Motivating Effort with Information about Future Rewards

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Abstract

This paper studies the optimal mechanism to motivate effort in a dynamic principal-agent model without transfers. An agent is engaged in a task with uncertain future rewards and can choose to shirk at any time. The principal knows the reward of the task and provides information to the agent over time. The optimal information policy can be characterized in closed form, revealing two key conditions that make dynamic disclosure valuable: one is that the principal is impatient compared with the agent; the other is that the environment makes the agent become pessimistic over time without any information disclosure. In a stationary environment, the principal benefits from dynamic disclosure if and only if she is less patient than the agent. Maximum delayed disclosure is optimal for an impatient principal: the principal delays all disclosures up to the maximum time threshold and then fully discloses. By contrast, in a pessimistic environment, the principal always benefits from dynamic disclosure, but the level of patience is still a crucial determinant of the structure of the optimal policy.

Keywords: Dynamic information design, delayed disclosure, informational incentives

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1 Introduction

Monetary and material rewards are often used to motivate desired actions in long-term relationships. However, if such rewards are either out of control or not allowed to be used at will, information can become an additional tool for motivating behavior. This paper studies the optimal mechanism to motivate effort in a dynamic principal-agent problem without transfers, focusing on a specific question: When can the principal benefit from revealing information over time?

Consider the relationship between a principal investigator (PI, she) and a postdoctoral researcher (post-doc, he) in a research laboratory. The salaries received by the postdoc are usually not based on his efforts and projects, but he does obtain personal credit that is beneficial to his future career path by completing major breakthroughs in projects. As a predecessor who has worked longer in academia, in many cases the PI knows better than the postdoc how much reward it will bring if a particular project succeeds. By revealing information to the postdoc, the PI can influence his attitude towards the project and indirectly motivate his effort. Moreover, even in situations where the PI does not have additional prior knowledge of the project compared with the postdoc, she often has control over the timing and type of experiments that can improve the understanding of the quality of the project.

Within this example, the main theoretical question of this paper can be expressed as follows: How should the PI release information to motivate the postdoc’s effort? In particular, is it optimal for her to provide an initial disclosure and then keep silent, or does she benefit from some form of dynamic disclosures? In the latter case, is it better to delay all disclosures to a certain date, or to gradually provide small amounts of information over time? What determines the form of the optimal information policy?

It seems a natural idea that the more informed side in a relationship should strategically utilize this advantage. Moreover, rather than taking dynamic or delayed disclosure literally as concealment of information that the principal already possesses, an alternative understanding is to replicate the disclosure of information by conducting experiments at specific times. Examples include the timing of various tests (during scientific research) and reviews by external committees (within business or industry). There may be other justifications for these timings, but managing expectations could be one of them.

In the baseline model of this paper (Section 2), the agent works for the principal on a task and exerts effort continuously until he chooses to shirk. We assume that the shirk decision is irreversible. The principal (she) knows the reward of the task and wants to induce the agent (he) to work, while the agent faces a trade-off between the reward from completing the task and the cost of exerting effort. We refer to the future
reward to the agent as the *quality* of the task, which can take two possible values.\(^1\) Task completion is stochastic and arrives with some Poisson rate that may depend on the task quality. Both players observe when the task is completed, at which point the relationship ends. The principal can commit to an arbitrary dynamic information policy, specifying how and when to disclose information to the agent about the quality of the task. The agent knows the policy and understands the principal’s commitment, observes the realized disclosures, rationally updates his belief about the task quality, and best responds with effort choices that maximize his expected payoff.

The main result of this paper is to identify two key conditions that make dynamic disclosure valuable: One is that the principal is *impatient* compared with the agent, and the other is that the environment makes the agent become *pessimistic* over time without any information disclosure. This result conceptually contributes to the area of dynamic mechanism design by helping to answer “under what conditions are dynamic persuasion mechanisms better than static ones”. Even if delayed disclosure is relatively rare in real life, we believe it is still important to understand why it is rare. There may be two reasons behind this: (1) principals often cannot commit to dynamic information policies, or (2) such policies are seldom optimal. By characterizing the optimal information policy in closed form, our results shed light on this issue by showing that a static policy can be optimal among all dynamic ones, even under the assumption of full commitment. The analysis also identifies the necessary and sufficient conditions under which static information policies are outperformed by dynamic ones.

There is a natural static version of the principal’s problem, where she makes an initial disclosure and then the agent chooses how long to work. We argue that this static version of the problem is equivalent to the standard Bayesian persuasion problem studied by Kamenica and Gentzkow (2011), and hence the optimal static information policy is characterized by the concavification of the principal’s value function. The principal optimally sends two messages: an optimistic message that persuades the agent to start working, and a pessimistic message that convinces the agent that the task has low quality so he shirks immediately. The design of the pooling probabilities makes the agent exactly indifferent between working and shirking after receiving the optimistic message. This characterization of the optimal static information policy provides a useful benchmark to consider whether the principal benefits from being able to commit to a dynamic information policy; that is, whether the principal’s commitment power to the entire signal process necessarily brings her additional value.

We first analyze a stationary environment (Section 3) where the mere passage of time is not informative about the quality of the task, shutting down the second channel. This corresponds to the case where the rate

\(^1\)In Appendix B, we extend the baseline model to the cases where the task quality can take an arbitrary number of values.
of task completion does not depend on its quality. In this framework, the principal benefits from dynamic disclosure if and only if she is less patient\textsuperscript{2} than the agent (the first part of Proposition 2). To get the intuition behind this result, suppose that the principal decides to delay the disclosure for a short period of time. Since the optimal static information policy induces a belief just enough for the agent to start working, the delayed disclosure has to make the agent more confident that the task has high quality. The gain from this dynamic policy compared with the optimal static one is that the agent always starts working until the disclosure time regardless of the task quality, and the loss from it is that the agent shirks from the low-quality task with a higher probability afterward. Since the gain occurs earlier than the loss, an impatient principal benefits from this dynamic adjustment.

Moreover, in the stationary environment, the optimal information policy for an impatient principal takes the form of a maximum delayed disclosure: the principal delays all disclosures up to the maximum time threshold and then fully discloses the task quality (the second part of Proposition 2). To understand this result, note that an arbitrary information policy induces a lottery over the agent’s shirk time, and the principal and the agent have different preferences over such time lotteries. The incentives of the principal and the agent are aligned on the high-quality task, so any optimal information policy never informs the agent to shirk when the task quality is high. On the other hand, when the task quality is low, exponential discounting implies that the principal is risk averse over time lotteries, while the agent is risk seeking. The principal’s problem can be regarded as choosing a time lottery that minimizes the loss when the agent shirks, subject to a series of obedience constraints. Therefore, the optimal time lottery is determined by comparing the risk attitudes of the two parties. Under exponential discounting, this is just a comparison of the discount rates. If the principal is less patient than the agent, which means that the principal has a stronger risk attitude, the optimal lottery will have the lowest risk, i.e., degenerate at some time threshold. This lottery is exactly induced by maximum delayed disclosure.

Moving on to non-stationary environments (Section 4), we allow the task completion rate to vary with the task quality. The analysis boils down to two cases, an optimistic case and a pessimistic case, depending on whether the completion rate of the high-quality task is smaller than that of the low-quality task. This reveals the second channel that makes dynamic disclosures valuable: the agent becomes pessimistic over time in absence of any information disclosure. In the optimistic case, “no news is good news” because the high-quality task has a longer expected duration than the low-quality task. In this case, we show that the

\textsuperscript{2}One might argue that it is standard to assume the principal is more patient than the agent. Indeed, then the first part of the results shows that a non-stationary environment is essential for the principal to adopt any dynamic information policy. However, we also want to point out that the principal may be more impatient than the agent in some cases. In the previous example of a research lab, the PI may be facing a promotion review, and the postdoc may have just been hired.
results from the stationary environment directly apply (Proposition 2'). The agent has no incentive to shirk halfway through once he is persuaded to start working, because he is increasingly likely to be engaged in the high-quality task as time passes.

By contrast, in the pessimistic case, “no news is bad news” because the high-quality task has a shorter expected duration. Consequently, as long as the agent does not believe that the task has high quality for sure, if no additional information is provided, he will eventually be too pessimistic to continue working. We show that whether or not the principal is more patient than the agent, she always benefits from dynamic disclosure. This is because with any static information policy, the only way for the principal to ensure that the agent completes the high-quality task is to make an initial full disclosure, which turns out to be too costly. Under the optimal static information policy, the agent does not complete the high-quality task for sure, since he plans to shirk at a finite time if the task is still not completed. This situation is easily improved by a simple augmentation using dynamic disclosure. In addition to the initial disclosure that motivates the agent to start working, the principal can also fully disclose the task quality when the agent is about to shirk. If the agent is informed that the task has high quality at that moment, he will continue to complete the task for sure. Note that the agent is still willing to work until the second disclosure since this dynamic information policy provides strictly stronger incentives than the optimal static one. However, the level of patience is still a crucial determinant of the structure of the optimal information policy (Proposition 3). Although maximum delayed disclosure continues to be optimal for an impatient principal, the optimal information policy for a patient principal involves Poisson disclosure, where full disclosure arrives at a calibrated Poisson rate in order to restore the stationarity of the environment, i.e., to make “no news is no news”.

Finally, we consider two extensions of the baseline model to illustrate the robustness of the conclusion to richer alternative environments. First, we analyze the cases where the task quality takes more than two values (Appendix B) under the assumption of stationarity, and show that, just as in the binary case, the principal benefits from dynamic disclosure if and only if she is less patient than the agent. In addition, the optimal information policy for an impatient principal involves a series of cutoff disclosures, where the principal informs the agent at each instant whether the task quality is above or below some cutoff value. We then extend the baseline model to a framework that allows transfers (Appendix C), formalized by an information/wage contract offer from the principal to the agent. It turns out that as long as the agent’s effort and task completion are noncontractible, information disclosure is still valuable for the principal to motivate effort, and the optimal contracts contain the information policies that arise from the baseline model.

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3We did not study non-stationary general-quality cases because it seems that somewhat arbitrary assumptions about completion rates must be made.
Related Literature  The literature on information design and Bayesian persuasion, to which this paper contributes, was pioneered by the work of Aumann, Maschler, and Stearns (1995) and Kamenica and Gentzkow (2011). Within this broad category, there is a particularly relevant literature on dynamic information design, which focuses on the role of information control in dynamic environments in terms of motivating and coordinating actions.

Our model is closest to Ely and Szydlowski (2020), which also studies the optimal information policy for a principal to motivate the effort of an agent in a dynamic moral hazard framework without transfers. In Ely and Szydlowski (2020), the asymmetric information is about the duration of the task, and the principal strategically withholds information to keep the agent working even though she knows that the task is complete. In our model, both parties know when the task is complete, so it is impossible for the principal to retain the agent past project completion. Therefore, the two papers obtain different forms of optimal policies. Especially, in our model, dynamic policies may not do better than static ones, unless one of the two channels we point out exists.

This paper is closely related to Au (2015) and Guo and Shmaya (2018), with the same purpose to understand the conditions such that the optimal disclosure plan in a dynamic disclosure framework is static. However, the setting they consider involves the dynamic persuasion of a privately informed receiver, where the sender has the opportunity to reveal information sequentially in order to screen the type of the receiver. Also related are Ely (2017), Renault, Solan, and Vieille (2017) and Ball (2019); however, in their settings, the principal privately observes a state of the world that evolves according to some exogenous stochastic process, and the structure of optimal information policy depends on this process. Instead, the model in this paper characterizes the timing and form of information disclosure in a situation where the state is fixed. Moreover, in both Ely (2017) and Renault, Solan, and Vieille (2017) the agent acts myopically, whereas this paper studies the dynamic relationship between two long-lived players. In a different setup, Smolin (2021) analyzes the optimal performance evaluation for an agent of uncertain productivity and its relationship to the agent’s career concerns, as well as the associated wage profile.

A number of other recent papers study topics similar to our work, but focus on different assumptions about the information that the principal can control. In Kaya (2021), the principal has the commitment power within each period to any signal structure but lacks the commitment over time to a sequence of signals. Thus, the relevant solution concept in that paper is perfect Bayesian equilibrium. In the model of our paper, the principal is granted full commitment power over arbitrary signal processes, yet our result

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4The commitment assumption along this style also occurs in Orlov, Skrzypacz, and Zryumov (2020), Bizzotto, Rüdiger, and Vigier (2021), and Escudé and Sinander (2022).
that a static information policy can be dynamically optimal indicates that this assumption is stronger than necessary in some cases. Orlov, Skrzypacz, and Zryumov (2020) and Bizzotto, Rüdiger, and Vigier (2021) make a different assumption that the principal is not the only source of information. In their models, there is an exogenous flow of payoff-relevant information over which the principal exerts no control. Furthermore, Che, Kim, and Mierendorff (2021) introduce the real cost of generating and processing information, and obtain a version of the folk theorem as the cost vanishes.

The rest of the paper is organized as follows. Section 2 lays out the baseline model. The first main part, Section 3, assumes that the environment is stationary, and derives one condition that makes dynamic information disclosure valuable: the principal is impatient compared with the agent. The second main part, Section 4 then analyzes the non-stationary environments and reveals another channel: the agent becomes pessimistic over time without any information disclosure. Section 5 concludes. Appendix A contains the proofs of all results in the main text. Appendices B and C analyze two extensions of our baseline model and discuss the robustness of the main results.

2 Model

The baseline model is a continuous-time principal-agent problem without transfers. An agent works for a principal and exerts effort until he chooses to shirk. We assume that the shirk decision is irreversible. The agent exerts effort at flow cost $c$, earns a reward $q$ if he works until the task is completed, and discounts at rate $r$. We refer to the reward, $q$, as the quality of the task. The task either has low quality, $L > 0$, or high quality, $H > L$. The realized quality $q$ is the principal’s private information, and the agent begins with prior probability $\mu \in (0, 1)$ that the task has high quality.

Assume that both players observe when the task is completed, and the completion arrives with Poisson rate $\lambda_q$ when the task quality is $q$, at which point the relationship ends. This implies that when the task quality is $q$, the task duration $x$ is an exponential random variable with mean $1/\lambda_q$. Thus, with realized duration $x$, the agent’s ex post payoff from exerting effort until time $\tau$ is

$$e^{-\tau r} q \cdot 1(\tau \geq x) - \int_0^\tau e^{-\tau r} c \, dt = e^{-\tau r} q \cdot 1(\tau \geq x) - \frac{c}{r} (1 - e^{-\tau r}).$$

The principal receives a bonus $b(q)$ if the agent works until the task is completed, and discounts at rate $r_p$.
therefore, her ex post payoff when the agent shirks at \( \tau \) is

\[
e^{-r_{ff} \tau} b(q) \cdot 1 \left( \tau \geq x \right).
\]

We compute the following expected payoff functions useful for later analysis. The agent’s expected payoff from completing a task with quality \( q \) (i.e., shirk at \( \tau = \infty \)) is

\[
v(q) \equiv \mathbb{E}_{x|q} \left[ e^{-r_{ff} x} q - \frac{c}{r} \left( 1 - e^{-r_{ff} x} \right) \right] = \lambda q e^{-r_{ff} x} dx = \frac{\lambda q - c}{r + \lambda q}.
\]

Assume \( v(H) > 0 > v(L) \), so that the low-quality task is not individually rational for the agent.\(^5\) The agent may also plan to shirk at some time \( \tau \) if the task is still incomplete. This leads to expected payoff

\[
v(\tau, q) \equiv \mathbb{E}_{x|q} \left[ e^{-r_{ff} x} q \cdot 1 \left( x \leq \tau \right) - \frac{c}{r} \left( 1 - e^{-r_{ff} \min(x, \tau)} \right) \right] = \frac{\lambda q - c}{r + \lambda q} \left( 1 - e^{-r_{ff} (\tau + l_q)} \right) v(q).
\]

Similarly, the principal’s expected payoff when the agent chooses to shirk at \( \tau \) is

\[
w(\tau, q) \equiv \mathbb{E}_{x|q} \left[ e^{-r_{ff} x} b(q) \cdot 1 \left( x \leq \tau \right) \right] = \frac{\lambda q b(q)}{r_p + \lambda q} \left( 1 - e^{-r_p (\tau + l_q)} \right) v(q).
\]

In particular, \( w(q) = \frac{\lambda q b(q)}{r_p + \lambda q} \) denotes the principal’s expected payoff from a completed task with quality \( q \).

Assume that the principal can commit to an arbitrary rule that specifies how and when to disclose information to the agent about the quality of the task.\(^6\) She can make any number of disclosures at any time during the process. The agent knows the rule and understands the principal’s commitment, observes the realized disclosures, rationally updates his belief about the quality of the task, and best responds with effort choices that maximize his expected payoff. Adapting the proof of Proposition 1 in Kamenica and Gentzkow (2011), one can show that it is without loss of generality to restrict attention to a particular class of rules that produce “recommended actions”, an analog to the revelation principle. We refer to a rule governing these

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\(^5\)This assumption is innocuous when \( \lambda_H \geq \lambda_L \), but when \( \lambda_H < \lambda_L \), it means that the difference between \( \lambda_H \) and \( \lambda_L \) is not so large as to reverse the interpretation of “task quality” (i.e., it is required that \( \lambda_H H > \lambda_L L \)).

\(^6\)In organizations, commitment can be sustained formally through contracts (e.g., reviewed by an external evaluator), or informally through reputation. Moreover, commitment can also be understood as using existing instruments to generate signals that cannot be hidden. In the previous example of a research lab, suppose the project is to develop a new empirical method, and the PI has control over exclusive data sets that can improve the understanding of the value of completing a breakthrough through test runs. The PI can decide which data set to provide to the postdoc and when to provide it, but once the data set is provided, the results of the test run will be publicly observed. Finally, as we show in the following sections, when the principal is patient and the environment either is stationary or makes the agent become optimistic over time, a static information policy turns out to be dynamically optimal. In these cases, full commitment power over arbitrary signal processes is stronger than necessary.
disclosures as an information policy.

Definition 1 (Information policy). An information policy \( \sigma = \{ \sigma_t \}_{t \geq 0} \) consists of a signal process \( \sigma_t(q) \in \Delta([0, 1]) \), which informs the agent whether to continue (\( \sigma_t = 1 \)) or shirk (\( \sigma_t = 0 \)) at any time.

In the following sections, we derive the optimal information policy in closed form, so as to obtain the necessary and sufficient conditions for the principal to benefit from a dynamic information policy.

3 Stationary Case

We begin our analysis with the stationary case, where the completion rate does not depend on the quality of the task, i.e., \( \lambda_H = \lambda_L = \lambda \). In this case, the mere passage of time is not informative about the quality of the task. We fully characterize the optimal dynamic information policy, and, through this analysis, reveal the first channel that makes dynamic information disclosure valuable: the principal is impatient compared with the agent.

The main result for the stationary case is Proposition 2, which shows that the principal benefits from dynamic disclosure if and only if she is less patient than the agent, i.e., \( r_p > r \). Moreover, if \( r_p > r \), maximum delayed disclosure is optimal: the principal delays all disclosure up to the maximum time and then fully discloses the task quality.

3.1 Static Disclosure

Consider the static version of the problem in which the principal makes a single disclosure, and then the agent chooses how long to work. We first show that this static version of the problem is equivalent to a standard Bayesian persuasion problem as studied by Kamenica and Gentzkow (2011).

Let \( \bar{\mu} \) denote the smallest belief at which the agent begins working; that is, the belief is such that the agent’s expected payoff from completing the task

\[
\bar{\mu} v(H) + (1 - \bar{\mu}) v(L) = 0. \tag{3}
\]

Stationarity of Poisson arrivals implies that if the task is still incomplete at time \( t > 0 \), the agent’s expected continuation value from completing the task is the same as the one at time zero. Formally, the memoryless property of the exponential distribution implies that

\[
\mathbb{E}_{t,q} \left[ e^{-r(x-t)}q - \frac{c}{r} \left( 1 - e^{-r(x-t)} \right) \right]_{x > t, q} = \frac{\lambda q - c}{r + \lambda} = v(q).
\]
Therefore, the agent’s decision problem at any time $t > 0$ is equivalent to the one at time zero.

This implies that once the agent starts working, he will not choose to shirk halfway through. As a result, if the agent is convinced that the task has high quality with at least probability $\bar{\mu}$, then he will optimally choose to complete the task ($\tau = \infty$). A slightly pessimistic belief will prompt the agent to shirk immediately ($\tau = 0$) and get a zero payoff. Hence, this static version of the problem is equivalent to a standard Bayesian persuasion problem studied by Kamenica and Gentzkow (2011). The optimal static information policy aims to minimize the probability of the agent becoming so pessimistic.

**Proposition 1.** For $\mu < \bar{\mu}$, the optimal static information policy KG satisfies

1. **The agent is told to continue ($\sigma_0 = 1$)** whenever the task has high quality ($q = H$), and also with positive probability when the task has low quality ($q = L$).

2. **The agent exactly holds belief $\bar{\mu}$** conditional on observing $\sigma_0 = 1$, i.e., $P(q = H|\sigma_0 = 1) = \bar{\mu}$.

**Proof.** All proofs of the results in the main text are in Appendix A. $\square$

For $\mu \geq \bar{\mu}$, no persuasion is necessary. When $\mu < \bar{\mu}$, the principal sends two messages in the optimal static information policy KG: an optimistic message that persuades the agent to start working, and a pessimistic message that convinces the agent that the task has low quality so he immediately shirks. The optimistic message is sent whenever the task quality is high, and also with positive probability when the task quality is low. The pooling probabilities are designed such that the agent’s belief is exactly $\bar{\mu}$ after receiving the optimistic message, so he is exactly indifferent between working and shirking.

The principal’s value function from KG with respect to the agent’s belief, $W_{KG}(\mu)$, is depicted in Figure 1. The solid line segments represent the benchmark without persuasion, and the dashed line segment, formed by the concavification of the solid line segments, represents the principal’s gain obtained from persuasion.

### 3.2 Delayed Disclosure

The previous subsection characterizes the optimal static information policy. Now we consider whether dynamic information policies can do better for the principal.

Suppose the principal delays all disclosure until a certain time $t > 0$, and then fully discloses the task quality. Again, due to the memoryless property of the exponential distribution, the agent’s expected continuation value at time $t$ (when the task is incomplete) is the same as the one at time zero. Thus, the agent will shirk when he is informed that the task has low quality, and will otherwise continue to work until the task is completed. We refer to this information policy as *delayed disclosure* at time $t$, denoted by DD($t$).
Definition 2 (Delayed disclosure). DD(t) denotes the information policy of delayed disclosure at time t, where the principal tells the agent nothing prior to the time t, and fully discloses the task quality at t. Formally, \( \sigma_s = 1 \) for all \( s < t \), \( \sigma_t(H) = 1 \), and \( \sigma_t(L) = 0 \).

Now we consider the value of DD(t) to the agent at time zero. The agent always completes the high-quality task, and plans to shirk at time \( \tau = t \) when the task has low quality. This leads to ex ante value

\[
\mu v(H) + (1 - \mu) v(t, L).
\]

Let \( \tilde{\mu}(t) \) denote the smallest prior at which DD(t) is sufficient to motivate the agent, i.e., the solution to

\[
\tilde{\mu}(t) v(H) + (1 - \tilde{\mu}(t)) v(t, L) = 0.
\]

(4)

Since \( \bar{\mu} \) is the smallest belief at which the agent would start working without any incentive, we have \( \tilde{\mu}(t) < \bar{\mu} \) for all \( t > 0 \).

