -0.0024 x_1^2 - 0.0237 x_2^2 - 0.4400 x_2^2 \\
&\quad -0.0322 x_1 x_2 + 0.1914 x_1 x_3 + 0.1984 x_2 x_3
\end{align*}
$$

it can be represented in the form of (6) with

$$
A_1 = \begin{bmatrix}
0.0962 & -0.0085 & -0.0024 \\
-0.0085 & 0.0291 & -0.0161 \\
-0.0024 & -0.0161 & 0.0957
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
0.2921 & 0.1844 & 0.0237 \\
-0.0161 & 0.0237 & 0.0992 \\
-0.0024 & -0.0161 & 0.0957
\end{bmatrix},
$$

$$
A_3 = \begin{bmatrix}
0.0957 & 0.0992 & -0.4402 \\
-0.0161 & 0.0237 & 0.0992 \\
-0.0024 & -0.0161 & 0.0957
\end{bmatrix},
$$

such that $A$ is odeco. Thus, we can write down the explicit solution of the HPDS according to Proposition 1, which is given by

$$
x(t) = \frac{\alpha_1}{1 + 0.5\alpha_1 t} \begin{bmatrix}
-0.1990 \\
-0.1953 \\
0.9603
\end{bmatrix} + \frac{\alpha_2}{1 + 0.2\alpha_2 t} \begin{bmatrix}
-0.1218 \\
-0.9674 \\
-0.2220
\end{bmatrix} + \frac{\alpha_3}{1 - 0.1\alpha_3 t} \begin{bmatrix}
0.9724 \\
-0.1612 \\
0.1687
\end{bmatrix},
$$

where $\alpha_r$ can be determined by initial conditions. The results are shown in Table 1, in which we compute the state coordinates for the initial condition $x_0 = [0.6516 -1.3239 0.9070]^T$ with $\alpha_r = 1$ for $r = 1, 2, 3$ at $t = 2, 4, 6, 8$. The domain of the solution is given by $[0, 10]$. It is evident that the state coordinates are very close at
of degree three

Given a following 2-dimensional odeco HPDS of degree three

\[
\begin{align*}
\dot{x}_1 &= -1.2593 x_1^2 + 1.6630 x_1^2 x_2 - 1.5554 x_1 x_2^2 - 0.1386 x_2^3, \\
\dot{x}_2 &= 0.5543 x_1^2 - 1.5554 x_1^2 x_2 - 0.4158 x_1 x_2^2 - 0.7037 x_2^3,
\end{align*}
\]

it can be represented in the form of (6) with

\[
A_{:11} = \begin{bmatrix} -1.2593 & 0.5543 \\ 0.5543 & -0.5185 \end{bmatrix}, \quad A_{:12} = \begin{bmatrix} 0.5543 & -0.5185 \\ -0.5185 & -0.1386 \end{bmatrix},
\]

\[
A_{:21} = \begin{bmatrix} 0.5543 & -0.5185 \\ -0.5185 & -0.1386 \end{bmatrix}, \quad A_{:22} = \begin{bmatrix} -0.5185 & -0.1386 \\ -0.1386 & -0.7037 \end{bmatrix},
\]

such that \( A \) is odeco. The two \( Z \)-eigenvalues in the orthogonal decomposition of \( A \) are \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \). Therefore, according to Corollary 1, the odeco HPDS is asymptotically stable for arbitrary initial conditions. The results are shown in Table 2, in which we compute the Frobenius norm of \( x(t) \) for five random initial conditions at \( t = 0, 10, 10^2, 10^3, 10^4, 10^5, 10^6 \). The five initial conditions are given by

\[
\text{IC 1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{IC 2} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \quad \text{IC 3} = \begin{bmatrix} 100 \\ 30 \end{bmatrix},
\]

\[
\text{IC 4} = \begin{bmatrix} -40 \\ -200 \end{bmatrix}, \quad \text{and IC 5} = \begin{bmatrix} -1000 \\ 800 \end{bmatrix}.
\]

It is clear to see that all the trajectories of the odeco HPDS with the five initial conditions converge to the origin. In addition, we plot the vector field of the dynamical system, which also indicates that the equilibrium point \( x = 0 \) is asymptotically stable, see Figure 2.

### C. Stability Using the Upper Bound

In this example, we apply the upper bound of the largest \( Z \)-eigenvalue defined in (10) to determine the stability of the odeco HPDS defined in the last example. The unfolded matrix \( \psi(A) \) is given by

\[
\psi(A) = \begin{bmatrix} -1.2593 & 0.5543 & 0.5543 & -0.5185 \\ 0.5543 & -0.5185 & -0.5185 & -0.1386 \\ 0.5543 & -0.5185 & -0.5185 & -0.1386 \\ -0.5185 & -0.1386 & -0.1386 & -0.7037 \end{bmatrix}.
\]

The maximum eigenvalue of the unfolded matrix is \( \mu_{\text{max}} = 0 \). Therefore, according to Corollary 2, the odeco HPDS is stable at the equilibrium point \( x = 0 \) (although we know that the system is actually asymptotically stable from the last example).
TABLE III
Trajectories of the Polynomial dynamical system using the implicit solution equations of \( c_r(t) \) and the MATLAB ODE45 Solver. We also report the relative errors between the two trajectories.