Next, we examine whether DD(t) enables the principal to increase her payoff relative to KG. The answer depends crucially on patience, i.e., the relationship between \( r_p \) and \( r \). The gain from DD(t) compared with KG is that the agent always starts working regardless of the task quality, and the loss from DD(t) is that the agent shirks from the low-quality task after the disclosure at time \( t \). Since the gain occurs earlier than the loss, an impatient principal benefits from DD(t). The formal result is summarized in the following lemma.

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\[\text{It follows from equation (1) that } v(\tau, L) = \left(1 - e^{-(\tau-p)t}\right) v(L) \text{ with } v(L) < 0, \text{ so } v(\tau, L) \text{ strictly decreases from } 0 \text{ to } v(L) \text{ as } \tau \text{ increases from } 0 \text{ to infinity. Comparing equations (3) and (4) that define the threshold priors } \bar{\mu} \text{ and } \tilde{\mu}(t), \text{ we can see that (i) } \tilde{\mu}(t) \text{ is strictly increasing in } t, \text{ (ii) } \lim_{t \to 0} \tilde{\mu}(t) = 0, \text{ and (iii) } \lim_{t \to \infty} \tilde{\mu}(t) = \bar{\mu}.\]
Lemma 1. Fix a time $t > 0$. $\text{DD}(t)$ increases the principal’s payoff relative to KG for some range of priors if and only if the principal is less patient than the agent, i.e., $r_p > r$.

Figure 2 illustrates the pattern in Lemma 1. In particular, when the principal is less patient than the agent, i.e., $r_p > r$, $\text{DD}(t)$ is better for the principal than KG for some priors (Figure 2a); when the principal is more patient than the agent, i.e., $r_p < r$, $\text{DD}(t)$ is worse for the principal than KG for all priors (Figure 2b).

![Figure 2: The principal’s payoff comparison between DD(t) and KG.](image)

Suppose $r_p > r$, so that $\text{DD}(t)$ enables the principal to increase her payoff relative to static disclosure when the agent’s prior is exactly $\tilde{\mu}(t)$. Taking one step further, we can augment $\text{DD}(t)$ by allowing the time of disclosure to depend on the agent’s prior $\mu$. For any $\mu < \bar{\mu}$, let $\tilde{t}(\mu)$ denote the maximum disclosure time, i.e., the latest time of disclosure subject to the constraint that the agent’s ex ante payoff is at least zero:  

$$
\mu v(H) + (1 - \mu) v(\tilde{t}(\mu), L) = 0.
$$

Given any $\mu < \bar{\mu}$, $\text{DD}(t)$ is enough to motive the agent as long as $t \leq \tilde{t}(\mu)$. Moreover, choosing any $t < \tilde{t}(\mu)$ yields a positive ex ante payoff to the agent, and the principal can improve by further delay. We refer to the information policy that provides the maximum delay $\tilde{t}(\mu)$ for each $\mu$ as maximum delayed disclosure, denoted by MDD.

Definition 3 (Maximum delayed disclosure). MDD denotes the information policy of maximum delayed disclosure, where for $\mu < \bar{\mu}$, the principal tells the agent nothing prior to the time $\tilde{t}(\mu)$, and fully discloses the task quality at $\tilde{t}(\mu)$. Formally, $\sigma_s = 1$ for all $s < \tilde{t}(\mu)$, $\sigma_{\tilde{t}(\mu)}(H) = 1$, and $\sigma_{\tilde{t}(\mu)}(L) = 0$.

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8Note from equations (4) and (5) that $\tilde{t}(\mu)$ is the inverse of $\tilde{\mu}(t)$, so it satisfies (i) $\tilde{t}(\mu)$ is strictly increasing in $\mu$ within the range $(0, \bar{\mu})$, (ii) $\tilde{t}(0) = 0$, $\lim_{\mu \uparrow \bar{\mu}} \tilde{t}(\mu) = \infty$, (iii) $\tilde{t}(\mu) = \infty$ for all $\mu \geq \bar{\mu}$. 
The relationship among the information policies DD($t$), MDD and KG when $r_p > r$ is depicted in Figure 3. In particular, for all $\mu \in (0, \mu)$, MDD strictly increases the principal’s payoff relative to the optimal Principal’s value, $W$

![Figure 3: The principal’s payoff comparison among DD($t$), MDD and KG (when $r_p > r$).](image)

static policy KG. Moreover, $W^{MDD}$ is the concave closure of the family of value functions $\{W^{DD(t)} : t > 0\}$. Hence, $W^{MDD}$ itself is a concave function, and it follows that MDD dominates all information policies in which the principal provides a less informative disclosure. The reason is that any such policy corresponds to a line segment that connects two points on $W^{MDD}$ and hence lies below it. In Proposition 2, we will establish the optimality of MDD within the set of all dynamic information policies, which may involve multiple (or even infinite) disclosure times.

Conversely, consider the case $r_p < r$. The relationship among the information policies DD($t$), MDD and KG are shown in Figure 4. MDD still dominates the family $\{DD(t) : t > 0\}$ of information policies, but $W^{MDD}$ is no longer concave, and its concavification is precisely $W^{KG}$. In this case, Proposition 2 below also establishes the optimality of the static policy KG within all dynamic information policies. In particular, when the principal is more patient than the agent, her commitment power to the whole signal process has no additional value.

3.3 Optimal Information Policy

We are now ready to state the main result for this section, Proposition 2, which establishes the optimality of the information policies mentioned above. Among all dynamic information policies, KG is optimal when $r_p \leq r$, and MDD is optimal when $r_p > r$.\(^9\)

\(^9\)When $r_p = r$, a range of policies including KG and MDD are optimal, as long as they are sufficient to motivate the agent and such that the agent’s expected payoff is exactly zero (e.g., non-maximum delayed partial disclosure).
Figure 4: The principal’s payoff comparison among DD(\(t\)), MDD and KG (when \(r_p < r\)).

Proposition 2. In the stationary case,

1. If \(r_p \leq r\), then KG is optimal among all dynamic information policies.

2. If \(r_p > r\), then MDD is optimal among all dynamic information policies.

The statements in Proposition 2 are about the optimality within the entire set of dynamic information policies, including those that involve multiple (even infinite) disclosures, which may induce shirk time other than zero or \(\tilde{t}(\mu)\).

To understand the intuition behind Proposition 2, note that an arbitrary information policy, \(\sigma\), induces a lottery over the agent’s shirk time, \(\tau\). The principal’s problem can be regarded as choosing a time lottery that minimizes the loss when the agent shirks, subject to a series of obedience constraints.

The incentives of the principal and the agent are aligned on the high-quality task, so an optimal information policy never informs the agent to shirk when the task quality is high. On the other hand, when the task quality is low, exponential discounting implies that the principal is risk averse over time lotteries, while the agent is risk seeking.\(^{10}\) To formally see this, we can apply Jensen’s inequality to the principal’s and the agent’s payoff functions, equations (1) and (2), and obtain

\[
\mathbb{E}_\sigma[w(\tau, L)] = w(L) \cdot \mathbb{E}_\sigma\left[1 - e^{-(r_p+1)\tau}\right] \leq w(L) \cdot \left(1 - e^{-(r_p+1)\mathbb{E}_\sigma[\tau]}\right) = w(\mathbb{E}_\sigma[\tau], L),
\]

\[
\mathbb{E}_\sigma[v(\tau, L)] = v(\mathbb{E}_\sigma[\tau], L) \geq v(L) \cdot \left(1 - e^{-(r_p+1)\mathbb{E}_\sigma[\tau]}\right) = v(\mathbb{E}_\sigma[\tau], L).
\]

\(^{10}\)The connection between exponential discounting and risk preferences over time lotteries already features in the analysis in Section IV.F of Ely and Szydlowski (2020).
Therefore, optimal time lottery $\tau$ is determined by the comparison between the risk attitudes of the two parties. If $r_p < r$, the agent’s risk attitude is stronger, and it follows that the optimal lottery has the highest risk in the sense of second-order stochastic dominance, i.e., $\tau$ takes support on $\{0, \infty\}$. This lottery is induced by KG. On the other hand, if $r_p > r$, the principal’s risk attitude is stronger, and the optimal lottery has the lowest risk, i.e., $\tau$ is degenerate at $\tilde{t}(\mu)$. This lottery is induced by MDD.

4 Optimistic and Pessimistic Case

In the previous section, we have focused on a stationary environment and identified the first feature that makes dynamic disclosure valuable, namely an impatient principal. This section analyzes the full model in Section 2, allowing the completion rate to vary with the task quality (i.e., $\lambda_H \neq \lambda_L$). The analysis boils down to an optimistic case and a pessimistic case, and reveals the second channel that makes dynamic disclosures valuable: the agent becomes pessimistic over time.

The optimistic case refers to $\lambda_H < \lambda_L$, where “no news is good news” because the high-quality task has a longer expected duration. In this case, we show in Proposition 2’ that the results from Section 3 directly apply: the principal benefits from dynamic disclosure if and only if she is less patient than the agent, i.e., $r_p > r$, and MDD is optimal for an impatient principal.

On the other hand, in the pessimistic case, $\lambda_H > \lambda_L$, “no news is bad news” because the high-quality task has a shorter expected duration. In this case, the principal benefits from dynamic disclosure regardless of the relationship between $r_p$ and $r$. However, as we show in Proposition 3, the structure of the optimal information policy still depends on the level of patience. Although MDD continues to be optimal for an impatient principal ($r_p > r$), the optimal information policy for a patient principal ($r_p \leq r$) involves Poisson disclosure, where full disclosure arrives at the calibrated Poisson rate ($\lambda_H - \lambda_L$) when the task has low quality. Under this rate of Poisson disclosure, the agent will neither become optimistic nor pessimistic over time when he does not receive any disclosure in the process. Effectively, Poisson disclosure restores the stationarity of the problem, because under this policy “no news is no news”.

4.1 Optimistic Case

We first analyze the optimistic case, where $\lambda_H < \lambda_L$. Consider the decision problem of the agent if the principal provides no information. Bayes’ rule implies that if the task is not completed at time $t \geq 0$, the
agent’s belief $\mu_t$ is given by

$$
\mu_t = \mathbb{P}(q = H|x > t) = \frac{\mathbb{P}(x > t|q = H)\mathbb{P}(q = H)}{\mathbb{P}(x > t|q = H)\mathbb{P}(q = H) + \mathbb{P}(x > t|q = L)\mathbb{P}(q = L)} = \frac{\mu}{\mu + e^{\lambda_H - \lambda_L}(1 - \mu)}.
$$

(6)

In particular, $\lambda_H < \lambda_L$ indicates that the expected duration of the high-quality task is longer than that of the low-quality task, so the agent becomes increasingly optimistic over time ($\mu_t$ strictly rises in $t$), i.e., “no news is good news”. As an implication, if the task is not completed at time $t \geq 0$, the agent’s expected continuation payoff from working till completion,

$$
\mu_t v(H) + (1 - \mu_t) v(L),
$$

strictly increases in $t$, so the agent will not choose to shirk halfway through once he starts working. As a result, if the agent’s prior belief is at least $\bar{\mu}$, as defined by equation (3) in Section 3, he will choose to complete the task ($\tau = \infty$); otherwise, the agent will shirk immediately ($\tau = 0$) and earn zero. Therefore, the optimal static information policy is exactly KG, as established in Proposition 1. Furthermore, in terms of dynamic information policies, the derivations and intuitions in Section 3 are directly applicable, thus extending the characterization of the optimal policy, Proposition 2, to the optimistic case here.

**Proposition 2’.** In the optimistic case with $\lambda_H \leq \lambda_L$,

1. If $r_p \leq r$, then KG is optimal among all dynamic information policies.

2. If $r_p > r$, then MDD is optimal among all dynamic information policies.

### 4.2 Pessimistic Case

Now we turn to the pessimistic case, where $\lambda_H > \lambda_L$. Again, we first consider the decision problem of the agent if the principal does not provide any information. There still exists a smallest belief at which the agent starts working. However, as is shown in Lemma 2 below, if the principal does not provide information, the agent will optimally choose a finite shirk time, rather than complete the task for sure.

Let $\hat{\mu}$ denote threshold prior such that

$$
\hat{\mu} (r + \lambda_H) v(H) + (1 - \hat{\mu}) (r + \lambda_L) v(L) = 0.
$$

(7)
Lemma 2 establishes that the agent chooses to start working if and only if his prior is above \( \hat{\mu} \). Moreover, Bayes’ rule implies that the agent’s belief evolves according to equation (6), so he becomes pessimistic over time because \( \lambda_H > \lambda_L \). For any \( \mu \in (\hat{\mu}, 1) \), the agent will eventually be too pessimistic to continue. Indeed, the threshold belief that makes the agent indifferent between continuing and shirking is exactly \( \hat{\mu} \). Denote by \( \bar{t}(\mu) \) the time when the agent’s belief falls to \( \hat{\mu} \), i.e., the solution to

\[
\frac{\mu}{\mu + e^{(\lambda_H - \lambda_L) \bar{t}(\mu)} (1 - \mu)} = \hat{\mu} \quad \Leftrightarrow \quad \bar{t}(\mu) = \frac{1}{\lambda_H - \lambda_L} \left( \log \frac{\mu}{1 - \mu} - \log \frac{\hat{\mu}}{1 - \hat{\mu}} \right).
\]

It is shown in Lemma 2 that for any \( \mu > \hat{\mu} \), the agent plans to shirk at time \( \bar{t}(\mu) \) if the task is not completed.

**Lemma 2.** Suppose that the principal does not provide any information to the agent.

1. If \( \mu \leq \hat{\mu} \), then the agent shirks immediately.
2. If \( \mu > \hat{\mu} \), then the agent starts working, but plans to shirk at time \( \bar{t}(\mu) \) if the task is not completed.

Figure 5 plots the agent’s shirk time without any information, \( \bar{t}(\mu) \).

![Figure 5: Agent’s shirk time without any information.](image)

As \( \mu \) rises from \( \hat{\mu} \) to one, \( \bar{t}(\mu) \) rises from zero to infinity. Thus, as long as the agent does not believe that the task has high quality for sure, he will choose a finite shirk time if no additional information is provided. Using any static information policy, the only way for the principal to ensure that the agent completes the high-quality task is to make an initial full disclosure, and this turns out to be too costly.

Figure 6 illustrates the optimal static information policy. The solid curve represents the principal’s value without persuasion. For \( \mu \leq \hat{\mu} \), the agent shirks immediately, and the principal’s value is zero. As \( \mu \)
rises from $\hat{\mu}$ to one, the agent works increasingly longer, so the principal’s value also rises. The dashed line segment, formed by the concavification of the solid curve, represents the principal’s gain from persuasion. In particular, the optimal static information policy KG “targets” at some belief $\mu^* \in (\hat{\mu}, 1)$. This implies that the agent does not complete the task for sure even if he starts working, since he plans to shirk at a finite time $\bar{t}(\mu^*)$ if the task is not completed.

This situation is easily improved using dynamic information policies. For example, suppose that in addition to the initial disclosure that motivates the agent to start working, the principal also fully discloses the task quality at time $\bar{t}(\mu^*)$, when the agent was about to shirk. If the agent is informed that the task quality is high, he will continue to complete the task for sure. Note that the agent is still willing to work until time $\bar{t}(\mu^*)$ if he starts working since this dynamic information policy provides strictly stronger incentives than the optimal static one. For the same reason, following this information policy yields a strictly positive ex ante payoff for the agent, so the principal has room for improvement.

Suppose that instead of setting a deterministic disclosure time, the principal discloses the task quality at Poisson rate $(\lambda_H - \lambda_L)$ if the task has low quality. We first show that this Poisson disclosure makes the agent neither optimistic nor pessimistic over time. Consider the event that the agent neither receives disclosure nor completes the task after a tiny $\Delta t$ amount of time. This event occurs with probability $(1 - \lambda_H \Delta t)$ if the task quality is high, and with probability

\[(1 - \lambda_L \Delta t)(1 - (\lambda_H - \lambda_L) \Delta t) = 1 - \lambda_H \Delta t\]

if the task quality is low. Since the event occurs with the same probability under the two task qualities,
Bayes’ rule implies that the agent will not update his belief accordingly. In this way, “no news is no news”, and the stationarity of the problem will be restored. That is, the Poisson disclosure plays the role of keeping silence in the stationary case analyzed in Section 3.

If $\mu < \hat{\mu}$, Poisson disclosure at rate $(\lambda_H - \lambda_L)$ is not enough to motivate the agent. The principal can enhance it with an initial announcement that raises the agent’s belief to $\hat{\mu}$, using a design similar to KG. Then, if the agent starts working, his belief will be constant at $\hat{\mu}$ if no additional disclosure arrives, and he will optimally choose to continue working. He will shirk when informed that the task has low quality. We refer to this information policy as initial & Poisson disclosure, denoted by IPD.

**Definition 4 (Initial & Poisson disclosure).** For $\mu \leq \hat{\mu}$, IPD denotes the information policy of initial & Poisson disclosure where

1. At time zero, the agent is told to continue ($\sigma_0 = 1$) whenever $q = H$, and also with positive probability when $q = L$, such that $\mathbb{P}(q = H | \sigma_0 = 1) = \hat{\mu}$.

2. Disclosure arrives at Poisson rate $(\lambda_H - \lambda_L)$ afterwards when $q = L$. Formally, at any time $t > 0$, $\sigma_t(H) = 1$, and $\mathbb{P}(\sigma_s(L) = 1 \text{ for all } s \in [0, t]) = e^{-(\lambda_H - \lambda_L)t}$.

Part (1a) of Proposition 3 below shows that, when the principal is more patient than the agent ($r_p \leq r$), IPD is optimal among all dynamic information policies for low priors ($\mu \leq \hat{\mu}$).

If $\mu > \hat{\mu}$, the agent will follow the recommendations given by Poisson disclosure at rate $(\lambda_H - \lambda_L)$. However, doing so gives the agent a positive ex ante payoff. In this case, the principal can further improve by delaying the time of the first disclosure. In fact, the agent’s belief is above $\hat{\mu}$ at any time before $\tilde{t}(\mu)$. Therefore, the principal does not need to provide any incentives prior to $\tilde{t}(\mu)$. This gives rise to the information policy delayed Poisson disclosure, denoted by DPD, where Poisson disclosure only begins to arrive after $\tilde{t}(\mu)$.

**Definition 5 (Delayed Poisson disclosure).** For $\mu > \hat{\mu}$, DPD denotes the information policy of delayed Poisson disclosure where the principal tells the agent nothing prior to the time $\tilde{t}(\mu)$, after which disclosure arrives at Poisson rate $(\lambda_H - \lambda_L)$ when $q = L$. Formally, $\sigma_t = 1$ for all $t \leq \tilde{t}(\mu)$; at any time $t > \tilde{t}(\mu)$, $\sigma_t(H) = 1$ and $\mathbb{P}(\sigma_s(L) = 1 \text{ for all } s \in [\tilde{t}(\mu), t]) = e^{-(\lambda_H - \lambda_L)(t - \tilde{t}(\mu))}$.

Similarly, Part (1b) of Proposition 3 shows that when the agent’s prior is high ($\mu > \hat{\mu}$), DPD is optimal for a patient principal ($r_p \leq r$).

Note that the information policies that take the form of a full delayed disclosure (e.g., DD($t$) and MDD in Section 3) are still available to the principal. The second part of Proposition 3 establishes the optimality
of MDD for an impatient principal \( (r_p > r) \), analogous to the second part of Proposition 2. For any \( \mu \leq \hat{\mu} \), the definition of the maximum disclosure time, \( \bar{t}(\mu) \), is the same as before, given by equation (5). However, for \( \mu > \hat{\mu} \), since the agent’s belief is higher than \( \hat{\mu} \) at any time before \( \bar{t}(\mu) \), persuasion only takes effect after time \( \bar{t}(\mu) \), when the agent’s belief is about to fall below \( \hat{\mu} \). Therefore, the effective part of delay in MDD is \( \bar{t}(\mu) - \bar{t}(\mu) \), and this difference must be equal to \( \bar{t}(\hat{\mu}) \), i.e., the maximum disclosure time when the agent starts with prior \( \hat{\mu} \). As a result, we have

\[
\bar{t}(\mu) = \bar{t}(\mu) + \bar{t}(\hat{\mu}) \quad \text{for} \quad \mu > \hat{\mu}.
\] (9)

With \( \bar{t}(\mu) \) now well-defined for any \( \mu \in (0, 1) \), we adapt the definition of MDD as follows:

**Definition 3’** (Maximum delayed disclosure). MDD denotes the information policy of maximum delayed disclosure, where for any \( \mu \in (0, 1) \), the principal tells the agent nothing prior to the time \( \bar{t}(\mu) \), and fully discloses the task quality at \( \bar{t}(\mu) \). Formally, \( \sigma_s = 1 \) for all \( s < \bar{t}(\mu) \), \( \sigma_{\bar{t}(\mu)}(H) = 1 \), and \( \sigma_{\bar{t}(\mu)}(L) = 0 \).

We are ready to present the main result of this section, Proposition 3, which establishes the optimality of the aforementioned information policies.\(^{11}\) When \( r_p \leq r \), depending on the range of priors, either IPD or DPD is optimal among all dynamic information policies. When \( r_p > r \), MDD is optimal.

**Proposition 3.** In the pessimistic case with \( \lambda_H > \lambda_L \),

1. If \( r_p \leq r \), then the optimal policy involves Poisson disclosure.

   (a) If \( \mu \leq \hat{\mu} \), then IPD is optimal among all dynamic information policies.

   (b) If \( \mu > \hat{\mu} \), then DPD is optimal among all dynamic information policies.

2. If \( r_p > r \), then MDD is optimal among all dynamic information policies.

The key step for this characterization is to identify the incentive constraint that must bind. In particular, the proof in Appendix A.2 shows that if \( \mu > \hat{\mu} \), then the agent’s incentive constraint at time \( \bar{t}(\mu) \) implies all earlier ones regardless of the selected information policy. As a corollary, it is optimal for the principal to inform the agent nothing and “free-ride” until time \( \bar{t}(\mu) \), when the agent’s belief falls to \( \hat{\mu} \). Therefore, once

\(^{11}\)We thank an anonymous referee for pointing out the connection between Proposition 3 (part 1) and Proposition 8 in Section VI of Ely and Szydlowski (2020), as both results feature exponentially distributed time lotteries with calibrated parameter. However, neither result directly nests the other. Section VI of Ely and Szydlowski (2020) considers the problem where the principal can both design the distribution of task durations and observe its realization before choosing an optimal information policy. In our model, the principal can neither choose nor ex ante observe the task duration. Nonetheless, in the special case of \( \lambda_L = 0 \) and \( \mu = \hat{\mu} \), one can deduce from Lemma 3 and Proposition 8 in Ely and Szydlowski (2020) that if \( r_p \leq r \), then the time lottery induced by Poisson disclosure at rate \( \lambda_H \) is optimal in our model.
we obtain the optimal information policy for the prior \( \mu = \hat{\mu} \), the principal’s problem for any \( \mu > \hat{\mu} \) can be solved using a translation on the time axis, where we relabel time \( t(\mu) \) as time zero and adapt that optimal policy naturally.

The intuition for the optimal information policy is very similar to the stationary case (Subsection 3.3); that is, it is determined by the risk attitude of the two parties towards time lotteries. The only difference here is that the agent becomes pessimistic in absence of any information disclosure. Therefore, any information policy that induces the agent to complete the high-quality task for sure ultimately needs to reveal the task quality at a rate no less than \( (\lambda_H - \lambda_L) \), in order to offset the downward drift in the agent’s belief (i.e., to make “no news no news”). On the other hand, the task quality can be revealed at any faster rate, and full disclosure at some fixed time can be regarded as the limit case of revealing the task quality at an infinite rate.

The relationship among the information policies IPD/DPD, MDD and KG when \( r_p < r \) is depicted in Figure 7a. In particular, for all \( \mu \in (0, 1) \), both IPD/DPD and MDD strictly increase the principal’s payoff relative to the optimal static policy KG. In the case \( r_p < r \), the agent’s risk attitude is stronger, so the optimal lottery has the highest risk. Within all information policies that make the agent follow the recommendations, the one that includes Poisson disclosure at a rate of exactly \( (\lambda_H - \lambda_L) \) has the highest risk in the sense of second-order stochastic dominance, as the induced time lottery has the largest total mass over an interval that extends to infinity. The initial announcement in IPD or the delay in DPD ensures that the incentives are calibrated just enough to motivate the agent to start working.

Conversely, in the case \( r_p > r \), the principal’s risk attitude is stronger, and the optimal time lottery has the lowest risk, i.e., it is degenerate at a fixed time \( \tilde{t}(\mu) \). This lottery is induced by MDD. As is shown in Figure 7b, both IPD/DPD and MDD still strictly increase the principal’s payoff relative to the optimal static policy KG. However, in contrast to the case \( r_p < r \) (Figure 7a), now IPD/DPD becomes worse than MDD due to the additional risk it brings.

Figure 7: The principal’s payoff comparison among IPD/DPD, MDD and KG.
5 Conclusion

In this paper, we study the optimal mechanism to motivate effort in a dynamic moral hazard model without transfers, in order to identify the conditions required for the principal to benefit from the dynamic utilization of information. We characterize the optimal information policy in closed form, thus revealing two key conditions that make dynamic disclosure valuable: one is that the principal is impatient compared with the agent, and the other is that the environment makes the agent become pessimistic over time. In a stationary environment, the principal can benefit from dynamic disclosure if and only if she is less patient than the agent. By contrast, in a pessimistic environment, the principal always benefits from dynamic disclosure, but the structure of the optimal policy still relies on the level of patience.

Our baseline model describes settings that exclude transfers completely. In the research lab example discussed in the Introduction, this reflects that the salaries of the postdoc are not based on his efforts and projects. In some other applications, it may be a strong assumption that the principal cannot use any form of transfers. To address this concern, we discuss how our analysis extends to a framework that allows transfers (Appendix C), where the principal offers an information/wage contract to the agent. The extension demonstrates that information disclosure is still valuable for the principal to motivate effort as long as the agent’s effort and task completion are noncontractible, and the optimal contracts contain the information policies that arise from the baseline model.

We regard one contribution of this paper as emphasizing the relationship between the optimal information policy and the players’ risk attitude over time lotteries. Under exponential discounting, the risk attitude is fully captured by the discount rate. However, this is not the case when the players have other time preferences. We hope this connection between dynamic information design and time preference will be further explored in future work.

References


A Proofs of Results in the Main Text

For any information policy \( \sigma \), let \( \tau \) denote the agent’s random shirk schedule it induces, and let \( \tau (q) \) denote the conditional distribution of \( \tau \) provided that \( q \) is the quality of the task.

A.1 Proofs for Section 3

Proof of Proposition 1. The shirk schedule induced by any static information policy is specified by two parameters: \( \alpha = P (\tau (H) = 0) \) and \( \beta = P (\tau (L) = 0) \). With the remaining probabilities, the corresponding task is completed (\( \tau = \infty \)). The expected payoffs to the principal and agent are, respectively,

\[
W = \mu (1 - \alpha) w(H) + (1 - \mu) (1 - \beta) w(L), \quad (A.1)
\]
\[
V = \mu (1 - \alpha) v(H) + (1 - \mu) (1 - \beta) v(L). \quad (A.2)
\]

For \( \mu < \bar{\mu} \), the principal maximizes \( W \) subject to the individual rationality constraint that \( V \geq 0 \).

Note that the principal’s objective (A.1) is strictly decreasing in both \( \alpha \) and \( \beta \). Moreover, reducing \( \alpha \) can both increase the objective and relax the constraint. Therefore, it is optimal to set \( \alpha = 0 \), i.e., the agent never shirks when \( q = H \). It follows that the agent is told to continue (\( \sigma_0 = 1 \)) for sure when \( q = H \).

By the definition of \( \bar{\mu} \), equation (3), \( \alpha = \beta = 0 \) is not feasible. It is thus optimal to set \( \alpha = 0 \) and \( \beta > 0 \) such that equation (A.2) is exactly equal to zero. Hence,

\[
\beta = 1 - \frac{\mu v(H)}{(1 - \mu) [-v(L)]} = 1 - \frac{\mu}{1 - \mu} \cdot \frac{1 - \bar{\mu}}{\bar{\mu}}. \quad (A.3)
\]

This implies that when \( q = L \), the agent is also told to continue with probability \( 1 - \beta > 0 \). Together, it follows from Bayes’ rule that

\[
P(q = H|\sigma_0 = 1) = \frac{P(\sigma_0 = 1|q = H) P(q = H)}{P(\sigma_0 = 1|q = H) P(q = H) + P(\sigma_0 = 1|q = L) P(q = L)} = \frac{1 \cdot \mu}{1 \cdot \mu + (1 - \beta) (1 - \mu)} = \bar{\mu}.
\]

This completes the proof. \( \square \)

Proof of Lemma 1. We first write the principal’s payoff from KG as a function of the agent’s prior \( \mu \). For \( \mu < \bar{\mu} \), substituting equation (A.3) into equation (A.1), we get

\[
W_{KG}^{KG} (\mu) = \mu w(H) + \mu \cdot \frac{1 - \bar{\mu}}{\bar{\mu}} w(L) \quad \text{if } \mu < \bar{\mu}.
\]

For \( \mu \geq \bar{\mu} \), the agent always starts out completing the task, yielding

\[
W_{KG}^{KG} (\mu) = \mu w(H) + (1 - \mu) w(L) \quad \text{if } \mu \geq \bar{\mu}.
\]

Now we consider the principal’s payoff from DD(\( t \)). If the agent’s prior is too low (below \( \bar{\mu} (t) \)), he will shirk immediately and the principal’s payoff is zero, strictly worse than KG. If the prior exceeds \( \bar{\mu} (t) \), DD(\( t \)) induces \( \tau (H) = \infty \) and \( \tau (L) = t \). Thus, the principal’s payoff in this range is given by

\[
W_{DD(t)}^{DD(t)} (\mu) = \mu w(H) + (1 - \mu) w(t, L) \quad \text{if } \mu \geq \bar{\mu} (t),
\]

where \( \bar{\mu} (t) < \bar{\mu} \). Since \( w(t, L) < w(L) \) for all finite \( t \), we have \( W_{DD(t)}^{DD(t)} (\mu) < W_{KG}^{KG} (\mu) \) if \( \mu \geq \bar{\mu} \). The only
range where DD(t) can be better than KG is \([\tilde{\mu}(t), \bar{\mu})\). Comparing equation (A.5) with equation (A.4), we see that for any \(\mu \in [\tilde{\mu}(t), \bar{\mu})\),

\[
W^{DD(t)}(\mu) > W^{KG}(\mu) \iff (1 - \mu)w(t, L) > \mu \cdot \frac{1 - \bar{\mu}}{\bar{\mu}}w(L)
\]

\[
\iff (1 - \mu)\left(1 - e^{-(r_0 + \lambda)t}\right) > \mu \cdot \frac{1 - \bar{\mu}}{\bar{\mu}},
\]

where the last equivalence follows from the principal’s payoff function, equation (2). The last inequality yields an upper bound on \(\mu\), so in particular we need the strict inequality to hold when \(\mu = \tilde{\mu}(t)\), i.e.,

\[
(1 - \tilde{\mu}(t))\left(1 - e^{-(r_0 + \lambda)t}\right) > \tilde{\mu}(t) \cdot \frac{1 - \bar{\mu}}{\bar{\mu}}.
\]

Comparing equations (3) and (4) that define the threshold priors \(\tilde{\mu}\) and \(\tilde{\mu}(t)\), we get

\[
(1 - \tilde{\mu}(t))\left(1 - e^{-(r_0 + \lambda)t}\right) = \tilde{\mu}(t) \cdot \frac{v(H)}{-v(L)} = \tilde{\mu}(t) \cdot \frac{1 - \bar{\mu}}{\bar{\mu}}.
\]

Therefore, strictly inequality (A.6) holds if and only if

\[
1 - e^{-(r_0 + \lambda)t} > 1 - e^{-(r_0 + \lambda)t} \iff r_\rho > r.
\]

This completes the proof. \(\square\)

**Proof of Proposition 2.** For \(\mu \geq \bar{\mu}\), both KG and MDD provide no information, and the agent sets out to complete the task with probability one without any persuasion. This is clearly optimal.

Henceforth, we assume \(\mu < \bar{\mu}\). For any information policy \(\sigma\), let \(\alpha\) denote the CDF of \(\tau(H)\), i.e., the agent’s shirk time given that the task has high quality. In particular, \(\alpha(0) \geq 0\) is the probability that the agent shirks immediately, and \(1 - \lim_{s \to \infty} \alpha(s) \geq 0\) is the probability that the agent never shirks. Similarly, let \(\beta\) denote the CDF of \(\tau(L)\), i.e., the agent’s shirk time given that the task has low quality.

The expected payoffs to the principal and agent are, respectively,

\[
W(\alpha, \beta) \equiv \mathbb{E}_\sigma[w(\tau, q)] = \mu \int_0^\infty w(t, H)\,d\alpha(t) + (1 - \mu) \int_0^\infty w(t, L)\,d\beta(t), \quad (A.7)
\]

\[
V(\alpha, \beta) \equiv \mathbb{E}_\sigma[v(\tau, q)] = \mu \int_0^\infty v(t, H)\,d\alpha(t) + (1 - \mu) \int_0^\infty v(t, L)\,d\beta(t). \quad (A.8)
\]

A necessary condition for a pair of CDFs \((\alpha, \beta)\) to be induced by some information policy is the individual rationality constraint \(V(\alpha, \beta) \geq 0\), since the agent can simply ignore all information provided by the principal and shirk immediately. Although this condition is far from sufficient, it turns out to be enough to characterize the optimal information policy. The following program can be seen as a relaxation of the principal’s problem of choosing the optimal information policy:

\[
\max_{\alpha, \beta: [0, \infty) \mapsto [0, 1]} W(\alpha, \beta)
\]

s.t. \(V(\alpha, \beta) \geq 0\),

\(\alpha, \beta\) non-decreasing and right-continuous, \(\alpha(\infty) = \beta(\infty) = 1\),

where the principal is choosing the optimal CDFs \((\alpha, \beta)\) subject only to the constraint of individual rationality. If the solution to the above program (A.9) turns out to be induced by some particular information policy,
then we can conclude that the candidate information policy is optimal. The rest of the proof proceeds with four steps, following this rationale.

**Step 1** Any solution to the relaxed program (A.9) satisfies \( \alpha (t) = 0 \) for all \( t \in [0, \infty) \), i.e., the agent never shirks when the task has high quality.

Let \( \tilde{\alpha} (t) = 0 \) for all \( t \in [0, \infty) \). We will show that (i) \( V (\tilde{\alpha}, \beta) \geq V (\alpha, \beta) \), and (ii) \( W (\tilde{\alpha}, \beta) \geq W (\alpha, \beta) \) where equality holds if and only if \( \alpha (s) = 0 \) for all \( s \in [0, \infty) \).

From equation (A.8) and the fact that \( \nu (t, H) \leq \nu (H) \),

\[
V (\tilde{\alpha}, \beta) = \mu \nu (H) + (1 - \mu) \int_0^{\infty} \nu (t, L) \, d\beta (t)
\]

\[
\geq \mu \int_0^{\infty} \nu (t, H) \, d\alpha (t) + (1 - \mu) \int_0^{\infty} \nu (t, L) \, d\beta (t) = V (\alpha, \beta).
\]

Similarly, from equation (A.7) and the fact that \( w (t, H) \leq w (H) \),

\[
W (\tilde{\alpha}, \beta) = \mu \nu (H) + (1 - \mu) \int_0^{\infty} w (t, L) \, d\beta (t)
\]

\[
\geq \mu \int_0^{\infty} w (t, H) \, d\alpha (t) + (1 - \mu) \int_0^{\infty} w (t, L) \, d\beta (t) = W (\alpha, \beta),
\]

and equality holds if and only if \( \alpha (t) = 0 \) for all \( t \in [0, \infty) \).

This shows that any solution to the relaxed program (A.9) satisfies \( \alpha (t) = 0 \) for all \( t \in [0, \infty) \).

**Step 2** Reformulation of the principal’s relaxed program (A.9).

From Step 1, the principal’s relaxed program (A.9) reduces to

\[
\max_{\beta \in [0, \infty] \to [0, 1]} \mu \nu (H) + (1 - \mu) \int_0^{\infty} w (t, L) \, d\beta (t)
\]

\[
\text{s.t.} \quad \mu \nu (H) + (1 - \mu) \int_0^{\infty} \nu (t, L) \, d\beta (t) \geq 0,
\]

\( \beta \) non-decreasing and right-continuous, \( \beta (\infty) = 1 \).

Note that \( \nu (t, L) = (1 - e^{-(r+\lambda)t}) \nu (L) \), so we can rewrite the constraint as

\[
\beta (0) + \int_0^{\infty} e^{-(r+\lambda)t} \, d\beta (t) \geq 1 - \frac{\mu \nu (H)}{(1 - \mu) [-\nu (L)]} \equiv C_1.
\]

Similarly, \( w (t, L) = (1 - e^{-(r+\lambda)t}) w (L) \), so we can rewrite the objective as

\[
\mu \nu (H) + (1 - \mu) w (L) + (1 - \mu) w (L) \left[ \beta (0) + \int_0^{\infty} e^{-(r+\lambda)t} \, d\beta (t) \right].
\]

Thus, maximizing the objective is equivalent to minimizing \( \beta (0) + \int_0^{\infty} e^{-(r+\lambda)t} \, d\beta (t) \). Hence, we can further
rewrite the principal’s program as

\[
\begin{align*}
\min_{\beta \in [0, \infty) \cap [0, 1]} & \quad P(\beta) \equiv \beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) \\
\text{s.t.} & \quad C(\beta) \equiv \beta(0) + \int_0^\infty e^{-(r + \lambda)t} d\beta(t) \geq C_1,
\end{align*}
\]

where \(C_1 \equiv 1 - \frac{\mu(H)}{1 - \mu([-v(L)])} < 1\). Moreover, the assumption \(\mu < \bar{\mu}\) implies that \(C_1\) is positive. Thus, it is infeasible to have \(\beta(t) = 0\) for all \(t \in [0, \infty)\).

**Step 3** If \(r_p \leq r\), then the CDF \(\beta^{KG}\) induced by KG solves the program (A.10).

Under KG, according to equation (A.3),

\[
\tau(L) = \begin{cases} 
0, & \text{w.p. } 1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]}, \\
\infty, & \text{w.p. } \frac{\mu v(H)}{(1 - \mu)[-v(L)]}.
\end{cases}
\]

Since \(C_1 = 1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]}\), \(\beta^{KG}(t) = C_1\) for all \(t \in [0, \infty)\). Thus, we have \(P(\beta^{KG}) = \beta^{KG}(0) = C_1\).

Now take any CDF \(\beta\) that satisfies the constraint in program (A.10). It suffices to show that \(P(\beta) = P(\beta^{KG}) = C_1\). Since \(r_p \leq r\),

\[
P(\beta) - C(\beta) = \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) - \int_0^\infty e^{-(r + \lambda)t} d\beta(t) = \int_0^\infty (e^{-(r_p + \lambda)t} - e^{-(r + \lambda)t}) d\beta(t) \geq 0.
\]

Therefore, \(P(\beta) \geq C(\beta) \geq C_1\), as desired. This confirms that one solution to the program (A.10) is induced by KG, implying that KG is optimal among all dynamic information policies.

**Step 4** If \(r_p > r\), then the CDF \(\beta^{MDD}\) induced by MDD solves the program (A.10).

Under MDD, \(\tau(L)\) is degenerate at \(\bar{\tau}(\mu)\), i.e., \(\beta^{MDD}(t) = 1(t \geq \bar{\tau}(\mu))\). Thus, we have \(P(\beta^{MDD}) = e^{-(r_p + \lambda)\tau(\mu)}\). It follows from the definition of \(\bar{\tau}(\mu)\), equation (5), that

\[
\mu v(H) + (1 - \mu)(1 - e^{-(r + \lambda)\bar{\tau}(\mu)}) v(L) = 0,
\]

which implies that

\[
\bar{\tau}(\mu) = -\frac{1}{r + \lambda} \log \left(1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]}\right) = -\frac{1}{r + \lambda} \log C_1
\]

\[
\Rightarrow P(\beta^{MDD}) = e^{-(r_p + \lambda)\bar{\tau}(\mu)} = (C_1)^{r_p + \lambda}.
\]

Now take any CDF \(\beta\) that satisfies the constraint in program (A.10). It suffices to show that \(P(\beta) \geq P(\beta^{MDD}) = (C_1)^{r_p + \lambda}\).

If \(\beta(0) \geq C_1\), since \(C_1 < 1\) and \(\frac{r_p + \lambda}{r + \lambda} \geq 1\), it follows that \(P(\beta) \geq \beta(0) \geq C_1\), and the proof is complete.
Henceforth, we assume $\beta(0) < C_1$. Since $r_p > r$, Hölder’s inequality\(^{12}\) implies that

\[
[P(\beta) - \beta(0)]^{\frac{p+1}{p-r}} \geq \int_0^\infty \left( e^{-(r+1)t} \right)^{\frac{p+1}{p-r}} d\beta(t) \geq \int_0^\infty \left(1 - \beta(0)\right)^{\frac{p+1}{p-r}} d\beta(t) = C(\beta) - \beta(0) \geq C_1 - \beta(0),
\]

and thus

\[
P(\beta) \geq \beta(0) + \left[ C_1 - \beta(0) \right]^{\frac{p+1}{p-r}} \left[ 1 - \beta(0) \right]^{\frac{p+1}{p-r}}.
\]

Let $y = \beta(0) \leq C_1$ and $\theta = \frac{r_p + 1}{r_p} > 1$. We now show that for any $0 \leq y < C_1 \leq 1$,

\[
y + \frac{(C_1 - y)^\theta}{(1 - y)^{\theta-1}} \geq (C_1)^\theta.
\]

Note that when $C_1 = 1$, we indeed have

\[
y + \frac{(1 - y)^\theta}{(1 - y)^{\theta-1}} = y + (1 - y) = 1.
\]

Hence, it suffices to show that the function $h(C_1) \equiv (C_1)^\theta - \frac{(C_1 - y)^\theta}{(1 - y)^{\theta-1}}$ is nondecreasing in $C_1$, so that $h(C_1) \leq h(1) = y$. Taking first-order derivative, since $\theta > 1$, we have

\[
h'(C_1) = \theta \left( (C_1)^{\theta-1} - \frac{C_1 - y}{1 - y} \right) \geq 0 \iff C_1 \geq \frac{C_1 - y}{1 - y} \iff (1 - C_1)y \geq 0,
\]

which holds by assumption. This confirms that for any CDF $\beta$ that satisfies the constraint in program (A.10), $P(\beta) \geq (C_1)^{\frac{p+1}{p-r}} = P(\beta^{MDD})$. Therefore, one solution to the program (A.10) is induced by MDD, implying that MDD is optimal among all dynamic information policies. \( \square \)

\[\text{A.2 Proofs for Section 4} \]

\textit{Proof of Proposition 2}. We follow the same rationale of the proof of Proposition 2. Specifically, we will show that if $r_p \leq r$, then the solution to the principal’s relaxed program (A.9) is induced by KG, and if $r_p > r$, then the solution to program (A.9) is induced by MDD. Since the program (A.9) remains unchanged, its solution will be the same as the one given in the proof of Proposition 2. Therefore, it suffices to show that the agent will follow the recommendations given by KG and MDD, respectively.

Under KG, $\sigma_0 = 0$ reveals that the task quality is low, so it is optimal for the agent to shirk. If the agent is informed to continue, by construction, he holds belief $\tilde{P}(q = H|\sigma_0 = 1) = \tilde{\mu}$. At any $t > 0$ the agent is informed to take

\[
1^{12}\text{Here we apply Hölder’s inequality }
\]

\[
\left( \int_S |f|^p \, dv \right)^\frac{1}{p} \left( \int_S |g|^q \, dv \right)^\frac{1}{q} \geq \int_S |fg| \, dv
\]

to the measure space $(S, \mathcal{B}, \nu)$, where $S = [0, \infty]$, $\mathcal{B}$ is the Borel $\sigma$-algebra on $S$, $\nu((s, t]) = \beta(t) - \beta(s)$, $\rho = \frac{r_p + 1}{r_p} > 1$, $\rho' = \frac{r_p + 1}{p-r} > 1$ are such that $\frac{1}{p} + \frac{1}{p'} = 1$, $f(t) = e^{-(r+1)\beta}$, $g(t) = 1$.
Since \( \lambda_H \leq \lambda_L \), \( \mu_t \) is nondecreasing in \( t \), implying that \( \mu_t \geq \bar{\mu} \) for all \( t > 0 \). Therefore, it is indeed optimal for the agent to continue when he is informed to do so.