<table>
<thead>
<tr>
<th>Time</th>
<th>( t = 0 )</th>
<th>( t = 0.05 )</th>
<th>( t = 0.1 )</th>
<th>( t = 0.15 )</th>
<th>( t = 0.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete Sol.</td>
<td>-0.4105</td>
<td>-0.5165</td>
<td>-0.6965</td>
<td>-1.0941</td>
<td>-3.3331</td>
</tr>
<tr>
<td>ODE45 Solver</td>
<td>-0.4105</td>
<td>-0.5165</td>
<td>-0.6965</td>
<td>-1.0941</td>
<td>-3.3360</td>
</tr>
<tr>
<td>Relative Error</td>
<td>0</td>
<td>4.1 \times 10^{-7}</td>
<td>6.0 \times 10^{-7}</td>
<td>1.2 \times 10^{-9}</td>
<td>6.7 \times 10^{-4}</td>
</tr>
</tbody>
</table>

D. HPDS with Constant Control

In this example, we solve the complete solution of an odeco HPDS with constant control, which is given by

\[
\begin{align*}
\dot{x}_1 &= 1.2593x_1^3 - 1.6630x_2^2 + 1.5554x_1x_2^2 + 0.1386x_2^3 - 0.4105 \\
\dot{x}_2 &= -0.5543x_1^3 + 1.5554x_1^2x_2 + 0.4158x_1x_2^2 + 0.7037x_2^3 + 1.3533 \\
\end{align*}
\]

The above dynamics can be represented in the form of (11) such that \( \mathbf{A} \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \) is odeco, and \( \mathbf{b} = \begin{bmatrix} -0.4105 \\ 1.3533 \end{bmatrix} \). The Z-eigenvalues in the orthogonal decomposition of \( \mathbf{A} \) are \( \lambda_1 = 2 \) and \( \lambda_2 = 1 \) with the corresponding Z-eigenvectors \( v_1 = \begin{bmatrix} -0.8819 \\ 0.4717 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 0.4714 \\ 0.8819 \end{bmatrix} \). Suppose that the initial condition is \( \mathbf{x}_0 = \begin{bmatrix} -0.4105 \\ 1.3533 \end{bmatrix} \).

Then the differential equations for the coefficient functions \( c_r(t) \) are given by

\[
\begin{align*}
\dot{c}_1(t) &= 2c_1(t)^3 + 1 \\
\dot{c}_2(t) &= c_2(t)^3 + 1 \\
\end{align*}
\]

with \( c_1(0) = c_2(0) = 1 \). Thus, the coefficient functions \( c_1(t) \) and \( c_2(t) \) can be solved implicitly by

\[
t = -\frac{g\left(\frac{2}{3}, -\frac{1}{2}c_1(t)^3\right)}{4c_1(t)^2} + \frac{g\left(\frac{2}{3}, -\frac{1}{2}\right)}{4},
\]

\[
t = -\frac{g\left(\frac{2}{3}, -\frac{1}{2}c_2(t)^3\right)}{2c_2(t)^2} + \frac{g\left(\frac{2}{3}, -\frac{1}{2}\right)}{4},
\]

respectively. The trajectories of \( c_1(t) \) and \( c_2(t) \) increase rapidly as \( t \) approaches to \( g\left(\frac{2}{3}, -\frac{1}{2}\right)/4 \) and \( g\left(\frac{2}{3}, -\frac{1}{2}\right)/2 \), respectively. This implies that the original polynomial dynamical system is unstable. In addition, we compare the complete solution solved from the implicit equations of \( c_r(t) \) to the trajectory using the MATLAB ODE45 solver, in which the solution is defined over \( t \in [0, g\left(\frac{2}{3}, -\frac{1}{2}\right)/4] \).

It is clear that the state coordinates are very close at each time point between the two approaches, see in Table 3.

VII. CONCLUSION

This paper investigated the explicit solutions and stability properties of certain continuous-time HPDS that can be represented by odeco tensors. We derived an explicit solution formula using the Z-eigenvalues and Z-eigenvectors from the orthogonal decomposition of the corresponding dynamic tensors. By utilizing the form of the explicit solutions, the stability properties of such HPDS could be formalized. In particular, the Z-eigenvalues can offer can offer necessary and sufficient stability conditions. Furthermore, we explored the complete solutions of such HPDS with constant inputs. Finally, we provided criteria to determine if a general HPDS can be transformed to an odeco HPDS. Future work will explore the potential of tensor algebra-based computations for Lyapunov equations and Lyapunov stability. This will be particularly important for applications in the robotic context [23]–[25]. Stabilizability and detectability are also important for future research.

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