Under MDD, \( \sigma_t^{\mu} = 0 \) reveals that the task quality is low, so it is optimal for the agent to shirk; \( \sigma_t^{\hat{\mu}} = 1 \) reveals that the task quality is high, so it is optimal for the agent to continue. It suffices to show that the agent chooses to continue at any time prior to \( \bar{\tau}(\mu) \). Suppose that the task is not completed at \( t < \bar{\tau}(\mu) \). Since the agent was told nothing, his belief evolved according to equation (6), leading to

\[
\mu_t = \frac{\bar{\mu}}{\bar{\mu} + e^{(\lambda_H - \lambda_L)r} (1 - \bar{\mu})}.
\]

Following the recommendations given by MDD, the agent always completes the high-quality task, and shirks at \( \tau(L) = \bar{\tau}(\mu) \) when the task has low quality. Thus, the agent’s expected continuation payoff at time \( t \) is

\[
\mu_t v(H) + (1 - \mu_t) v(\bar{\tau}(\mu) - t, L).
\]

Since \( \lambda_H \leq \lambda_L \), \( \mu_t \) is nondecreasing in \( t \), implying that \( \mu_t \geq \mu \) for all \( t > 0 \). Moreover, since \( v(L) < 0 \), the agent’s payoff from working on the low quality task \( v(\bar{\tau}(\mu) - t, L) \) is negative and decreases in its absolute value as \( t \) gets closer to \( \bar{\tau}(\mu) \). Together, we show that for any \( t < \bar{\tau}(\mu) \),

\[
\mu_t v(H) + (1 - \mu_t) v(\bar{\tau}(\mu) - t, L) \geq \mu v(H) + (1 - \mu) v(\bar{\tau}(\mu), L) = 0,
\]

where the last equality is just the definition of \( \bar{\tau}(\mu) \), equation (5). This proves that the agent will follow MDD prior to \( \bar{\tau}(\mu) \). \( \square \)

**Proof of Lemma 2.** The agent’s expected payoff from choosing any shirk time \( \tau \) is given by

\[
h(\tau) = \mu v(\tau, H) + (1 - \mu) v(\tau, L) = \mu \left( 1 - e^{-(r + \lambda_H)\tau} \right) v(H) + (1 - \mu) \left( 1 - e^{-(r + \lambda_L)\tau} \right) v(L).
\]

Thus,

\[
h'(\tau) = \mu e^{-(r + \lambda_H)\tau} (r + \lambda_H) v(H) + (1 - \mu) e^{-(r + \lambda_L)\tau} (r + \lambda_L) v(L).
\]

First assume \( \mu \leq \bar{\mu} \). Since \( \lambda_H > \lambda_L \), we have \( e^{-(r + \lambda_H)\tau} < e^{-(r + \lambda_L)\tau} \) for any \( \tau > 0 \). Therefore,

\[
h'(\tau) < \mu e^{-(r + \lambda_H)\tau} (r + \lambda_H) v(H) + (1 - \mu) e^{-(r + \lambda_L)\tau} (r + \lambda_L) v(L) = e^{-(r + \lambda_L)\tau} \left[ \mu (r + \lambda_H) v(H) + (1 - \mu) (r + \lambda_L) v(L) \right] \leq 0,
\]

where the last inequality follows from the definition of \( \bar{\mu} \), equation (7). This shows that \( h(\tau) \) is strictly decreasing in \( \tau \), and hence the optimal shirk time \( \tau = 0 \).

Now assume \( \mu > \bar{\mu} \). We argue that \( h(\tau) \) attains its maximum at \( \tau = \bar{\tau}(\mu) \). From equation (A.11) and the
As a corollary, the principal should inform the agent nothing before time $\mu > \text{optimal information policy}$.

Proof of Proposition 3. We follow the same rationale of the proof of Proposition 2. For any information policy $\sigma$, let $\alpha$ denote the CDF of $H$ (i.e., the agent’s shirk time given that the task has high quality, and let $\beta$ denote the CDF of $L$).

The expected payoffs to the principal and agent are given by equations (A.7) and (A.8), respectively. A necessary condition for a pair of CDFs $(\alpha, \beta)$ to be induced by some information policy is the individual rationality constraint $V(\alpha, \beta) \geq 0$, since the agent can simply ignore all information provided by the principal and shirk immediately. However, it turns out that this condition is not sufficient to characterize the optimal information policy in the case $\lambda_H > \lambda_L$. Going one step further, we can consider the incentives of the agent at any point in time $s > 0$. The agent can ignore any further messages from the principal and shirk. If $(\alpha, \beta)$ is induced by some information policy, doing so would not be better than the continuation value promised by $(\alpha, \beta)$, defined formally as follows:

$$V_s(\alpha, \beta) \equiv \mathbb{E}_\sigma [v(\tau - s, q)|x > s, \tau > s]$$

$$= \mu_s(\alpha, \beta) \int_s^\infty \frac{v(t - s, H) \alpha(t)}{1 - \alpha(s)} + (1 - \mu_s(\alpha, \beta)) \int_s^\infty \frac{v(t - s, L) \beta(t)}{1 - \beta(s)},$$

where $\mu_s(\alpha, \beta)$ denotes the agent’s belief if the task is not completed at $s$ and he is asked to continue:

$$\mu_s(\alpha, \beta) \equiv \mathbb{P}_\sigma (q = H|x > s, \tau > s) = \frac{(1 - \alpha(s))\mu}{(1 - \alpha(s))\mu + e^{(\lambda_H - \lambda_L)/s}(1 - \beta(s))(1 - \mu)}.$$  (A.13)

Here, equation (A.13) is derived from Bayes’ rule, and is a natural extension of equation (6), the belief evolution when information is not provided. As is discussed above, $V_s(\alpha, \beta) \geq 0$ is also necessary for $(\alpha, \beta)$ to be induced by some information policy. The following program can be seen as a relaxation of the principal’s problem of choosing the optimal information policy:

$$\max_{\alpha, \beta: \text{[0, } \infty) \to \text{[0, 1]}} \quad W(\alpha, \beta)$$

s.t.  
$$V_s(\alpha, \beta) \geq 0, \quad \forall s \geq 0,$$

$$\alpha, \beta \text{ non-decreasing and right-continuous, } \alpha(\infty) = \beta(\infty) = 1,$$

(A.14)

where the principal is choosing the optimal CDFs $(\alpha, \beta)$ subject to a series of incentive constraints indexed by $s$. If the solution to the above program (A.14) turns out to be induced by some particular information policy, then we can conclude that the candidate information policy is optimal.

For $\mu \leq \tilde{\mu}$, we will solve program (A.14), and verify that its solution is induced by the candidate optimal information policy. For $\mu > \tilde{\mu}$, we will show that the time-$\hat{\tau}(\mu)$ constraint implies all earlier ones. As a corollary, the principal should inform the agent nothing before time $\hat{\tau}(\mu)$, when the agent’s belief falls
to \( \hat{\mu} \). Therefore, the principal’s problem for any \( \mu > \hat{\mu} \) can be solved using a translation on the time axis, where we relabel time \( \tilde{t}(\mu) \) as time zero, and adapt the derivation for the case \( \mu = \hat{\mu} \).

The proof proceeds with six steps, similar to the proof of Proposition 2.

**Step 1** Any solution to the relaxed program (A.14) satisfies \( \alpha(t) = 0 \) for all \( t \in [0, \infty) \), i.e., the agent never shirks when the task has high quality.

This step is exactly the same as the one in the proof of Proposition 2, so it is omitted here.

**Step 2** Reformulation of the principal’s relaxed program (A.14).

From Step 1, the principal’s relaxed program (A.14) reduces to

\[
\max_{\beta[0,\infty] \rightarrow [0,1]} \mu w(H) + (1 - \mu) \int_0^\infty w(t, L) \, d\beta(t)
\]

s.t. \( \mu_s(0, \beta) v(H) + (1 - \mu_s(0, \beta)) \frac{\int_s^\infty v(t - s, L) \, d\beta(t)}{1 - \beta(s)} \geq 0, \quad \forall s \geq 0, \)

\( \beta \) non-decreasing and right-continuous, \( \beta(\infty) = 1 \).

Note that \( v(t, L) = \left(1 - e^{-r \cdot \lambda t, L} \right)v(L) \), so substituting in equation (A.13), the definition for \( \mu_s \), we can rewrite the constraint at \( s \) as

\[
e^{(\lambda t - \lambda L)s} \int_s^\infty \left(1 - e^{-(r + \lambda L)(t-s)}\right) \, d\beta(t) \leq \frac{\mu v(H)}{(1 - \mu)[-v(L)]} \equiv C_2
\]

\[\Leftrightarrow \int_s^\infty \left(e^{-(r + \lambda L)s} - e^{-(r + \lambda L)t}\right) \, d\beta(t) \leq C_2 e^{-(r + \lambda L)s}.
\]

Since \( w(t, L) = \left(1 - e^{-(r + \lambda L)t}\right)w(L) \), we can rewrite the objective as

\[
\mu w(H) + (1 - \mu) w(L) - (1 - \mu) w(L) \left[\beta(0) + \int_0^\infty e^{-(r + \lambda L)t} \, d\beta(t)\right].
\]

Thus, maximizing the objective is equivalent to minimizing \( \beta(0) + \int_0^\infty e^{-(r + \lambda L)t} \, d\beta(t) \). Hence, we can further rewrite the principal’s program as

\[
\min_{\beta[0,\infty] \rightarrow [0,1]} P(\beta) \equiv \beta(0) + \int_0^\infty e^{-(r + \lambda L)t} \, d\beta(t)
\]

s.t. \( K_s(\beta) \equiv \int_s^\infty \left( e^{-(r + \lambda L)s} - e^{-(r + \lambda L)t} \right) \, d\beta(t) \leq C_2 e^{-(r + \lambda L)s}, \quad \forall s \geq 0, \quad (A.15) \)

\( \beta \) non-decreasing and right-continuous, \( \beta(\infty) = 1 \).

where \( C_2 = \frac{\mu v(H)}{(1 - \mu)[-v(L)]}. \)

First assume \( \mu \leq \hat{\mu} \). It follows from the definition of \( \hat{\mu} \), equation (7), that

\[
\mu \leq \hat{\mu} \quad \Leftrightarrow \quad C_2 = \frac{\mu v(H)}{(1 - \mu)[-v(L)]} \leq \frac{r + \lambda_L}{r + \lambda_H}. \quad (A.16)
\]

**Step 3** If \( r_p \leq r \) and \( \mu \leq \hat{\mu} \), then the CDF \( \beta^{\text{IPD}} \) induced by IPD solves the program (A.15).
Thus, it follows from the time-disclosure arrives at Poisson rate (\( \lambda_H - \lambda_L \)) afterwards, it follows that for \( s > 0 \),

\[
\beta^{\text{IPD}} (s) = \mathbb{P} (\tau (L) \leq s) = \mathbb{P} (\tau (L) = 0) + \mathbb{P} (\tau (L) \in (0, s)) = \beta^{\text{IPD}} (0) + \left( 1 - \beta^{\text{IPD}} (0) \right) \mathbb{P} (\sigma_s (L) = 0)
\]

Thus, in program (A.15) we have

\[
P \left( \beta^{\text{IPD}} \right) = \sum_{s \geq 0} \left( 1 - C_2 \frac{r + \lambda_H}{r + \lambda_L} + C_2 \frac{r + \lambda_H}{r + \lambda_L} \left( 1 - e^{-2(r + \lambda_L) s} \right) \right) dt
\]

and for all \( s \geq 0 \),

\[
K_s \left( \beta^{\text{IPD}} \right) = \int_s^\infty \left( e^{-(r + \lambda_L) u} s - e^{-(r + \lambda_L) u} t \right) dt = \int_s^\infty \left( \int_s^t \lambda_H - \lambda_L \right) e^{-(r + \lambda_L) u} dt
\]

Now take any CDF \( \beta \) that satisfies the constraints in program (A.15). It suffices to show that \( P (\beta) \geq P (\beta^{\text{IPD}}) \). Rewrite the formula of \( K_s \) using Fubini's theorem:

\[
K_s (\beta) = \int_s^\infty \left( e^{-(r + \lambda_L) u} s - e^{-(r + \lambda_L) u} t \right) dt = \int_s^\infty \left( \int_s^t \lambda_H - \lambda_L \right) e^{-(r + \lambda_L) u} du
\]

Thus, it follows from the time-\( s \) constraint in program (A.15) that

\[
K_s (\beta) \leq C_2 e^{-(r + \lambda_L) s} = K_s (\beta^{\text{IPD}})
\]

\[
\Leftrightarrow \int_s^\infty e^{-(r + \lambda_L) u} \left( 1 - \beta (u) \right) du \leq \int_s^\infty e^{-(r + \lambda_L) u} \left( 1 - \beta^{\text{IPD}} (u) \right) du
\]

\[
\Leftrightarrow \int_s^\infty e^{-(r + \lambda_L) u} \left( \beta (u) - \beta^{\text{IPD}} (u) \right) du \geq 0, \quad \forall s \geq 0.
\]
Similarly, rewrite the formula of $P(\beta)$ using Fubini’s theorem:

$$P(\beta) = \beta(0) + \int_{0}^{\infty} \left( \int_{t}^{\infty} (r_p + \lambda_L) e^{-(r_p + \lambda_L)u} du \right) d\beta(t)$$

$$= \beta(0) + (r_p + \lambda_L) \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} \left( \int_{0}^{u} d\beta(t) \right) du$$

$$= \beta(0) + (r_p + \lambda_L) \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} (\beta(u) - \beta(0)) du = \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} \beta(u) du,$$

which implies that

$$P(\beta) \geq P(\beta_{\text{IPD}}) \iff \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} \beta(u) du \geq \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} \beta_{\text{IPD}}(u) du$$

$$\iff \int_{0}^{\infty} e^{-(r_p + \lambda_L)u} (\beta(u) - \beta_{\text{IPD}}(u)) du \geq 0.$$  \hspace{1cm} (A.19)

Denote by

$$h_1(s) \equiv - \int_{s}^{\infty} e^{-(r_p + \lambda_L)u} \left( \beta(u) - \beta_{\text{IPD}}(u) \right) du.$$

Since $\beta$ and $\beta_{\text{IPD}}$ are CDFs, we have

$$|h_1(s)| \leq \int_{s}^{\infty} e^{-(r_p + \lambda_L)u} du = \frac{e^{-(r_p + \lambda_L)s}}{r_p + \lambda_L}. \hspace{1cm} (A.20)$$

To show equation (A.19), we need $h_1(0) \leq 0$. Leibniz integral rule implies that

$$h_1'(s) = e^{-(r_p + \lambda_L)s} (\beta(s) - \beta_{\text{IPD}}(s)) \Rightarrow \beta(s) - \beta_{\text{IPD}}(s) = e^{(r_p + \lambda_L)s} h_1'(s).$$

Substituting into equation (A.18) and using integration by parts, we get

$$0 \leq \int_{s}^{\infty} e^{-(r_p + \lambda_L)u} \cdot e^{(r_p + \lambda_L)s} h_1(u) du = \int_{s}^{\infty} e^{-(r_p - r_p)p} du h_1(u)$$

$$= e^{-(r_p - r_p)p} h_1(u) \bigg|_{u=s}^{u=\infty} - \int_{s}^{\infty} h_1(u) \frac{d}{du} \left( e^{-(r_p - r_p)p} u \right) du$$

$$= -e^{-(r_p - r_p)p} h_1(s) + (r - r_p) \int_{s}^{\infty} h_1(u) e^{-(r_p - r_p)p} u du$$

$$\iff e^{-(r_p - r_p)p} h_1(s) \leq (r - r_p) \int_{s}^{\infty} e^{-(r_p - r_p)p} u h_1(u) du. \hspace{1cm} (A.21)$$

Denote by

$$H_1(s) \equiv - \int_{s}^{\infty} e^{-(r_p - r_p)p} u h_1(u) du.$$

From equation (A.20), we get

$$|H_1(s)| \leq \int_{s}^{\infty} e^{-(r_p - r_p)p} \cdot \frac{e^{-(r_p + \lambda_L)s}}{r_p + \lambda_L} du = \frac{e^{-(r_p + \lambda_L)s}}{(r_p + \lambda_L)(r + \lambda_L)} \Rightarrow \lim_{s \to \infty} e^{-(r_p - r_p)p} H_1(s) = 0.$$
Using Leibniz integral rule again and substituting in equation (A.21), we have

\[ H'_1(s) = e^{-(r-r_p)s}h_1(s) \leq -\left(r - r_p\right)H_1(s). \tag{A.22} \]

Thus, Grönwall’s inequality\(^{13}\) implies that

\[ H_1(s) \leq H_1(0)e^{-(r-r_p)s}, \quad \forall s \geq 0. \]

Letting \( s \to \infty \), we obtain

\[ H_1(0) \geq \lim_{s \to \infty} e^{(r-r_p)s}H_1(s) = 0. \]

Substituting back into equation (A.22), we have

\[ H_1(0) \leq H_1(0)e^{-(r-r_p)s}, \quad \forall s \geq 0. \]

Therefore, one solution to the program (A.15) is induced by IPD, implying that IPD is optimal among all dynamic information policies.

**Step 4** If \( r_p > r \) and \( \mu \leq \hat{\mu} \), then the CDF \( \tilde{p}^{MDD} \) induced by MDD solves the program (A.15).

We first check that the agent will follow the recommendations given by MDD. Under MDD, \( \sigma_{\tilde{r}(\mu)} = 0 \) reveals that the task quality is low, so it is optimal for the agent to shirk; \( \sigma_{\tilde{r}(\mu)} = 1 \) reveals that the task quality is high, so it is optimal for the agent to continue. It suffices to show that the agent chooses to continue at any time prior to \( \tilde{r}(\mu) \). Suppose that the task is not completed at \( t < \tilde{r}(\mu) \). Since the agent was told nothing, his belief evolved according to equation (6), leading to

\[ \mu_t = \frac{\mu}{\mu + e^{(\lambda_H - \lambda_L)t}(1 - \mu)}. \tag{A.23} \]

Following the recommendations given by MDD, the agent always completes the high-quality task, and shirks at \( \tau(L) = \tilde{r}(\mu) \) when the task has low quality. Thus, the agent’s expected continuation payoff at time \( t \) is

\[ \mu_tv(H) + (1 - \mu_t)v(\tilde{r}(\mu) - t, L), \tag{A.24} \]

and we need to check that this value is nonnegative for any \( t < \tilde{r}(\mu) \). Substituting equation (A.23) into equation (A.24), we know that this is equivalent to

\[ h_2(t) \equiv \mu_v(H) + e^{(\lambda_H - \lambda_L)t}(1 - \mu)v(\tilde{r}(\mu) - t, L) \geq 0, \quad \forall t < \tilde{r}(\mu). \]

By the definition of \( \tilde{r}(\mu) \), equation (5), we have \( h_2(0) = \mu_v(H) + (1 - \mu)v(\tilde{r}(\mu), L) = 0 \). Now we show that \( h_2'(t) \geq 0 \) for all \( t < \tilde{r}(\mu) \). Since \( v(t, L) = \left(1 - e^{-(r+\lambda_L)t}\right)v(L) \), we have

\[
\begin{align*}
    h'_2(t) &= (1 - \mu)v(L) \frac{d}{dt} \left[e^{(\lambda_H - \lambda_L)t}\left(1 - e^{-(r+\lambda_L)(\tilde{r}(\mu) - t)}\right)\right] \\
    &= (1 - \mu)v(L)\left((\lambda_H - \lambda_L)e^{(\lambda_H - \lambda_L)t} - (r + \lambda_H)e^{-(r+\lambda_L)(\tilde{r}(\mu) - t)}\right).
\end{align*}
\]

\(^{13}\)Here we apply Grönwall’s inequality to bound a function that is known to satisfy a differential inequality by the solution of the corresponding differential equation. Formally, let \( \gamma \) and \( u \) be continuous functions defined on interval \( I \). Then

\[ u'(t) \leq \gamma(t)u(t), \quad \forall t \in I \implies u(t) \leq u(a)\exp\left(\int_a^t \gamma(s) \, ds\right), \quad \forall a, t \in I, \, a < t. \]
Since \( v(L) < 0, h'_2(t) \geq 0 \) if and only if

\[
(\lambda_H - \lambda_L) e^{(\lambda_H - \lambda_L)t} - (r + \lambda_H) e^{-(r+\lambda_L)\mu} e^{(r+\lambda_H)t} \leq 0
\]

\[
\Leftrightarrow e^{-(r+\lambda_L)\mu} e^{(r+\lambda_H)t} \geq \frac{\lambda_H - \lambda_L}{r + \lambda_H} \Rightarrow e^{-(r+\lambda_L)\mu} \geq \frac{\lambda_H - \lambda_L}{r + \lambda_H}.
\]

It follows from the definition of \( \bar{t}(\mu) \), equation (5), that

\[
(1 - \mu) \left( 1 - e^{-(r+\lambda_L)\mu} \right) v(L) = 0.
\]

Together with equation (A.16), we have

\[
e^{-(r+\lambda_L)\mu} = 1 - C_2 \geq 1 - \frac{r + \lambda_L}{r + \lambda_H} = \frac{\lambda_H - \lambda_L}{r + \lambda_H},
\]

as desired. Therefore, we have shown that \( h'_2(t) \geq 0 \) for all \( t < \bar{t}(\mu) \), implying that \( h(t) \geq h(0) = 0 \). This proves that the agent will follow MDD prior to \( \bar{t}(\mu) \) as well.

Under MDD, \( \tau(L) \) is degenerate at \( \bar{t}(\mu) \), i.e., \( \beta(t) = 1 \) \( t \geq \bar{t}(\mu) \). Thus, in program (A.15) we have \( P(\beta) = e^{-(r+\lambda_L)\mu} = 1 - C_2 \). Now take any CDF \( \beta \) that satisfies the constraints in program (A.15). It suffices to show that \( P(\beta) \geq P(\beta_{MDD}) \). We ignore the time-\( s \) constraints for all \( s > 0 \), and only use the time-zero constraint \( K_0(\beta) \leq C_2 \). This is equivalent to

\[
(1 - \beta(0)) - \int_0^\infty e^{-(r+\lambda_L)t} \beta(t) \leq C_2 \iff \beta(0) + \int_0^\infty e^{-(r+\lambda_L)t} \beta(t) \geq 1 - C_2 = 1,
\]

the constraint in program (A.10). Since the two programs (A.10) and (A.15) have the same objective, it follows from Step 4 in the proof of Proposition 2 that \( P(\beta) \geq P(\beta_{MDD}) \), as desired. Therefore, one solution to the program (A.15) is induced by MDD, implying that MDD is optimal among all dynamic information policies.

Now we turn to the case \( \mu > \bar{\mu} \). It follows from equation (A.16) that \( C_2 > \frac{r + \lambda_L}{r + \lambda_H} \). We first show that the time-\( \bar{t}(\mu) \) constraint implies all earlier ones.

**Step 5** If \( \mu > \bar{\mu} \), then the time-\( \bar{t}(\mu) \) constraint in program (A.15) implies all earlier ones. As a corollary, any solution to program (A.15) satisfies \( \beta(t) = 0 \) for all \( t \in [0, \bar{t}(\mu)] \).

Suppose that CDF \( \beta \) satisfies the time-\( \bar{t}(\mu) \) constraint in program (A.15), \( K_{\bar{t}(\mu)}(\beta) \leq C_2 e^{-(r+\lambda_H)\bar{t}(\mu)} \). We now show that for any \( s < \bar{t}(\mu) \), \( \beta \) satisfies the time-\( s \) constraint in program (A.15), \( K_s(\beta) \leq C_2 e^{-(r+\lambda_H)s} \). It follows from the alternative representation of \( K_s(\beta) \), equation (A.17), that

\[
K_s(\beta) = (r + \lambda_L) \int_s^{\bar{t}(\mu)} e^{-(r+\lambda_L)u} (1 - \beta(u)) \, du + (r + \lambda_L) \int_s^\infty e^{-(r+\lambda_L)u} (1 - \beta(u)) \, du
\]

\[
\leq (r + \lambda_L) \int_s^{\bar{t}(\mu)} e^{-(r+\lambda_L)u} \, du + K_{\bar{t}(\mu)}(\beta) \leq e^{-(r+\lambda_L)s} e^{-(r+\lambda_L)\bar{t}(\mu)} + C_2 e^{-(r+\lambda_H)\bar{t}(\mu)},
\]

so it suffices to show that

\[
e^{-(r+\lambda_L)s} - e^{-(r+\lambda_L)\bar{t}(\mu)} + C_2 e^{-(r+\lambda_H)\bar{t}(\mu)} \leq C_2 e^{-(r+\lambda_H)s}
\]

\[
\Leftrightarrow e^{-(r+\lambda_L)s} - C_2 e^{-(r+\lambda_H)s} \leq e^{-(r+\lambda_L)\bar{t}(\mu)} - C_2 e^{-(r+\lambda_H)\bar{t}(\mu)}.
\]

(A.25)
Denote by $h_3(s) \equiv e^{-(r+\lambda_L)s} - C_2 e^{-(r+\lambda_H)s}$. We will show that $h_3'(s) \geq 0$ for all $s < \bar{t}(\mu)$. Note that
\[
h_3'(s) = -(r + \lambda_L) e^{-(r+\lambda_L)s} + C_2 (r + \lambda_H) e^{-(r+\lambda_H)s} = (r + \lambda_H) e^{-(r+\lambda_L)s} \left( C_2 e^{-(\lambda_H-\lambda_L)s} - \frac{r + \lambda_L}{r + \lambda_H} \right).
\]
Thus, substituting in the definitions for $\hat{\mu}$ and $\bar{t}(\mu)$, equations (7) and (8), $h_3'(s) \geq 0$ if and only if
\[
C_2 e^{-(\lambda_H-\lambda_L)s} \geq \frac{r + \lambda_L}{r + \lambda_H} \Leftrightarrow s \leq \frac{1}{\lambda_H - \lambda_L} \log \left( C_2 \cdot \frac{r + \lambda_H}{r + \lambda_L} \right) = \frac{1}{\lambda_H - \lambda_L} \log \left( \frac{\mu v(H)}{(1-\mu)(-v(L))} \cdot \frac{r + \lambda_H}{r + \lambda_L} \right) = \frac{1}{\lambda_H - \lambda_L} \left( \log \frac{\mu}{1-\mu} - \log \frac{1-\mu}{1-\mu} \right) = \bar{t}(\mu),
\]
which holds by assumption. As a result, $h_3(s)$ is increasing in $s$, implying that $h_3(s) \leq h_3(\bar{t}(\mu))$, i.e., equation (A.25) holds. Therefore, for any $s < \bar{t}(\mu)$, $\beta$ also satisfies the time-$s$ constraint in program (A.15).

Now we argue that any solution to program (A.15) satisfies $\beta(t) = 0$ for all $t \in [0, \bar{t}(\mu))$. Take any solution $\beta$, and let
\[
\tilde{\beta}(t) = \begin{cases} 0, & t \in [0, \bar{t}(\mu)), \\ \beta(t), & t \in [\bar{t}(\mu), \infty). \end{cases}
\]
For any $s \geq \bar{t}(\mu)$, we have $K_s(\tilde{\beta}) = K_s(\beta) \leq C_2 e^{-(r+\lambda_H)s}$. For any $s < \bar{t}(\mu)$, we have shown above that the time-$s$ constraint is implied by the time-$\bar{t}(\mu)$ one. Putting together, we see that $\tilde{\beta}$ satisfies all constraints in program (A.15). Moreover, the difference in the objective
\[
P(\beta) - P(\tilde{\beta}) = \beta(0) + \int_0^{\bar{t}(\mu)} e^{-(r+\lambda_L)t} d\beta(t) \geq 0,
\]
and equality holds if and only if $\beta(t) = 0$ for all $t \in [0, \bar{t}(\mu))$.

**Step 6** Assume $\mu > \hat{\mu}$. If $r_p \leq r$, then the CDF induced by DPD solves the program (A.15); if $r_p > r$, then the CDF induced by MDD solves the program (A.15).

From Step 5, any solution to program (A.15) satisfies $\beta(t) = 0$ for all $t \in [0, \bar{t}(\mu))$. This implies that the agent’s belief evolves according to equation (6) prior to the time $\bar{t}(\mu)$, and thus he holds belief $\hat{\mu}$ at time $\bar{t}(\mu)$. Therefore, we can adapt the derivation for the case $\mu = \hat{\mu}$ from then on, relabeling time $\bar{t}(\mu)$ as time zero, i.e., letting $s' = s - \bar{t}(\mu)$.

If $r_p \leq r$, Step 3 shows that $\beta^{\text{IPD}}$ solves the program (A.15) if $\mu = \hat{\mu}$. Substituting in equation (A.16), the formula for $\beta^{\text{IPD}}$ is given by
\[
\beta^{\text{IPD}}(s') = 1 - C_2 \frac{r + \lambda_H}{r + \lambda_L} e^{-(\lambda_H-\lambda_L)s'} = 1 - e^{-(\lambda_H-\lambda_L)s'}.
\]
Thus, the following CDF is a solution to program (A.15) for $\mu > \hat{\mu}$:
\[
\beta(s) = \beta^{\text{IPD}}(s - \bar{t}(\mu)) = 1 - e^{-(\lambda_H-\lambda_L)(s-\bar{t}(\mu))}, \quad \forall s \geq \bar{t}(\mu),
\]
where disclosure arrives at Poisson rate $(\lambda_H - \lambda_L)$ any time after $\bar{t}(\mu)$. Therefore, $\beta$ is induced by DPD,
implying that DPD is optimal among all dynamic information policies if \( r_p \leq r \) and \( \mu > \hat{\mu} \).

If \( r_p > r \), Step 4 shows that \( \beta^{\text{MDD}} \) solves the program (A.15) if \( \mu = \hat{\mu} \). The formula for \( \beta^{\text{MDD}} \) is given by \( \beta^{\text{MDD}}(s') = 1(s' \geq \hat{t}(\hat{\mu})) \). Thus, the following CDF is a solution to program (A.15) for \( \mu > \hat{\mu} \):

\[
\beta(s) = \beta^{\text{MDD}}(s - \hat{t}(\hat{\mu})) = 1 \left( s \geq \hat{t}(\mu) + \hat{t}(\hat{\mu}) \right).
\]

It follows from the definition of \( \hat{t}(\mu) \), equation (9), that \( \hat{t}(\mu) = \hat{t}(\mu) + \hat{t}(\hat{\mu}) \) for \( \mu > \hat{\mu} \). Therefore, \( \beta \) is induced by MDD, implying that MDD is optimal among all dynamic information policies if \( r_p > r \) and \( \mu > \hat{\mu} \). \( \square \)
B General Distributions

This appendix presents a general characterization of the optimal dynamic information policy when the task quality takes more than two values, assuming that the environment is stationary \((\lambda_q \equiv \lambda)\). In other words, it is a multi-state extension of the binary stationary case (Section 3).

Assume that the agent’s prior is given by a continuous distribution \(F\) that has a full-support density over \([0, \infty)\). This assumption implies that, conditional on the realized task quality, the agent’s shirk time \(\tau(q)\) will be deterministic.\(^{14}\) In addition, assume that the agent’s ex ante payoff from completing the task is negative, i.e., \(\mathbb{E}_q[\nu(q)] = \int_0^\infty \nu(q) dF(q) < 0\); otherwise, no persuasion is necessary. Finally, assume that the principal’s bonus \(b(q)\) is nondecreasing in the task quality \(q\).\(^{15}\)

The main result for this section is that, analogous to the binary stationary case (Section 3), the principal benefits from dynamic disclosure if and only if she is less patient than the agent \(r_p > r\). If \(r_p \leq r\), Proposition B.1 shows that the optimal information policy is an adaptation of KG. If \(r_p > r\), the optimal information policy involves a series of cutoff disclosures, as is shown in Propositions B.2 and B.3.

B.1 Static Disclosure

As in the binary case, we consider the static version of the principal’s problem, where she provides a one-shot disclosure, and then the agent chooses effort.

Because we assume that the distribution \(F\) has full support and no atoms, there exists a unique task quality \(q^* > 0\) such that

\[
\int_{q^*}^{\infty} v(q) \, dF(q) = 0. \tag{B.1}
\]

Let \(\bar{q}\) denote the minimum individually rational quality for the agent; that is, \(\bar{q}\) solves

\[
v(\bar{q}) = \frac{\lambda \bar{q} - c}{r + \lambda} = 0, \quad \text{i.e.,} \quad \bar{q} = \frac{c}{\lambda}.
\]

It follows from equation (B.1) that \(q^* < \bar{q}\).

Consider the following static information policy, which is a counterpart of KG: the principal informs the agent whether the task has high quality \((q \geq q^*)\) or low quality \((q < q^*)\). Again, the stationarity of Poisson arrivals implies that if the task is still incomplete at time \(t > 0\), the agent’s decision problem is equivalent to the one at time zero. Therefore, equation (B.1) indicates that if the agent is informed that the task has high quality, he will optimally choose to complete the task \((\tau = \infty)\). Otherwise, the agent shirks immediately \((\tau = 0)\) and earns zero. Slightly abusing notation, we also use KG to represent this information policy.

**Definition B.1** (Static information policy KG). KG denotes the static information policy that informs the agent whether the task has high quality \((q \geq q^*)\) or low quality \((q < q^*)\). Formally, \(\sigma_0(q) = 1 (q \geq q^*)\). The

---

\(^{14}\)In the baseline model, when the task quality is low, the induced shirk time \(\tau(L)\) under KG is not deterministic. The probabilities of the agent’s immediate shirk \((\tau(L) = 0)\) and completion of the task \((\tau(L) = \infty)\) are both positive. This is because when the task quality takes discrete values, such mixing between working and shirking is required to adjust the agent’s expected payoff to zero. With a continuous quality space, the same adjustment can be achieved by continuously changing the interval of quality that the agent shirks immediately.

\(^{15}\)This assumption applies to many situations. One example is that the principal receives the same bonus at completion regardless of the task quality \((b(q) = b_0)\). Another is that the principal and the agent have the same reward from the task \((b(q) = q)\), so the difference between their payoffs is only that the principal does not incur the costs. Furthermore, the results for this section would go through even if \(b(q)\) may be decreasing, as long as the speed is not too fast. Formally, we need the ratio \([-\nu(q)] / b(q)\) to nonincreasing over the range of individually irrational tasks \(q \in [0, \bar{q}]\).
induced shirk time, $\tau^{KG}$, is given by

$$\tau^{KG}(q) = \begin{cases} 0, & q < q^*, \\ \infty, & q \geq q^*. \end{cases}$$

Indeed, the initial disclosure motivates the agent to complete some tasks with lower quality than the minimum individually rational level (those in the range $q^* \leq q < \tilde{q}$). As an analog of the first part of Proposition 2, we show below that if $r_p \leq r$, the static information policy $KG$ is optimal among all dynamic ones.

**Proposition B.1.** If $r_p \leq r$, then $KG$ is optimal among all dynamic information policies.

*Proof.* All proofs for this Appendix are in Appendix B.5. $\square$

### B.2 Cutoff Disclosures

Henceforth, we consider the case where the principal is less patient than the agent, $r_p > r$. Although it is true as in the binary case that dynamic disclosure policies can indeed do strictly better than static ones, the counterpart of MDD (Definition 3) cannot be characterized that easily. The key complication here is that the principal can customize the disclosure times for different realized quality $q$, and the principal strictly benefits from this design.

As a starting point, we define the following family of information policies:

**Definition B.2 (Cutoff disclosures).** An information policy features cutoff disclosures if there exists a function of cutoff quality, $\tilde{q}(s)$, such that at each instant $s \geq 0$ if the task is still incomplete, the principal tells the agent whether the task has relatively high quality ($q \geq \tilde{q}(s)$) or relatively low quality ($q < \tilde{q}(s)$), such that the agent shirks when he is informed that the task has relatively low quality, and continues otherwise. Formally, $\sigma_t(q) = 1$ ($q \geq \tilde{q}(s)$).

There is no guarantee that the agent will follow the recommendations given by an information policy with cutoff disclosures. One necessary condition is the individual rationality constraint that the agent obtains at least zero ex ante expected payoff; otherwise, he can simply ignore all the information provided by the principal, shirk immediately, and ensure a zero payoff. Lemma B.1 shows that individual rationality is not only necessary but also sufficient for the agent to follow any information policy with cutoff disclosures, as long as the policy is specified by a nondecreasing function of cutoff quality $\tilde{q}(s)$ that takes value in $[0, \tilde{q}]$.

**Lemma B.1.** Consider any information policy $\sigma$ specified by a function of cutoff quality $\tilde{q}(s)$, where $\tilde{q}(s)$ is nondecreasing and ranges in $[0, \tilde{q}]$. The agent’s best response is to follow $\sigma$ if and only if $\sigma$ is individually rational; that is, following $\sigma$ yields a nonnegative ex ante payoff for the agent.

Figure 8 plots some examples of cutoff disclosures. Note that KG can be viewed as a one-shot cutoff disclosure, specified by the constant function of cutoff quality $\tilde{q}(s) = q^*$, as shown in Figure 8a. Figure 8b plots the information policy of a one-shot delayed disclosure at time $t$, informing the agent whether the task is individually rational, i.e., the direct adaptation of the information policy DD($t$) (Definition 2).

Figure 8c plots an information policy of an initial disclosure with cutoff quality $q_0$, followed by a series of cutoff disclosures. Such an information policy is an extension of static disclosure (Figure 8a), where information is not only released at time zero, but also gradually disclosed over time. In Appendix B.3 below, we refer to a specific member within this class of dynamic disclosure policies as initial & gradual disclosure with cutoff quality $q_0$, IGD($q_0$), and show in Proposition B.2 that when $r_p$ is slightly larger than $r$, an optimal information policy takes the form of IGD($q^{**}$) for some optimally chosen $q^{**}$.

Figure 8d plots an information policy of a series of delayed cutoff disclosures starting at time $t$. Analogously, such an information policy can be viewed as an extension of a one-shot delayed disclosure at time...
(a) Static disclosure with cutoff quality $q^*$. (b) One-shot delayed disclosure at time $t$. (c) Initial disclosure with cutoff quality $q_0$, followed by a series of gradual disclosures. (d) Delayed gradual disclosure starting at time $t$.

Figure 8: Examples of cutoff disclosures.

$t$ (Figure 8b), where no information is released at any time before $t$, and information is gradually disclosed at and after time $t$. In Appendix B.4 below, we refer to a specific member within this class of delayed disclosure policies as delayed gradual disclosure starting at time $t$, DGD($t$), and show in Proposition B.3 that when $r_p$ is much larger than $r$, an optimal information policy takes the form of DGD($\tilde{t}$) for some optimally chosen $\tilde{t}$.

### B.3 Initial & Gradual Disclosure

Denote by $u(q) \equiv [-v(q)]/w(q)$ the ratio between the agent’s loss and the principal’s gain from any individually irrational task $q \in [0, \bar{q}]$. Since the principal’s bonus $b(q)$ is assumed to be nondecreasing, so is $w(q) = \frac{b(q)}{r_p + \lambda}$. In addition, because the denominator $[-v(q)]$ drops from the positive value $[-v(0)]$ to zero as $q$ rises from zero to $\bar{q}$, it follows that the ratio $u(q)$ also drops from the positive value $u(0)$ to zero.

The definition of initial & gradual disclosure with cutoff quality $q_0$, IGD($q_0$) is as follows.

**Definition B.3** (Initial & gradual disclosure). IGD($q_0$) denotes the information policy of initial & gradual
disclosure with cutoff quality \(q_0\), specified by the following function of cutoff quality \(\tilde{q}_{\text{IGD}(q_0)}\):

\[
\tilde{q}_{\text{IGD}(q_0)}(s) = u^{-1} \left( u(q_0) e^{-(r_p-r)s} \right),
\]

where \(u^{-1}(\cdot)\) denotes the inverse function of \(u\), and \(q_0 \in [0, q]\).

As \(s\) rises from zero to infinity, \(u(q_0) e^{-(r_p-r)s}\) drops from the positive value \(u(q_0)\) to zero. Therefore, the cutoff quality specified by IGD \((q_0)\) rises from \(u^{-1}(u(q_0)) = q_0\) to \(u^{-1}(0) = \tilde{q}\), just like the pattern shown in Figure 8c.

According to IGD \((q_0)\), at each instant \(s \geq 0\) if the task is still not completed, the agent shirks when he is informed that the task quality \(q < \tilde{q}_{\text{IGD}(q_0)}(s)\), and continues otherwise. Thus, the shirk time induced by IGD \((q_0)\), \(\tau_{\text{IGD}(q_0)}\), is determined by inverting the function of cutoff quality \(\tilde{q}_{\text{IGD}(q_0)}(s)\), leading to

\[
\tau_{\text{IGD}(q_0)}(q) = \begin{cases} 
0, & q < q_0, \\
\frac{1}{r_p-r} \log \left( \frac{u(q_0)}{u(q)} \right), & q_0 \leq q < \tilde{q}, \\
\infty, & q \geq \tilde{q}.
\end{cases}
\] (B.2)

Note that if the principal is to set \(q_0 = q^*\), the cutoff quality in KG, she will disclose strictly more information compared with KG. Thus, the agent’s ex ante expected payoff from IGD \((q^*)\) is strictly positive. Lemma B.1 indicates that the principal can implement the associated shirk time \(\tau_{\text{IGD}(q^*)}\), but doing so gives the agent a positive surplus. The principal can improve within the family of IGD \((q_0)\) by lowering the initial cutoff quality \(q_0\), but the lower bound of \(q_0\) is zero. Therefore, the analysis is divided into two cases, depending on whether IGD \((q^*)\) is individually rational.

If IGD\((0)\) is not individually rational, by continuity, the principal can choose an optimal initial cutoff quality \(q^{**} \in (0, q^*)\) such that the value of the agent from IGD \((q^{**})\) is exactly zero.

**Definition B.4** (Optimal initial & gradual disclosure). OIGD denotes the information policy of optimal initial & gradual disclosure, specified by the following function of cutoff quality \(\tilde{q}_{\text{OIGD}}\):

\[
\tilde{q}_{\text{OIGD}}(s) = \tilde{q}_{\text{IGD}(q^{**})}(s) = u^{-1}(u(q^{**}) e^{-(r_p-r)s}),
\]

where \(u^{-1}(\cdot)\) denotes the inverse function of \(u\), and \(q^{**} \in (0, \tilde{q})\) is chosen such that the agent’s ex ante expected payoff equals zero.

We show in Proposition B.2 that for impatient principal \((r_p > r)\), if IGD\((0)\) is not individually rational, OIGD is optimal among all dynamic information policies.

**Proposition B.2.** If \(r_p > r\) and IGD\((0)\) is not individually rational, then OIGD is optimal among all dynamic information policies.

The relationship between the information policies KG and OIGD is shown in Figure 9, which corresponds to the case where \(r_p\) is slightly larger than \(r\), and OIGD is optimal.

Consider a comparative statics in the principal’s discount rate \(r_p\). In the limit case, as \(r_p\) approaches \(r\) from above, the cutoff quality of the initial disclosure \(q^{**}\) approaches \(q^*\) (the one in KG) from below. Moreover, for any \(q \in (q^*, \tilde{q})\), the shirk time \(\tau_{\text{OIGD}}(q) = \frac{1}{r_p-r} \log \left( \frac{u(q^{**})}{u(q)} \right)\) tends to infinity. Therefore, the optimal dynamic information policy OIGD converges to the optimal static one, KG. This exercise shows the continuity of the optimal information policy, as a function of the principal’s discount rate \(r_p\).
B.4 Delayed Gradual Disclosure

When IGD(0) is individually rational, Lemma B.1 indicates that the principal can implement the associated shirk time \( \tau_{\text{IGD}(0)} \). However, doing so may bring a strictly positive ex ante payoff to the agent. In this case, the principal can further improve by delaying the time of the first disclosure. Suppose that the principal delays any disclosure until time \( t \), and then follow IGD(0). This gives rise to the information policy: *delayed gradual disclosure* starting at time \( t \), DGD(\( t \)) for any delay time \( t \geq 0 \). The formal definition is as follows:

**Definition B.5** (Delayed gradual disclosure). DGD(\( t \)) denotes the information policy of delayed gradual disclosure starting at time \( t \), specified by the following function of cutoff quality \( \tilde{q} \):

\[
\tilde{q}_{\text{DGD}(t)}(s) = \begin{cases} 
0, & s < t, \\
 u^{-1}(u(0) e^{-(r_p - r)(s-t)}), & s \geq t,
\end{cases}
\]

where \( u^{-1}(\cdot) \) denotes the inverse function of \( u \).

For \( s \) smaller than \( t \), the cutoff quality specified by DGD(\( t \)) is zero, and hence the agent is always asked to continue prior to time \( t \). As \( s \) rises from \( t \) to infinity, \( u(0) e^{-(r_p - r)(s-t)} \) drops from the positive value \( u(0) \) to zero. Therefore, the cutoff quality specified by DGD(\( t \)) rises from \( u^{-1}(u(0)) = 0 \) to \( u^{-1}(0) = \tilde{q} \), just like the pattern shown in Figure 8d.

According to DGD(\( t \)), the agent will not shirk prior to time \( t \), and at each instant \( s \geq t \) if the task is still not completed, he shirks when he is informed that the task quality \( q < \tilde{q}_{\text{DGD}(t)}(s) \), and continues otherwise. Thus, the shirk time induced by DGD(\( t \)), \( \tau_{\text{DGD}(t)} \), is determined by inverting the function of cutoff quality \( \tilde{q}_{\text{DGD}(t)}(s) \), leading to

\[
\tau_{\text{DGD}(t)}(q) = \begin{dcases} 
 t + \frac{1}{r_p - r} \log \left( \frac{u(0)}{u(q)} \right), & q < \tilde{q}, \\
 \infty, & q \geq \tilde{q}.
\end{dcases}
\]

In particular, as \( q \) rises from 0 to \( \tilde{q} \), the agent’s shirk time \( \tau_{\text{DGD}(t)}(q) \) continuously rises from \( t \) to infinity.

Denote by \( \tilde{t} \) the maximum time of disclosure, i.e., the latest time such that IGD(\( \tilde{t} \)) is individually rational for the agent. Since we assume that the agent needs some incentive to start working, and the agent’s expected payoff changes continuously in the time of the first disclosure, \( \tilde{t} \) is unique and finite.
Definition B.6 (Maximum delayed gradual disclosure). MDGD denotes the information policy of maximum delayed gradual disclosure, specified by the following function of cutoff quality $q^{MDGD}$:

$$q^{MDGD}(s) = q^{DGD}(i) = \begin{cases} 
0, & s < \tilde{i}, \\
 u^{-1}(u(0)e^{-(r_p-r)(s-\tilde{i})}), & s \geq \tilde{i}, 
\end{cases}$$

where $u^{-1}(\cdot)$ denotes the inverse function of $u$, and $\tilde{i} \geq 0$ is chosen such that the agent’s ex ante expected payoff equals zero.

We show in Proposition B.3 that for impatient principal ($r_p > r$), when IGD(0) is individually rational, MDGD is optimal among all dynamic information policies.

**Proposition B.3.** If $r_p > r$ and IGD(0) is individually rational, then MDGD is optimal among all dynamic information policies.

The relationship among the information policies KG, OIGD and MDGD is shown in Figure 10.

**Figure 10:** Relationship among KG, OIGD and MDGD.

10a is a reproduction of Figure 9, showing the situation where $r_p$ is slightly larger than $r$, and OIGD is optimal among all information policies. Continue the comparative statics in the principal’s discount rate $r_p$ at the end of Appendix B.3, but this time it is in the opposite direction, i.e., assume that $r_p$ rises. Equation (B.2) implies that, holding fixed the initial cutoff quality $q^{**}$, the shirk time $\tau^{OIGD}(q) = \frac{1}{r_p-r} \log \left( \frac{u(q^{**})}{u(q)} \right)$ specified for any $q \in (q^{**}, \bar{q})$ strictly reduces, and hence OIGD becomes more appealing to the agent. Therefore, if the principal were to keep $q^{**}$ at its status quo level, the agent would have a positive ex ante value, and the principal can improve by lowering the value of $q^{**}$. This implies that the optimal $q^{**}$ is strictly decreasing in $r_p$. Eventually, when $r_p$ is large enough, the optimal $q^{**}$ reaches zero and cannot continue to decrease. After that point, the principal begins to use the other method of improving information policy, namely delayed disclosure. This pattern is displayed in Figure 10b, showing that MDGD is optimal when $r_p$ is much larger than $r$. The notable difference from the binary case (Section 3) is that even in the presence of delay, the optimal information policy still involves gradual disclosure rather than a one-shot delayed disclosure. The principal customizes the disclosure times for different realized quality $q$ and strictly benefits from this design.

**B.5 Proofs of Results in Appendix B**

As discussed at the start of this appendix, because we assume that the task quality follows a full-support and atomless distribution, it is without loss to assume that each $\tau(q)$ is pure; that is, $\tau(q)$ assigns probability one...
Lemma B.2. A shirk schedule \( \tau \) solves the principal’s program (B.5) if and only if (i) \( \tau (q) \equiv \infty \) on \([\bar{q}, \infty)\), and (ii) its restriction to \((0, \bar{q})\) solves the following program

\[
\begin{align*}
\min_{\tau :(0,\bar{q})\rightarrow [0,\infty]} & \quad P(\tau) \equiv \int_{0}^{\bar{q}} e^{-(r+\lambda)\tau(q)}w(q) \, dF(q) \\
\text{s.t.} & \quad C(\tau) \equiv \int_{0}^{\bar{q}} e^{-(r+\lambda)\tau(q)}[-v(q)] \, dF(q) \geq C_3,
\end{align*}
\]

where \( C_3 \equiv -\int_{0}^{\infty} v(q) \, dF(q) > 0 \). In particular, it is infeasible to have \( \tau (q) = \infty \) for all \( q \in (0, \bar{q}) \).

Proof of Lemma B.2. We proceed with the following two steps.

Step 1 Any solution to the relaxed program (B.5) satisfies \( \tau (q) \equiv \infty \) on \([\bar{q}, \infty)\), i.e., the agent completes any individually rational task.

For any shirk schedule \( \tau \), consider the associated shirk time \( \tilde{\tau} \) that coincides with \( \tau \) when the task quality \( q < \bar{q} \), and \( \tilde{\tau} (q) = \infty \) for all \( q \geq \bar{q} \). We will show that (i) \( V(\tilde{\tau}) \geq V(\tau) \), and (ii) \( W(\tilde{\tau}) \geq W(\tau) \) where equality holds if and only if and only if \( \tau (q) \equiv \infty \) on \([\bar{q}, \infty)\).

From equation (B.4) and the fact that \( v(\tau(q), q) \leq v(q) \) for \( q \geq \bar{q} \),

\[
\begin{align*}
V(\tilde{\tau}) &= \int_{0}^{\bar{q}} v(\tau(q), q) \, dF(q) + \int_{\bar{q}}^{\infty} v(q) \, dF(q) \\
&\geq \int_{0}^{\bar{q}} v(\tau(q), q) \, dF(q) + \int_{\bar{q}}^{\infty} v(q) \, dF(q) = V(\tau).
\end{align*}
\]

to a deterministic shirk time.

The expected payoffs to the principal and agent from shirk schedule \( \tau \) are, respectively,

\[
\begin{align*}
W(\tau) &= \mathbb{E}_\sigma [w(\tau, q)] = \int_{0}^{\infty} w(\tau(q), q) \, dF(q), \quad (B.3) \\
V(\tau) &= \mathbb{E}_\sigma [v(\tau, q)] = \int_{0}^{\infty} v(\tau(q), q) \, dF(q). \quad (B.4)
\end{align*}
\]

A necessary condition for \( \tau \) to be induced by some information policy is the individual rationality constraint \( V(\tau) \geq 0 \), since the agent can ignore all information provided by the principal. Although this condition is not sufficient, it turns out to be enough to characterize the optimal information policy. The following program can be seen as a relaxation of the principal’s problem of choosing the optimal information policy:

\[
\begin{align*}
\max_{\tau :(0,\infty)\rightarrow [0,\infty]} & \quad W(\tau) \\
\text{s.t.} & \quad V(\tau) \geq 0,
\end{align*}
\]

where the principal is choosing the optimal shirk schedule \( \tau \) subject only to the constraint of individual rationality. If the solution to the above program (B.5) turns out to be induced by some particular information policy, then we can conclude that the candidate information policy is optimal. All proofs for Appendix B follow this rationale. We begin with a reformulation of the principal’s program.
Similarly, from equation (B.3) and the fact that \( w(\tau(q), q) \leq w(q) \),

\[
W(\tilde{\tau}) = \int_0^{q_0} w(\tau(q), q) \, dF(q) + \int_{q_0}^{\infty} w(q) \, dF(q)
\]

\[
\geq \int_0^{q_0} w(\tau(q), q) \, dF(q) + \int_{q_0}^{\infty} w(\tau(q), q) \, dF(q) = W(\tau),
\]

and by the full support assumption, equality holds if and only if \( \tau(q) \overset{a.e.}{=} \infty \) on \([q, \infty)\).

This shows that any solution to the relaxed program (B.5) satisfies \( \tau(q) \overset{a.e.}{=} \infty \) on \([q, \infty)\).

**Step 2** Reformulation of the principal’s relaxed program.

From Step 1, the principal’s relaxed problem reduces to

\[
\max_{\tau: (0, q) \to [0, \infty]} \int_0^{q_0} w(\tau(q), q) \, dF(q) + \int_{q_0}^{\infty} w(q) \, dF(q)
\]

s.t. \( \int_0^{q_0} v(\tau(q), q) \, dF(q) + \int_{q_0}^{\infty} v(q) \, dF(q) \geq 0. \)

Note that \( v(t, q) = (1 - e^{-(r+\lambda)t}) v(q) \), so we can rewrite the constraint as

\[
\int_0^{q_0} e^{-(r+\lambda)t} [-v(q)] \, dF(q) \geq -\int_{q_0}^{\infty} v(q) \, dF(q) \equiv C_3.
\]

Similarly, \( w(t, q) = (1 - e^{-(r+\lambda)t}) w(q) \), so we can rewrite the objective as

\[
\int_0^{\infty} w(q) \, dF(q) - \int_0^{q_0} e^{-(r+\lambda)t} w(q) \, dF(q).
\]

Thus, maximizing the objective is equivalent to minimizing the integral \( \int_0^{q_0} e^{-(r+\lambda)t} w(q) \, dF(q) \). Therefore, we can further rewrite the principal’s program as the form in (B.6). \( \square \)

**Proof of Proposition B.1.** Suppose \( r_p \leq r \). It suffices to show that \( \tau^{KG} \) solves the principal’s relaxed program (B.5). Lemma B.2 above implies that it is equivalent to show that (i) \( \tau^{KG}(q) \overset{a.e.}{=} \infty \) on \([q, \infty)\), and (ii) its restriction to \((0, q)\) solves program (B.6).

Since \( q' < q \), it follows from the definition of \( \tau^{KG} \),

\[
\tau^{KG}(q) = \begin{cases} 0, & q < q^*; \\ \infty, & q \geq q^*; \end{cases}
\]

that \( \tau^{KG}(q) = \infty \) on \([q, \infty)\). Moreover,

\[
P(\tau^{KG}) = \int_0^{q'} w(q) \, dF(q).
\]

Now we show that \( P(\tau) \geq P(\tau^{KG}) \) for any \( \tau \) that satisfies the constraint in program (B.6). Note that
\[ e^{-(r+\lambda)\tau(y)} \leq 1 \text{ implies that} \]
\[ \int_0^\tau [-v(y)] \; dF(q) \geq \int_0^\tau e^{-(r+\lambda)\tau(y)} [-v(y)] \; dF(q). \]

It follows from the intermediate value theorem that there exists \( y \in [0, \tau] \) such that
\[ \int_0^y [-v(y)] \; dF(q) = C(\tau) = \int_0^\tau e^{-(r+\lambda)\tau(y)} [-v(y)] \; dF(q). \]  \hfill (B.8)

If \( y = \tau \), equation (B.8) implies that \( \tau(y) \leq 0 \) on \([0, \tau]\), and thus
\[ P(\tau) = \int_0^\tau w(q) \; dF(q) \geq \int_0^\tau w(q) \; dF(q) = P(\tau). \]

and the proof is complete.

Henceforth, we assume that \( y < \tau \), so in particular \([-v(y)] > 0\). We derive the following lower bound on the principal’s objective \( P(\tau) \), as an implication of equation (B.8):
\[ \int_0^y w(q) \; dF(q) \leq P(\tau) = \int_0^\tau e^{-(r+\lambda)\tau(y)} w(q) \; dF(q). \]  \hfill (B.9)

Since \( r_p \leq r \), it suffices to show that
\[ \int_0^y w(q) \; dF(q) \leq \int_0^\tau w(q) \; dF(q). \]

which, after rearranging, is equivalent to
\[ \int_0^y \left(1 - e^{-(r+\lambda)\tau(y)}\right) w(q) \; dF(q) \leq \int_y^\tau e^{-(r+\lambda)\tau(y)} w(q) \; dF(q). \]  \hfill (B.10)

Since the principal’s bonus \( b(q) \) is assumed to be nondecreasing, so is \( w(q) = \frac{b(q)}{r_p + \lambda} \). Thus,
\[
\text{LHS of (B.10)} \leq w(y) \int_0^y \left(1 - e^{-(r+\lambda)\tau(y)}\right) \; dF(q),
\]
and
\[
\text{RHS of (B.10)} \geq w(y) \int_y^\tau e^{-(r+\lambda)\tau(y)} \; dF(q).
\]

Therefore, in order to prove equation (B.9) (and its equivalence (B.10)), it suffices to show
\[ \int_0^y \left(1 - e^{-(r+\lambda)\tau(y)}\right) \; dF(q) \leq \int_y^\tau e^{-(r+\lambda)\tau(y)} \; dF(q). \]  \hfill (B.11)

Similarly, rearranging terms in equation (B.8), we get
\[ \int_0^y \left(1 - e^{-(r+\lambda)\tau(y)}\right) [-v(y)] \; dF(q) = \int_y^\tau e^{-(r+\lambda)\tau(y)} [-v(y)] \; dF(q). \]  \hfill (B.12)
Since $[-v(q)]$ is positive and strictly decreasing for $q \leq \bar{q}$, it follows that

$$\text{LHS of (B.12)} \geq [-v(y)] \int_0^y \left(1 - e^{-(r+\lambda)\tau(q)}\right) dF(q),$$

and

$$\text{RHS of (B.12)} \leq [-v(y)] \int_y^\infty e^{-(r+\lambda)\tau(q)} dF(q).$$

Cancelling the positive constant $[-v(y)]$ out from both sides and rearranging terms, we precisely get (B.11). Therefore, we establish the lower bound on the principal’s objective $P(\tau)$, equation (B.9), as desired.

We conclude by proving that the lower bound in equation (B.9) is larger than $P(\tau_{ KG})$. Comparing equations (B.7) and (B.9), we see that this is equivalent to $y \geq q^*$. Since $\tau$ satisfies the constraint in program (B.6), together with the definition of $y$, equation (B.8), we obtain

$$\int_0^y [-v(q)] dF(q) = C(\tau) \geq C_3 = -\int_0^\infty v(q) dF(q),$$

$$\Rightarrow \int_y^\infty v(q) dF(q) \geq 0.$$

It then follows from the definition of $q^*$, equation (B.1), that $y \geq q^*$, as desired.

This establishes the optimality of KG.

\[\square\]

**Proof of Lemma B.1.** Denote by $\tau$ the agent’s shirk schedule according to $\sigma$. That is, at each instant $s \geq 0$ if the task is still not completed, he shirks when he is informed that the task quality $q < \bar{q}(s)$, and continues otherwise. Thus, $\tau(q)$ is determined by inverting the function of cutoff quality $\bar{q}(s)$, i.e.,

$$q = \bar{q}(s) \iff s = \tau(q).$$

If the individual rationality constraint is violated, i.e., $V(\tau) < 0$, the agent will ignore all information provided by the principal and shirk immediately. Thus, $V(\tau) \geq 0$ is necessary for the agent to follow $\sigma$.

Now suppose $V(\tau) \geq 0$. We show that the agent’s best response is to follow $\sigma$ at each time $s \geq 0$ when the task is not completed. If the agent is informed that the task has relatively low quality ($q < \bar{q}(s)$), his expected continuation value is bounded above by $v(\bar{q}(s)) \leq v(\bar{q}) = 0$, so it is optimal for him to shirk. If the agent is informed that the task has relatively high quality ($q \geq \bar{q}(s)$), his expected continuation value promised by DGD($t$) is given by

$$\frac{1}{1 - F(\bar{q}(s))} \int_{\bar{q}(s)}^\infty v(\tau(q) - s, q) dF(q).$$

We need to show that this value is nonnegative, so that it is optimal for the agent to continue when he is informed to do so.

Define

$$h(s) \equiv \int_{\bar{q}(s)}^\infty v(\tau(q) - s, q) dF(q).$$

Since $\tau$ is the inverse of $\bar{q}$, it follows that $\tau(\bar{q}(s)) = s$. Therefore, Leibniz integral rule implies that

$$h'(s) = \int_{\bar{q}(s)}^\infty \left[-\frac{\partial v}{\partial \tau}(\tau(q) - s, q)\right] dF(q).$$
Since \( v(t, q) = \left(1 - e^{-(r + \lambda)t}\right) v(q) \), we have \( \frac{\partial v}{\partial t}(t, q) = (r + \lambda)e^{-(r + \lambda)t}v(q) \). For \( q \leq \bar{q} \), \( v(q) \leq 0 \) and hence \( -\frac{\partial v}{\partial t} (\tau (q) - s, q) \geq 0 \). For \( q > \bar{q} \), since the agent is never asked to shirk \( (\bar{q} (s) \in [0, \bar{q}]) \), \( \tau (q) = \infty \), and hence \( -\frac{\partial v}{\partial t} (\tau (q) - s, q) = 0 \). Combining the two parts, we have \( -\frac{\partial v}{\partial t} (\tau (q) - s, q) \geq 0 \) for all \( q \), and thus \( h' (s) \geq 0 \). Note that \( h (0) = V (\tau) \geq 0 \). It follows that \( h (s) \geq 0 \) for all \( s \geq 0 \), completing the agent will follow \( \sigma \) at each time \( s \geq 0 \). \( \square \)

For the case where the principal is less patient than the agent, \( r_p > r \), we can further simplify the principal’s program (B.5), according to the following lemma.

**Lemma B.3.** If \( r_p > r \), any solution \( \tau \) to the principal’s program (B.5) takes the following form:

\[
\tau (q) = \begin{cases} 
0, & q < y, \\
\frac{1}{r + \lambda} \log (z_1/z_2) - \frac{1}{r_p - r} \log u(q), & y \leq q < \bar{q}, \\
\infty, & q \geq \bar{q}.
\end{cases}
\] (B.13)

As a result, the program (B.6) is equivalent to the following program of choosing three constants \((y, z_1, z_2)\):

\[
\min_{y, z_1, z_2} \int_0^y w(q) \, dF(q) + (z_1) \frac{r_p + 1}{r_p} \int_0^y \frac{1}{r - \lambda} \log (z_1/z_2) \, dF(q),
\]

\[
\text{s. t. } \int_0^y [-v(q)] \, dF(q) + z_1 \geq C_3,
\]

\[
z_2 = \int_y^{\bar{q}} [-v(q)] \frac{r_p + 1}{r_p} w(q) \frac{1}{r - \lambda} \, dF(q),
\]

\[
0 \leq y < \bar{q}, \quad z_1 \leq z_2 u(y) \frac{1}{r - \lambda},
\]

where \( C_3 \equiv -\int_0^{\bar{q}} v(q) \, dF(q) > 0 \).

**Proof of Lemma B.3.** Fix a solution \( \tau \) to program (B.6). Denote by \( y \equiv \inf \{ q \in [0, \bar{q}] : \tau (q) > 0 \} \). We have

\[
P(\tau) = \int_0^y w(q) \, dF(q) + \int_y^{\bar{q}} e^{-(r_p + \lambda)\tau(q)} w(q) \, dF(q),
\]

\[
C(\tau) = \int_0^y [-v(q)] \, dF(q) + \int_y^{\bar{q}} e^{-(r_p + \lambda)\tau(q)} [-v(q)] \, dF(q) \geq C_3.
\]

Let

\[
z_1 \equiv \int_y^{\bar{q}} e^{-(r_p + \lambda)\tau(q)} [-v(q)] \, dF(q) \quad \text{and} \quad z_2 \equiv \int_y^{\bar{q}} [-v(q)] \frac{r_p + 1}{r_p} \log (z_1/z_2) \, dF(q).
\]

We first establish that over \( q \in [y, \bar{q}] \),

\[
e^{-(r_p + \lambda)\tau(q)} \frac{r_p + 1}{r_p} \log (z_1/z_2) = (z_1/z_2) \frac{r_p + 1}{r_p} [-v(q)] \frac{r_p + 1}{r_p} \log (z_1/z_2) \, dF(q).
\] (B.15)
Hölder’s inequality\(^{16}\) implies that
\[
\left[ \int_y^\bar{y} e^{-(r_p + \lambda)q} w(q) \, dF(q) \right]^\frac{r_p + 1}{p + 1} (z_2)^\frac{r_p - r}{p - r} \\
= \left[ \int_y^\bar{y} \left( e^{-(r_p + \lambda)q} w(q) \right)^\frac{r_p + 1}{p + 1} \, dF(q) \right]^{\frac{p}{p - 1}} \left( \int_y^\bar{y} \left( \left[-v(q)\right] w(q)^{-\frac{r_p + 1}{p - r}} \right)^\frac{p}{p - 1} \, dF(q) \right)^\frac{p - r}{p}
\geq \int_y^\bar{y} e^{-(r_p + \lambda)q} \left[-v(q)\right] \, dF(q) = z_1,
\] (B.16)
where equality holds if and only if there exists constant \(\gamma > 0\) such that over \(q \in [y, \bar{y}]\),
\[
e^{-(r_p + \lambda)\gamma\left[-v(q)\right] w(q)^{\frac{r_p + 1}{p - r}}} = \gamma \left(\left[-v(q)\right] w(q)^{\frac{r_p + 1}{p - r}}\right) = \gamma u(q)^{\frac{r_p + 1}{p - r}}.
\] (B.17)
Raising both sides of equation (B.16) to the power of \(\frac{r_p + 1}{r + \lambda}\), we obtain
\[
\int_y^\bar{y} e^{-(r_p + \lambda)\tau(q)} w(q) \, dF(q) \geq \left\{ \frac{z_1}{(z_2)^{\frac{p}{p - 1}}} \right\}^{\frac{r_p + 1}{p - 1}} (z_2)^{\frac{r_p - r}{p - r}} = (z_1)^{\frac{r_p + 1}{p - 1}} (z_2)^{\frac{r_p - r}{p - r}},
\] (B.18)
and equality holds if and only if \(\tau(q)\) takes the form in equation (B.17). We can pin down the constant \(\gamma\) by substituting equation (B.17) into (B.18), giving
\[
(z_1)^{\frac{r_p + 1}{p - 1}} (z_2)^{\frac{r_p - r}{p - r}} = \int_y^\bar{y} \gamma \left(\left[-v(q)\right] w(q)^{\frac{r_p + 1}{p - r}}\right) \, dF(q) = \gamma z_2 \quad \Rightarrow \quad \gamma = (z_1/z_2)^{\frac{r_p + 1}{p - 1}},
\]
precisely the one given by equation (B.15). Taking logarithm on both sides of equation (B.15), we see that any solution to program (B.6) takes the form\(^{17}\)
\[
\tau(q) = \begin{cases} 0, & \text{if } q < y, \\ -\frac{1}{r + \lambda} \log (z_1/z_2) - \frac{1}{r_p - r} \log u(q), & \text{if } y \leq q < \bar{y}. \end{cases}
\]
for some constants \((y, z_1, z_2)\). This shows equation (B.13). Note that \(\tau(q)\) cannot be strictly negative, which

\(^{16}\)Here we apply Hölder’s inequality
\[
\left( \int_S |f|^p \, dv \right)^\frac{1}{p} \left( \int_S |g|^q \, dv \right)^\frac{1}{q} \geq \int_S |fg| \, dv
\]
to the measure space \((S, \mathcal{B}, v)\), where \(S = [y, \bar{y}], \mathcal{B}\) is the Borel \(\sigma\)-algebra on \(S\), \(v([q, q']) = F(q') - F(q)\), \(\rho = \frac{r_p + 1}{p} > 1, \rho' = \frac{r_p + 1}{p - r} > 1\) are such that \(\frac{1}{p} + \frac{1}{\rho'} = 1\), \(f(q) = e^{-(r_p + \lambda)q} w(q)^{\frac{r_p + 1}{p - r}}\), \(g(q) = \left[-v(q)\right] w(q)^{-\frac{r_p + 1}{p - r}}\).

\(^{17}\)Indeed, if equation (B.15) does not hold, we can consider an alternative shirk schedule \(\bar{\tau}\) given by
\[
\bar{\tau}(q) = \begin{cases} 0, & \text{if } q < y, \\ -\frac{1}{r + \lambda} \log (z_1/z_2) - \frac{1}{r_p - r} \log u(q), & \text{if } y \leq q < \bar{y}. \end{cases}
\]
By construction, \(\bar{\tau}\) would satisfy \(C(\bar{\tau}) = C(\tau)\) and \(P(\bar{\tau}) < P(\tau)\), contradicting the assumption that \(\tau\) is a solution to program (B.6).
implies that
\[- \frac{1}{r+A} \log(\frac{z_1}{z_2}) - \frac{1}{r_p-r} \log u(y) \geq 0 \iff z_1 \leq z_2 u(y)^{\frac{r+1}{r_p-r}}.
\]

Lemma B.2 implies that we can further simplify the program by plugging back the functional form into program (B.6). The result is exactly given by program (B.14). This completes the proof. □

**Proof of Proposition B.2.** Lemma B.1 indicates that the shirk time \(OIGD\) is induced by the information policy OIGD. It suffices to show that \(OIGD\) solves the principal’s relaxed program (B.5). By definition,

\[OIGD(q) = \begin{cases} 
0, & q < q^{**}, \\
\frac{1}{r_p-r} \log \left( \frac{u(q^{**})}{u(q)} \right), & q^{**} \leq q < \bar{q}, \\
\infty, & q \geq \bar{q}.
\end{cases}\]

This exactly has the form given by equation (B.13), with

\[y^* = q^{**}, \quad z_1^* = z_2^* u(q^{**})^{\frac{r+1}{r_p-r}}, \quad z_2^* = \int_{q^{**}}^{\bar{q}} \left[ -v(q) \right]^{\frac{r+1}{r_p-r}} \frac{dF(q)}{u(q)}.
\]

It follows from Lemma B.3 that we need to show \((y^*, z_1^*, z_2^*)\) solve program (B.14).

Take any constants \((y, z_1, z_2)\) that satisfy the constraint in program (B.14). We proceed with following two steps.

**Step 1** \(y \geq y^* = q^{**}\). That is, for any \(y < q^{**}\), the constraints

\[\int_0^y \left[ -v(q) \right] dF(q) + z_1 \geq C_3
\]

and \(z_1 \leq u(y)^{\frac{r+1}{r_p-r}} \int_y^{\bar{q}} \left[ -v(q) \right]^{\frac{r+1}{r_p-r}} w(q) \frac{dF(q)}{u(q)}\)

cannot hold simultaneously.

This amounts to

\[\int_0^y \left[ -v(q) \right] dF(q) + u(y)^{\frac{r+1}{r_p-r}} \int_y^{\bar{q}} \left[ -v(q) \right]^{\frac{r+1}{r_p-r}} w(q) \frac{dF(q)}{u(q)} < C_3.
\]

Let

\[h_1(y) \equiv \int_0^y \left[ -v(q) \right] dF(q) + u(y)^{\frac{r+1}{r_p-r}} \int_y^{\bar{q}} \left[ -v(q) \right]^{\frac{r+1}{r_p-r}} w(q) \frac{dF(q)}{u(q)}.
\]

We need to show that \(h_1(y) < C_3\) for all \(0 \leq y < q^{**}\).

Leibniz integral rule implies that

\[h'_1(y) = \left[ -v(y) \right] f(y) - u(y)^{\frac{r+1}{r_p-r}} \left[ -v(y) \right]^{\frac{r+1}{r_p-r}} w(y) \frac{dF(q)}{u(q)} f(y) + \frac{d}{dy} \left( u(y)^{\frac{r+1}{r_p-r}} \right) \int_y^{\bar{q}} \left[ -v(q) \right]^{\frac{r+1}{r_p-r}} w(q) \frac{dF(q)}{u(q)}.
\]
By definition $u(y) = [-v(y)] / w(y)$, and hence

$$ u(y)^{-\frac{1}{p+q}} [v(y)]^\frac{p+q}{p+q} w(y)^{-\frac{1}{p+q}} = [-v(y)]^{-\frac{1}{p+q}} [w(y)]^\frac{p+q}{p+q} = [-v(y)]^{-\frac{1}{p+q}} w(y)^{-\frac{1}{p+q}} \tau(y)^{-\frac{1}{p+q}}, $$

implying that the first two terms in $h'(y)$ cancel out. Moreover, $u(y)$ is positive and strictly decreasing in $y$ on $[0, \bar{q})$, implying that

$$ \frac{d}{dy} \left( u(y)^{-\frac{1}{p+q}} \right) > 0. $$

Together, we show that $h'_1(y) > 0$, i.e., $h$ is strictly increasing.

Since IGD(0) is not individually rational, the constants $(y, z_1, z_2)$ given by

$$ y = 0, \quad z_1 = z_2 u(0)^{-\frac{1}{p+q}}, \quad z_2 = \int_0^\bar{q} [-v(q)]^\frac{p+q}{p+q} w(q)^{-\frac{1}{p+q}} dF(q) $$

do not satisfy the constraint in program (B.14); that is

$$ h_1(0) = u(0)^{-\frac{1}{p+q}} \int_0^\bar{q} [-v(q)]^\frac{p+q}{p+q} w(q)^{-\frac{1}{p+q}} dF(q) < C_3. $$

Since $q^*$ is chosen such that $V(\tau^{OIGD}) = 0$, it follows from the equivalence of programs, Lemma B.3, that

$$ h_1(q^*) = \int_0^{q^*} [-v(q)] dF(q) + u(q^*)^{-\frac{1}{p+q}} \int_{q^*}^\bar{q} [-v(q)]^\frac{p+q}{p+q} w(q)^{-\frac{1}{p+q}} dF(q) = C_3. $$

Therefore, for any $y < q^*$, $h_1(y) < C_3$. This confirms that $y \geq y^* = q^*$ for any constants $(y, z_1, z_2)$ that satisfy the constraint in program (B.14).

**Step 2** Any $y > q^*$ is dominated by $y^* = q^*$.

Note that the objective in program (B.14) is strictly decreasing in $z_1$, it is optimal to choose the smallest feasible $z_1$, i.e.,

$$ \int_0^y [-v(q)] dF(q) + z_1(y) = C_3 \quad \Rightarrow \quad z_1(y) = C_3 - \int_0^y [-v(q)] dF(q). $$

Together with

$$ z_2(y) = \int_y^\bar{q} [-v(q)]^\frac{p+q}{p+q} w(q)^{-\frac{1}{p+q}} dF(q), $$

the objective in program (B.14) becomes

$$ h_2(y) \equiv \int_0^y w(q) dF(q) + (z_1(y))^\frac{p+q}{p+q} (z_2(y))^\frac{p+q}{p+q}. $$

We need to show that $h_2(y) \geq h_2(q^*)$ for all $y > q^*$. Indeed, we will prove that $h'_2(y) \geq 0$ for all $y < \bar{q}$.

Leibniz integral rule implies that

$$ z'_1(y) = -[-v(y)] f(y) \quad \text{and} \quad z'_2(y) = -[-v(y)]^\frac{p+q}{p+q} w(y)^{-\frac{1}{p+q}} f(y). $$

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It follows that

\[
\begin{align*}
   h'_2(y) &= w(y) f(y) + (z_1(y))^{\frac{r_p + \lambda}{r_p - r}} (z_2(y))^{\frac{r_p - r}{r_p + \lambda}} \left[ r_p + \lambda \cdot \frac{z'_1(y)}{z_1(y)} + r_p - r \cdot \frac{z'_2(y)}{z_2(y)} \right] \\
   &= w(y) + (z_1(y))^{\frac{r_p + \lambda}{r_p - r}} (z_2(y))^{\frac{r_p - r}{r_p + \lambda}} \left[ -\frac{r_p + \lambda}{r_p - r} \cdot \frac{v(y)}{z_1(y)} + \frac{r_p - r}{r_p + \lambda} \cdot \frac{v(y)}{z_2(y)} \right] f(y) \\
   &= 1 + (z_1(y))^{\frac{r_p + \lambda}{r_p - r}} (z_2(y))^{\frac{r_p - r}{r_p + \lambda}} \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \left[ 1 + \hat{z} \right] w(y) f(y) \\
   &= 1 + \frac{z_1(y)}{z_2(y)} \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \left[ 1 + \frac{r_p + \lambda}{r_p - r} \cdot \frac{z_1(y)}{z_2(y)} \right] w(y) f(y). \quad (B.19)
\end{align*}
\]

Denote by \( \hat{z} \equiv z_1(y)/z_2(y) \). From equation (B.19), \( h'_2(y) \geq 0 \) if and only if

\[
1 + \hat{z} \cdot \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \hat{z} \geq 0.
\]

The constraint of program (B.14) implies that \( \hat{z} \leq u(y)^{-\frac{r_p + \lambda}{r_p - r}} \). Let

\[
h_3(\hat{z}) \equiv 1 + \hat{z} \cdot \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \hat{z}.
\]

We have \( h_3(0) = 1 > 0 \) and \( h_3(u(y)^{-\frac{r_p + \lambda}{r_p - r}}) = 1 - u(y)^{-1} \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] = 0 \). Hence, to prove \( h'_2(y) \geq 0 \), it suffices to show that \( h_3(\hat{z}) \) is nonincreasing over \( \left[ 0, u(y)^{-\frac{r_p + \lambda}{r_p - r}} \right] \), i.e., \( h'_3(\hat{z}) \leq 0 \). We have

\[
\begin{align*}
h'_3(\hat{z}) &= \frac{r_p - r}{r_p + \lambda} \cdot \hat{z}^{-\frac{r_p + \lambda}{r_p - r} - 1} \cdot \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \hat{z} + \hat{z}^{-\frac{r_p + \lambda}{r_p - r}} \cdot \frac{r_p - r}{r_p + \lambda} \cdot u(y) \hat{z} \\
   &= \frac{r_p - r}{r_p + \lambda} \cdot \hat{z}^{-\frac{r_p + \lambda}{r_p - r} - 1} \cdot \left[ -\frac{r_p + \lambda}{r_p - r} \cdot u(y) + \frac{r_p - r}{r_p + \lambda} \cdot u(y) \right] \hat{z} + \hat{z}^{-\frac{r_p + \lambda}{r_p - r}} \cdot \frac{r_p - r}{r_p + \lambda} \cdot u(y) \hat{z} \\
   &= \frac{r_p - r}{r_p + \lambda} \cdot \hat{z}^{-\frac{r_p + \lambda}{r_p - r} - 1} \cdot \left[ -u(y) + u(y) \hat{z}^{-\frac{r_p + \lambda}{r_p - r}} \right] \leq 0,
\end{align*}
\]

as desired. This completes the proof that the objective in program (B.14) is strictly increasing in \( y \) for \( y \geq y^* = \hat{q}^* \).

Combining the two steps above, we conclude that \( (y^*, z_1^*, z_2^*) \) solve program (B.14).

\[\square\]

**Proof of Proposition B.3.** Lemma B.1 indicates that the shirk time \( T_{\text{MDGD}} \) is induced by the information policy MDGD. It suffices to show that \( T_{\text{MDGD}} \) solves the principal’s relaxed program (B.5). By definition,

\[
T_{\text{MDGD}}(q) = \begin{cases} 
\hat{t} + \frac{1}{r_p - r} \log \left( \frac{u(0)}{u(q)} \right), & q < \hat{q}, \\
\infty, & q \geq \hat{q}.
\end{cases}
\]
This exactly has the form given by equation (B.13), with

\[ y^* = 0, \quad z_1^* = z_2^* e^{-(\rho + \lambda)\tilde{t}} u(0) \frac{\text{e}^{\tilde{y} + \lambda}}{\rho + \tilde{y} + \lambda}, \quad z_2^* = \int_0^{\tilde{q}} \left[-v(q)\right] \frac{\text{e}^{\tilde{q} + \lambda}}{\rho + \tilde{q} + \lambda} \, dF(q). \]

Since \( \tilde{t} \geq 0 \) is chosen such that \( V(\tau^\text{MDGD}) = 0 \), it follows from the equivalence of two programs, Lemma B.3, that

\[ \int_0^{y^*} \left[-v(q)\right] \, dF(q) + z_1^* = C_3 \quad \Rightarrow \quad z_1^* = C_3. \]

It remains to show that the constants \( (y^*, z_1^*, z_2^*) \) solve program (B.14).

Take any constants \( (y, z_1, z_2) \) that satisfy the constraint in program (B.14). We first show that any \( y > 0 \) is dominated by \( y^* = 0 \). Note that the objective in program (B.14) is strictly decreasing in \( z_1 \), it is optimal to choose the smallest feasible \( z_1 \), i.e.,

\[ \int_0^{y} \left[-v(q)\right] \, dF(q) + z_1(y) = C_3 \quad \Rightarrow \quad z_1(y) = C_3 - \int_0^{y} \left[-v(q)\right] \, dF(q). \]

Together with

\[ z_2(y) = \int_y^{\tilde{q}} \left[-v(q)\right] \frac{\text{e}^{\tilde{q} + \lambda}}{\rho + \tilde{q} + \lambda} w(q) \, dF(q), \]

the objective in program (B.14) becomes

\[ h_2(y) = \int_0^{y} w(q) \, dF(q) + (z_1(y))^{\frac{\rho + \lambda}{\rho + \tilde{y} + \lambda}} (z_2(y))^{\frac{\rho + \lambda}{\rho + \tilde{q} + \lambda}}. \]

This is exactly the same expression as in Step 2 in the proof of Proposition B.2! Therefore, we have \( h_2(y) \geq h_2(0) \) for all \( y > 0 \), and hence \( y^* = 0 \) is optimal. It follows that

\[ z_1(y^*) = C_3 = z_1^*, \quad z_2(y^*) = \int_0^{\tilde{q}} \left[-v(q)\right] \frac{\text{e}^{\tilde{q} + \lambda}}{\rho + \tilde{q} + \lambda} w(q) \, dF(q) = z_2^*. \]

This completes the proof. \( \square \)
C A Framework with Transfers

In this appendix, we extend the baseline model in Section 2 to a framework that allows transfers, and show that information disclosure is still valuable for the principal to motivate the agent’s effort. The principal has full commitment power and offers an information/wage contract to the agent.\footnote{One interpretation is that the contracting stage happens before observing the quality of the task. This timing is different from the ones in Maskin and Tirole (1990, 1992), but is consistent with Bayesian persuasion models.} We allow arbitrary dynamic information disclosure and wage payments based on the task quality, but assume that the agent’s effort and task completion are noncontractible.

The main results for this Appendix are Propositions C.1 and C.2, which characterize the optimal contract in the stationary case. Analogous to Section 3, the principal benefits from a dynamic contract if and only if she is less patient than the agent, i.e., $r_p > r$. Moreover, the optimal contracts in different scenarios contain the optimal information policies that arise from the baseline model (e.g., KG and MDD).

Specifically, for a patient principal ($r_p \leq r$), Proposition C.1 shows that the optimal contract depends on whether the total surplus is maximized by completing the low-quality task (i.e., the sign of $w(L) + v(L)$). If $w(L) + v(L) < 0$, then the optimal contract involves an initial full disclosure in order to avoid wasting effort on the low-quality task, thereby saving payments. On the other hand, if $w(L) + v(L) > 0$, then it is indeed efficient to persuade the agent to work on the low-quality task, yet payments alone do not suffice to motivate the agent. The principal achieves her optimal value using the optimal static information policy KG. Regardless of the information policy chosen, it is optimal for the principal to make all payments upfront.

Conversely, for an impatient principal ($r_p > r$), Proposition C.2 shows that the optimal contract makes no payments upfront, and delays wage payments as much as possible. Holding fixed the agent’s present value of payments, the principal is able to choose a wage policy with an arbitrarily small present value from her perspective, as long as her financial constraint is sufficiently relaxed. Consequently, the principal achieves her optimal value using maximum delayed disclosure (MDD) regardless of the sign of $w(L) + v(L)$.

C.1 Model

In addition to the setting in Section 2, assume that both the principal’s and the agent’s payoffs are quasilinear in money. The principal has full commitment power, and offers a contract to the agent before time zero, specifying an information policy, a wage policy, and an upfront payment. The principal observes the realized quality of the task $q$ at time zero. Then she discloses information and pays wages to the agent according to the contract. Crucially, we assume that information disclosure and wage payments can depend on the quality of the task, but the agent’s effort and task completion are noncontractible. This leads to the following definition of a contract.

**Definition C.1 (Contract).** A contract $C = (\sigma, p, M)$ consists of an information policy $\sigma$, a wage policy $p$, and an upfront payment $M$, where

1. $\sigma$ is a signal process $\sigma_t(q) \in \Delta([0, 1])$, which informs the agent whether to continue ($\sigma_t = 1$) or shirk ($\sigma_t = 0$) at any time.

2. $p$ is a stream of (possibly random) wage payments to the agent $p_t(q) \in [0, B]$ at any time, where $B > 0$ is the principal’s flow budget constraint.

3. $M \geq 0$ specifies an upfront payment granted to the agent upon signing the contract.

Before signing the contract, the agent has access to an outside option $v$. After signing the contract, the agent can shirk at any time, and he will not incur flow costs from then on. However, the wage payments are
una
ffected by this decision. Therefore, given a contract C, the principal’s and agent’s expected payoffs are, respectively,

\[ W(C) = \mathbb{E}_C \left[ w(\tau, q) - \int_0^\infty e^{-rt} p_t(q) dt \right] - M, \]  

(C.1)

and

\[ V(C) = \mathbb{E}_C \left[ v(\tau, q) + \int_0^\infty e^{-rt} p_t(q) dt \right] + M, \]  

(C.2)

where \( \tau \) denotes the agent’s random shirk time induced by the contract C.

The principal’s problem is to maximize expected payoff \( W(C) \) over contracts \( C \) that satisfy the agent’s participation constraint \( V(C) \geq v \). Lemma C.1 below shows that the principal’s optimal value can be attained through contracts involving deterministic wage payments that do not depend on task quality.

**Lemma C.1.** For any contract \( C = (\sigma, p, M) \), there exists an associated contract \( \tilde{C} = (\tilde{\sigma}, \tilde{p}, M) \) satisfying:

1. Wage payments \( \tilde{p} = (\tilde{p}_t)_{t\geq 0} \) are deterministic and do not depend on task quality.
2. The induced shirk time, \( \tau \), is the same under both contracts \( C \) and \( \tilde{C} \).
3. The principal and the agent have the same payoffs under both contracts.

**Proof.** All proofs for this Appendix are in Appendix C.3.

The intuition behind Lemma C.1 is as follows. Since the agent’s effort and task completion are non-contractible, the wage payments only enter into the agent’s participation constraint and do not directly affect the agent’s effort decision. Therefore, information disclosure is still required to motivate the agent. The principal can without loss integrate any information disclosed through wage payments into the information policy.

Henceforth, we focus on the contracts that involve deterministic wage payments that do not depend on task quality.

### C.2 Stationary Case

Retain the stationarity assumption in Section 3, \( \lambda_H = \lambda_L = \lambda \), and suppose that the agent begins with prior \( \mu < \bar{\mu} \) that the task has high quality, so that some form of persuasion is necessary. Moreover, we denote the information policy of *initial full disclosure* by IFD, which provides the strongest incentive to the agent.

**Definition C.2** (Initial full disclosure). IFD denotes the information policy of initial full disclosure, where the principal fully discloses the task quality at time zero. Formally, \( \sigma_0(H) = 1 \), and \( \sigma_0(L) = 0 \).

Assume that the agent’s outside option \( v \) satisfies (i) \( v > \mu v(H) \), (ii) \( v < \mu [v(H) + w(H)] \), and (iii) \( v < B/(r + \lambda) \). The first inequality implies that even if the principal chooses IFD, the total payments for the agent to sign the contract have to be positive. The second inequality ensures that the principal can obtain a strictly positive payoff by contracting with the agent. One such contract is IFD with upfront payment \( M = v - \mu v(H) \). The third inequality ensures that the principal can avoid upfront payments, as long as she chooses the maximum wage policy \( p_t = B \) for all \( t \geq 0 \).

Now we state the main results for this Appendix, Propositions C.1 and C.2, which characterize the optimal contract in the binary case. If \( r_p \leq r \), it is optimal for the principal to make all payments upfront, and the optimal contract depends on the sign of \( w(L) + v(L) \).
Proposition C.1. If \( r_p \leq r \), then the optimal contract depends on the sign of \( w(L) + v(L) \).

1. If \( w(L) + v(L) < 0 \), then \((\sigma, p, M) = (\text{IFD}, 0, \hat{v} - \mu v(H))\) is optimal.

2. If \( w(L) + v(L) \geq 0 \), then \((\sigma, p, M) = (\text{KG}, 0, \hat{v})\) is optimal.

If \( w(L) + v(L) < 0 \), then the principal chooses IFD to save payments. The agent shirks immediately when the task has low quality, so no effort will be wasted. On the other hand, if \( w(L) + v(L) > 0 \), the principal adopts the optimal static information policy KG to motivate the agent to work on the low-quality task, while it is not enough to use payments alone.

Conversely, if \( r_p > r \), then it is optimal for the principal to make no payments upfront, and delay wage payments as much as possible. As \( B \) increases, the principal’s flow budget constraint is relaxed, so she can choose a wage policy with a vanishing present value from her perspective. Therefore, regardless of the sign of \( w(L) + v(L) \), the optimal contract involves MDD as its information policy for a sufficiently large \( B \).

Proposition C.2. If \( r_p > r \), then the following results hold regardless of the sign of \( w(L) + v(L) \).

1. The optimal contract makes no upfront payment \((M = 0)\), no wage payment until some time \( T \), and then pays the maximum wage \( B \) thereafter.

2. For a sufficiently large \( B \), the optimal contract involves MDD as its information policy.

C.3 Proofs of Results in Appendix C

Proof of Lemma C.1. For any contract \( C = (\sigma, p, M) \), denote by \( \overline{p}_t \), the expected wage payment at time \( t \), \( \overline{p}_t = \mathbb{E}_q[p_t(q)] \), and denote by \( \overline{\sigma}_t(q) \in \Delta([0, 1]) \) the signal that combines the information disclosed through \( \sigma_t \) and \( p_t \). Consider the contract \( \overline{C} = (\overline{\sigma}, \overline{p}, M) \). Then the induced shirk time, \( \tau \), is the same under both contracts \( C \) and \( \overline{C} \). Therefore, it follows from the payoff functions, equations (C.1) and (C.2), that the principal and the agent have the same payoffs under both contracts. \( \square \)

Before stating the proofs for Propositions C.1 and C.2, we begin with a lemma that characterize the optimal upfront and wage payments, holding fixed the agent’s present value.

Lemma C.2. Hold fixed the agent’s present value of total payments \((p, M)\) at \( PV \leq \hat{v} \).

1. If \( r_p \leq r \), then it is optimal to make any payments upfront, i.e., \( p = 0 \) and \( M = PV \).

2. If \( r_p > r \), then it is optimal to make no upfront payment \((M = 0)\), no wage payment until some time \( T \), and then pay \( B \) thereafter. That is, the optimal wage policy takes the form

\[
    p_t \overset{a.e.}{=} \begin{cases} 
        0, & t < T, \\
        B, & t \geq T.
    \end{cases}
\]  

(C.3)

Proof of Lemma C.2. We first use integration by parts to simplify the agent’s present value of any wage policy \( p \):

\[
    \mathbb{E}_x \left[ \int_0^x e^{-rt} p_t \, dt \right] = \int_0^\infty \left[ \int_0^x e^{-rt} p_t \, dt \right] \lambda e^{-Ax} \, dx = \int_0^\infty \left[ \int_t^\infty \lambda e^{-Ax} \, dx \right] e^{-rt} p_t \, dt = \int_0^\infty e^{-(r+A)t} p_t \, dt.
\]
That is, due to the Poisson arrival of task completion, the agent’s effective discount rate is \((r + \lambda)\). Similarly, the principal’s has effective discount rate \((r_p + \lambda)\). Holding fixed the agent’s present value of total payments at \(PV\), the principal’s problem is to minimize the present value from her perspective, i.e.,

\[
\min_{p,M} \int_0^\infty e^{-(r_p + \lambda)t} p_t dt + M
\]

\[
\text{s.t. } \int_0^\infty e^{-(r_p + \lambda)t} p_t dt + M = PV.
\]

(C.4)

1. If \(r_p \leq r\), then \(p = 0\) and \(M = PV\) solves the program (C.4).

Note that

\[
\int_0^\infty e^{-(r_p + \lambda)t} p_t dt - \int_0^\infty e^{-(r + \lambda)t} p_t dt = \int_0^\infty \left( e^{-(r_p + \lambda)t} - e^{-(r + \lambda)t} \right) p_t dt \geq 0.
\]

Thus, the principal’s objective in (C.4) is lower bounded by \(PV\). Moreover, this lower bound is achieved by \(p = 0\) and \(M = PV\). This establishes the optimality.

2. If \(r_p > r\), then the solution to program (C.4) features \(M = 0\) and \(p\) with the form in equation (C.3).

Fix an optimal wage policy \(p\). For any \(T > 0\), let

\[
\mathcal{A}(T) \equiv \{ t \in [0, T) : p_t > 0 \} \quad \text{and} \quad \mathcal{A}(T) \equiv \{ t \in [T, \infty) : p_t < B \}.
\]

We proceed with three steps.

**Step 1** For any \(T\), either \(m(\mathcal{A}(T)) = 0\), or \(m(\mathcal{A}(T)) = 0\), where \(m(\cdot)\) denotes the Lebesgue measure.

Suppose not. Then \(m(\mathcal{A}(T)) > 0\) and \(m(\mathcal{A}(T)) > 0\). Denote by \(\mathcal{A}^\varepsilon(T) = \{ t \in [0, T) : p_t \geq \varepsilon \}\) and \(\mathcal{A}^\varepsilon(T) = \{ t \in [T, 1/\varepsilon) : p_t \leq B - \varepsilon \}\). Continuity of the measure implies that there exists \(\varepsilon > 0\) such that \(m(\mathcal{A}^\varepsilon(T)) > 0\) and \(m(\mathcal{A}^\varepsilon(T)) > 0\). For a pair of constants \(\varepsilon_1, \varepsilon_2 \in (0, \varepsilon)\), consider the following perturbation of \(p\), resulting in another wage policy \(\tilde{p}\):

\[
\tilde{p}_t = \begin{cases} 
 p_t - \varepsilon_1, & \text{if } t \in \mathcal{A}(T) \\
 p_t + \varepsilon_2, & \text{if } t \in \mathcal{A}(T) \\
 p_t, & \text{otherwise.}
\end{cases}
\]

We want to pick the constants \(\varepsilon_1, \varepsilon_2\) such that the constraint in (C.4) still holds with equality, i.e.,

\[
\int_0^\infty e^{-(r_p + \lambda)t} \tilde{p}_t dt = \int_0^\infty e^{-(r + \lambda)t} p_t dt \iff \varepsilon_1 \int_{\mathcal{A}(T)} e^{-(r_p + \lambda)t} dt = \varepsilon_2 \int_{\mathcal{A}(T)} e^{-(r + \lambda)t} dt,
\]

(C.5)

which is feasible since \(m(\mathcal{A}(T)) > 0\) and \(m(\mathcal{A}(T)) > 0\). Now we argue that the principal strictly improves her objective in (C.4) using \(\tilde{p}\), i.e.,

\[
\int_0^T e^{-(r_p + \lambda)t} \tilde{p}_t dt < \int_0^T e^{-(r + \lambda)t} p_t dt \iff \varepsilon_1 \int_{\mathcal{A}(T)} e^{-(r_p + \lambda)t} dt > \varepsilon_2 \int_{\mathcal{A}(T)} e^{-(r + \lambda)t} dt
\]

\[
\iff \frac{\int_{\mathcal{A}(T)} e^{-(r_p + \lambda)t} dt}{\int_{\mathcal{A}(T)} e^{-(r + \lambda)t} dt} > \frac{\int_{\mathcal{A}(T)} e^{-(r_p + \lambda)t} dt}{\int_{\mathcal{A}(T)} e^{-(r + \lambda)t} dt},
\]

(C.6)
where the last equivalence is obtained by dividing both sides by equation (C.5). We will show a stronger version of equation (C.6), namely,

$$\int_{A(T)}^p e^{-(r+p)t} \, dt > e^{-(p-r)T} > \frac{\int_{A(T)}^p e^{-(r+p)t} \, dt}{\int_{A(T)}^p e^{-(r+p)t} \, dt}. \quad (C.7)$$

Since $A^e(T) \subseteq [0, T]$, it follows that

$$\int_{A^e(T)}^p e^{-(r+p)t} \, dt = \int_{A^e(T)}^p e^{-(r+p)t} \, dt \geq \int_{A^e(T)}^p e^{-(r+p)t} \, dt = e^{-(r+p)t} \int_{A^e(T)}^p e^{-(r+p)t} \, dt.$$

Moreover, from the assumption that $m(A^e(T)) > 0$, equality cannot hold. This proves the left inequality in (C.7). Similarly, since $A^e(T) \subseteq [7, 1/\epsilon]$, the right inequality in (C.7) also holds.

To sum up, we show that if $m(A(T)) > 0$ and $m(A(T)) > 0$, the principal strictly improves her objective in (C.4) using a perturbed wage policy $\tilde{p}$, a contradiction!

**Step 2** Denote by $T = \inf \{T : m(A(T)) > 0\}$. Then $p$ takes the form in equation (C.3).

By the definition of $T$, for any $T' < T$ it holds that $m(A(T')) = 0$, i.e., $p_{t_{A(T')}} = 0$ on $[0, T')$. Thus, it follows that $p_t \equiv 0$ on $[0, T)$.

Similarly, for any $T' > T$ it holds that $m(A(T')) > 0$ and $m(A(T')) = 0$, i.e., $p_{t_{T'}} \equiv B$ on $[T', \infty)$. It follows that $p_t \equiv B$ on $[T, \infty)$.

Together, we show that $p$ takes the form in equation (C.3).

**Step 3** It is optimal to have upfront payment $M = 0$.

Suppose not. Then $M > 0$. Step 2 implies that $p$ takes the form in equation (C.3) for some $T$. Since we assume that $PVy < B/(r + \lambda)$, the constraint in program (C.4) implies that $T > 0$. For $\epsilon$ that is sufficiently close to zero, consider the following perturbation of $p$ and $M$, resulting in another wage policy $\tilde{p}$ and upfront payment $\tilde{M} \geq 0$:

$$\tilde{p}_t = \begin{cases} 0, & t < T - \epsilon, \\ B, & t \geq T - \epsilon, \end{cases} \quad \text{and} \quad \tilde{M} = M - \int_{T-\epsilon}^T e^{-(r+\lambda)t} B \, dt.$$

By construction, the constraint in (C.4) still holds with equality, i.e.,

$$\int_0^\infty e^{-(r+\lambda)t} \tilde{p}_t \, dt + \tilde{M} = \int_0^\infty e^{-(r+\lambda)t} p_t \, dt + M.$$
Now we argue that the principal strictly improves her objective in (C.4) using \( \tilde{p} \) and \( \tilde{M} \), i.e.,

\[
\int_0^{\infty} e^{-(r_p + \lambda) t} \tilde{p} \, dt + \tilde{M} < \int_0^{\infty} e^{-(r_p + \lambda) t} p \, dt + M
\]

\[
\Leftrightarrow \int_{T^*}^{\infty} e^{-(r_p + \lambda) t} B \, dt > \int_{T^*}^{\infty} e^{-(r_p + \lambda) t} B \, dt
\]

\[
\Leftrightarrow \int_{T^*}^{\infty} \left( e^{-(r_p + \lambda) t} - e^{-(r_p + \lambda) t} \right) B \, dt > 0,
\]

which holds since \( r_p > r \). To sum up, we show that if \( M > 0 \), the principal strictly improves her objective in (C.4) using the perturbed \( \tilde{p} \) and \( \tilde{M} \), a contradiction!

Together, we establish the requirements for the optimal wage policy and upfront payment for the case \( r_p > r \), namely \( M = 0 \) and \( p \) with the form in equation (C.3).

Therefore, the proof is complete. \( \square \)

**Proof of Proposition C.1.** We follow the same rationale of the proof of Proposition 2. For any contract \( C = (\alpha, p, M) \), let \( \alpha \) denote the CDF of \( \tau (H) \), i.e., the agent’s shirk time given that the task has high quality, and let \( \beta \) denote the CDF of \( \tau (L) \).

From equations (C.1) and (C.2), we can rewrite the expected payoffs to the principal and agent as follows:

\[
W(C) = W^b (\alpha, \beta) - \int_0^{\infty} e^{-(r_p + \lambda) t} p \, dt - M,
\]

\[
V(C) = V^b (\alpha, \beta) + \int_0^{\infty} e^{-(r_p + \lambda) t} p \, dt + M,
\]

where \( W^b \) and \( V^b \) denote the principal’s and agent’s benchmark payoffs from the task without transfers, as functions of the CDFs \( (\alpha, \beta) \). They exactly take the form given by equations (A.7) and (A.8),

\[
W^b (\alpha, \beta) = \mu \int_0^{\infty} w(t, L) \, d\alpha (t) + (1 - \mu) \int_0^{\infty} w(t, H) \, d\beta (t),
\]

\[
V^b (\alpha, \beta) = \mu \int_0^{\infty} v(t, H) \, d\alpha (t) + (1 - \mu) \int_0^{\infty} v(t, L) \, d\beta (t).
\]

A necessary condition for a pair of CDFs \( (\alpha, \beta) \) to be induced by some contract is that the overall contract satisfies the agent’s participation constraint \( V(C) \geq \nu \). Moreover, since effort and task completion are noncontractible, payments do not directly affect the agent’s effort decision. Therefore, another necessary condition is \( V^b (\alpha, \beta) \geq 0 \), since the agent can ignore all information provided by the principal and shirk immediately. Although these two conditions are not sufficient, it turns out that they are enough to characterize the optimal contract. The following program can be seen as a relaxation of the principal’s problem of choosing the optimal contract:

\[
\max_{\alpha, \beta, p, M} \quad W^b (\alpha, \beta) - \int_0^{\infty} e^{-(r_p + \lambda) t} p \, dt - M
\]

s.t. \( V^b (\alpha, \beta) + \int_0^{\infty} e^{-(r_p + \lambda) t} p \, dt + M \geq \nu \).

\( V^b (\alpha, \beta) \geq 0, \)

\( \alpha, \beta \) non-decreasing and right-continuous, \( \alpha (\infty) = \beta (\infty) = 1. \)
If the solution to the above program (C.10) is induced by some particular contract, then we can conclude that the candidate contract is optimal. The rest of the proof proceeds with four steps.

**Step 1** Any solution to the relaxed program (C.10) satisfies \( \alpha (t) = 0 \) for all \( t \in [0, \infty) \), i.e., the agent never shirks when the task has high quality. Moreover, it is without loss to assume \( p = 0 \).

The proof of \( \alpha (t) = 0 \) for all \( t \in [0, \infty) \) is exactly the same as in the proof of Proposition 2, so it is omitted here. To show that it is without loss to assume \( p = 0 \), take any solution \((\alpha, \beta, p, M)\) to the program (C.10), and suppose that the agent’s present value of total payments \((p, M)\) is \( PV \). If \( PV > v \), then \( V^b (\alpha, \beta) + PV > 0 + v \), so the agent’s participation constraint will be slack. It follows that the principal can strictly improve the objective by paying less, a contradiction to optimality! Thus, we have \( PV \leq v \), and Lemma C.2 shows that it is optimal to have \( p = 0 \) and \( M = PV \).

**Step 2** Reformulation of the principal’s relaxed program (C.10).

From Step 1, the principal’s relaxed program (C.10) reduces to

\[
\begin{align*}
\max_{\beta, M} & \quad \mu w (H) + (1 - \mu) \int_0^\infty w (t, L) \, d\beta (t) - M \\
\text{s.t.} & \quad \mu v (H) + (1 - \mu) \int_0^\infty v (t, L) \, d\beta (t) + M \geq v, \\
& \quad \mu v (H) + (1 - \mu) \int_0^\infty v (t, L) \, d\beta (t) \geq 0, \\
& \quad \beta \text{ non-decreasing and right-continuous, } \beta (\infty) = 1.
\end{align*}
\]

Note that \( v (t, L) = \left( 1 - e^{-(r+\lambda)t} \right) v (L) \), so we can rewrite the second constraint as

\[
\beta (0) + \int_0^\infty e^{-(r+\lambda)t} \, d\beta (t) \geq 1 - \frac{\mu v (H)}{(1 - \mu) [v (L)]} \equiv C_1.
\]

If the first constraint is slack, then the principal can strictly improve the objective by paying less, a contradiction to optimality! Thus, we have

\[
M = v - \left[ \mu v (H) + (1 - \mu) \int_0^\infty v (t, L) \, d\beta (t) \right]
\]

at optimum. Together with \( w (t, L) = \left( 1 - e^{-(r+\lambda)t} \right) w (L) \), we can rewrite the objective as

\[
\mu (w (H) + v (H)) + (1 - \mu) \left[ w (L) \int_0^\infty \left( 1 - e^{-(r+\lambda)t} \right) \, d\beta (t) + v (L) \int_0^\infty \left( 1 - e^{-(r+\lambda)t} \right) \, d\beta (t) \right].
\]

Thus, maximizing the objective is equivalent to minimizing \( \beta (0) + \int_0^\infty e^{-(r+\lambda)t} \, d\beta (t) \). Hence, we can further rewrite the principal’s program as

\[
\begin{align*}
\max_{\beta : [0, \infty) \to [0, 1]} & \quad P(\beta) \equiv w (L) \int_0^\infty \left( 1 - e^{-(r+\lambda)t} \right) \, d\beta (t) + v (L) \int_0^\infty \left( 1 - e^{-(r+\lambda)t} \right) \, d\beta (t) \\
\text{s.t.} & \quad C(\beta) \equiv \beta (0) + \int_0^\infty e^{-(r+\lambda)t} \, d\beta (t) \geq C_1, \\
& \quad \beta \text{ non-decreasing and right-continuous, } \beta (\infty) = 1,
\end{align*}
\]

(C.12)
where \( C_1 \equiv 1 - \frac{\mu v(H)}{(1-\mu)\mu v(L)} < 1 \). Moreover, the assumption \( \mu < \bar{\mu} \) implies that \( C_1 \) is positive.

### Step 3

If \( w(L) + v(L) < 0 \), then the CDF \( \beta^{IFD} \) induced by IFD solves the program (C.12).

Under IFD, \( \tau(L) \) is degenerate at zero, i.e., \( \beta^{IFD}(t) = 1 \) for all all \( t \in [0, \infty) \). Thus, in program (C.12) we have \( P(\beta^{IFD}) = 0 \).

Now take any CDF \( \beta \) that satisfies the constraint in program (C.12). It suffices to show that \( P(\beta) \leq P(\beta^{IFD}) = 0 \). Since \( w(L) < [-v(L)] \) and \( r_p \leq r \),

\[
P(\beta) \leq [-v(L)] \int_0^\infty \left(1 - e^{-(r_p+\lambda)t}\right) d\beta(t) + v(L) \int_0^\infty \left(1 - e^{-(r_p+\lambda)t}\right) d\beta(t) \]

\[
= [-v(L)] \int_0^\infty \left(e^{-(r_p+\lambda)t} - e^{-(r_p+\lambda)t}\right) d\beta(t) \leq 0, \]

as desired. Therefore, \( \beta^{IFD} \) solves the program (C.12). Substituting back into equation (C.11), we obtain \( M = \bar{\nu} - \mu \nu(H) \). This confirms that the contract \((\sigma, p, M) = (IFD, 0, \bar{\nu} - \mu \nu(H)) \) is optimal.

### Step 4

If \( w(L) + v(L) \geq 0 \), then the CDF \( \beta^{KG} \) induced by KG solves the program (C.12).

Under KG, according to equation (A.3),

\[
\tau(L) = \begin{cases} 
0, & \text{w.p. } 1 - \frac{\mu v(H)}{(1-\mu)[-v(L)]}, \\
\infty, & \text{w.p. } \frac{\mu v(H)}{(1-\mu)[-v(L)]}.
\end{cases}
\]

Since \( C_1 = 1 - \frac{\mu v(H)}{(1-\mu)[-v(L)]}, \beta^{KG}(t) = C_1 \) for all \( t \in [0, \infty) \). Thus, we have

\[
P(\beta^{KG}) = (w(L) + v(L))\left(1 - \beta^{KG}(0)\right) = (w(L) + v(L))(1 - C_1). \]

Now take any CDF \( \beta \) that satisfies the constraint in program (C.12). It suffices to show that \( P(\beta) \leq P(\beta^{KG}) = (w(L) + v(L))(1 - C_1) \). Since \( r_p \leq r \),

\[
P(\beta) \leq w(L) \int_0^\infty \left(1 - e^{-(r_p+\lambda)t}\right) d\beta(t) + v(L) \int_0^\infty \left(1 - e^{-(r_p+\lambda)t}\right) d\beta(t)
\]

\[
= (w(L) + v(L)) \left[1 - \beta(0) - \int_0^\infty e^{-(r_p+\lambda)t} d\beta(t)\right]
\]

\[
= (w(L) + v(L)) (1 - C(\beta)) \leq (w(L) + v(L))(1 - C_1),
\]

as desired. Therefore, \( \beta^{KG} \) solves the program (C.12). Substituting back into equation (C.11), we obtain \( M = \bar{\nu} \). This confirms that the contract \((\sigma, p, M) = (KG, 0, \bar{\nu}) \) is optimal. \( \square \)

**Proof of Proposition C.2.** We follow the same procedure in the proof of Proposition C.1. In particular, if the solution to program (C.10) is induced by some particular contract, then we can conclude that the candidate contract is optimal. The rest of the proof proceeds with three steps.

### Step 1

Any solution to the relaxed program (C.10) satisfies \( \alpha(t) = 0 \) for all \( t \in [0, \infty) \), i.e., the agent never shirks when the task has high quality. Moreover, \( M = 0 \), and \( p \) takes the form in equation (C.3).

The proof of \( \alpha(t) = 0 \) for all \( t \in [0, \infty) \) is exactly the same as in the proof of Proposition 2, so it is omitted here. To show that \( M = 0 \) and \( p \) takes the form in equation (C.3), pick any solution \((\alpha, \beta, p, M) \) to
Thus, maximizing the objective is equivalent to minimizing $\text{MDD}$, as well as the $\text{C}$ implies that $\text{PV}$ can strictly improve the objective by paying less, a contradiction to optimality! Thus, we have $\text{PV} \leq \gamma$, and Lemma C.2 shows that it is optimal to have $M = 0$ and $p$ with the form in equation (C.3).

**Step 2** Reformulation of the principal’s relaxed program (C.10).

From Step 1, the principal’s relaxed program (C.10) reduces to

$$\max_{\beta,T} \mu w(H) + (1 - \mu) \int_0^\infty w(t, L) \, d\beta(t) - \int_T^\infty e^{-(r_p + \lambda)t} B \, dt$$

s.t. $$\mu v(H) + (1 - \mu) \int_0^\infty v(t, L) \, d\beta(t) + \int_T^\infty e^{-(r_p + \lambda)t} B \, dt \geq \gamma,$$

$$\mu v(H) + (1 - \mu) \int_0^\infty v(t, L) \, d\beta(t) \geq 0,$$

$\beta$ non-decreasing and right-continuous, $\beta(\infty) = 1$.

Note that $v(t, L) = (1 - e^{-r_p t}) v(L)$, so we can rewrite the first and second constraints together as one

$$\beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) \geq C_1 + \frac{1}{(1 - \mu)[-v(L)]} \max \left\{ \gamma - \frac{B}{r + \lambda} e^{-(r_p + \lambda)t}, 0 \right\},$$

where $C_1 \equiv 1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]}$. Similarly, $w(t, L) = (1 - e^{r_p t}) w(L)$, so we can rewrite the objective as

$$\mu w(H) + (1 - \mu) w(L) \left[ \beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) \right] - \frac{B}{r_p + \lambda} e^{-(r_p + \lambda)t}.$$

Thus, maximizing the objective is equivalent to minimizing

$$\beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) + \frac{B}{(1 - \mu)(r_p + \lambda) w(L)} e^{-(r_p + \lambda)t}.$$

Hence, we can further rewrite the principal’s program as

$$\min_{\beta,T} P(\beta, T) \equiv \beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) + \frac{B}{(1 - \mu)(r_p + \lambda) w(L)} e^{-(r_p + \lambda)t}$$

s.t. $$C(\beta) \geq C_1 + \frac{1}{(1 - \mu)[-v(L)]} \max \left\{ \gamma - \frac{B}{r + \lambda} e^{-(r_p + \lambda)t}, 0 \right\},$$

$\beta$ non-decreasing and right-continuous, $\beta(\infty) = 1$, (C.13)

where $C(\beta) \equiv \beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t)$ and $C_1 \equiv 1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]} < 1$. Moreover, the assumption $\mu < \bar{\mu}$ implies that $C_1$ is positive.

**Step 3** For sufficiently large $B$, the solution to the program (C.13) involves $\beta^{\text{MDD}}$, the CDF induced by MDD, as well as the $T^*$ such that

$$\frac{B}{r + \lambda} e^{-(r_p + \lambda)t} = \gamma,$$

(C.14)
Under MDD, $\tau(L)$ is degenerate at $\tilde{\tau}(\mu)$, i.e., $\beta^{MDD}(t) = 1$ ($t \geq \tilde{\tau}(\mu)$). Thus, we have

$$P(\beta^{MDD}, T^*) = e^{-(r_p + \lambda)\tilde{\tau}(\mu)} + \frac{B}{(1 - \mu)(r_p + \lambda)w(L)}e^{-(r_p + \lambda)T^*}.$$  

It follows from the definition of $\tilde{\tau}(\mu)$, equation (5), that

$$\mu v(H) + (1 - \mu) \left(1 - e^{-(r_p + \lambda)\tilde{\tau}(\mu)}\right)v(L) = 0,$$

which implies that

$$\tilde{\tau}(\mu) = -\frac{1}{r + \lambda} \log \left(1 - \frac{\mu v(H)}{(1 - \mu)[-v(L)]}\right) = -\frac{1}{r + \lambda} \log C_1 \Rightarrow e^{-(r_p + \lambda)\tilde{\tau}(\mu)} = (C_1)^{\frac{r_p + \lambda}{T^*}},$$

and thus

$$P(\beta^{MDD}, T^*) = (C_1)^{\frac{r_p + \lambda}{T^*}} + \frac{B}{(1 - \mu)(r_p + \lambda)w(L)}e^{-(r_p + \lambda)T^*}. \quad (C.15)$$

Now take any $(\beta, T)$ that satisfies the constraint in program (C.13). It suffices to show that $P(\beta, T) \geq P\left(\beta^{MDD}, T^*\right)$ when $B$ is large enough. If $T \leq T^*$, we have $C(\beta) \geq C_1$. Therefore, Step 4 the proof of Proposition 2 shows that

$$\beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) \geq (C_1)^{\frac{r_p + \lambda}{T^*}} \Rightarrow P(\beta, T) \geq (C_1)^{\frac{r_p + \lambda}{T^*}} + \frac{B}{(1 - \mu)(r_p + \lambda)w(L)}e^{-(r_p + \lambda)T^*},$$

as desired. Henceforth, we assume that $T > T^*$, which implies that

$$C(\beta) \geq C_1 + \frac{1}{(1 - \mu)[-v(L)]} \left(\frac{B}{r + \lambda} - e^{-(r_p + \lambda)T}\right) > C_1.$$  

Again, Step 4 of the proof of Proposition 2 implies that

$$\beta(0) + \int_0^\infty e^{-(r_p + \lambda)t} d\beta(t) \geq (C(\beta))^{\frac{r_p + \lambda}{T^*}} > (C_1)^{\frac{r_p + \lambda}{T^*}} \Rightarrow P(\beta, T) > (C_1)^{\frac{r_p + \lambda}{T^*}}.$$  

Substituting equation (C.14) into equation (C.15), we obtain

$$P(\beta^{MDD}, T^*) = (C_1)^{\frac{r_p + \lambda}{T^*}} + \frac{[r + \lambda]v}{(1 - \mu)(r_p + \lambda)w(L)}B^{\frac{r_p + \lambda}{T^*}} \rightarrow (C_1)^{\frac{r_p + \lambda}{T^*}} < P(\beta, T) \quad \text{as } B \rightarrow \infty.$$  

This confirms that for sufficient large $B$, the optimal contract involves MDD as its information policy. □