1 Implications of No Arbitrage, State Prices, and SDF (Notes 4)

1.1 Case 1: Finite State Space

- Prop 1: Complete markets $\Longrightarrow \varphi$ (state prices) unique
- Prop 2: Incomplete markets $\Longrightarrow \exists$ infinitely-many φ (state prices)
- Theo 1: No arbitrage $\Longleftrightarrow \exists$ strictly positive state price vector$\varphi \in \mathbb{R}^s_{++}$ s.t. $P = X\varphi$
- Corr 1: No arbitrage $\Longleftrightarrow \exists$ strictly positive SDF $m \in \mathbb{R}^s_{++}$ s.t. $P = E(mX)$
- Corr 2: No arbitrage in a complete market $\Longleftrightarrow \exists$ unique and strictly positive state price vector $\varphi \in \mathbb{R}^s_{++}$ (and SDF $m \in \mathbb{R}^s_{++}$)
- Corr 3: No arbitrage in an incomplete market $\Longrightarrow \exists$ infinitely-many φ (state prices) and SDF $m \in \mathbb{R}^s_{++}$ (Not everyone has to be str +, but at least one is str + under no arbitrage)

1.2 Case 2: General State Space

- Prop 1: LOOP $\Longleftrightarrow$ Linear Price Functional $[p(ax+by) = ap(x) + bp(y)]$
- Prop 2: LOOP $\Longleftrightarrow$ basis vector of random payoffs $\overrightarrow{x}$ is nondegereate (i.e. non-redundant)
- Prop 3: LOOP $\Longrightarrow \exists\ E\left(\overrightarrow{x}\,\overrightarrow{x}^T\right)^{-1}$ (i.e. $E\left(\overrightarrow{x}\,\overrightarrow{x}^T\right)$ invertible)
- Prop 4: No arbitrage $\Rightarrow$ LOOP (But LOOP $\nRightarrow$ No arbitrage)
- Theo 1: Under frictionless portfolio formation and additional technical assumptions, there is NO ARIBTRAGE $\Longleftrightarrow \exists$ a strictly positive $m \in L^2$ s.t. $\forall x \in \underline{X}$, $p(x) = E(mX)$
- Theo 2: Under frictionless portfolio formation, LOOP holds $\Longleftrightarrow \exists$ a SDF $x^* \in \underline{X}$ and it is unique. (so that $\forall x \in \underline{X}$, $p(x) = E(x^* x)$)
- Theo 3: Under LOOP (so that x^* exists), $m \in L^2$ is a SDF $\Longleftrightarrow \pi(m \mid \underline{X}) = x^*$

2 Mean Variance Frontier and CAPM (Notes 5)

- Prop 1: For a mean portfolio return of μ_p, the MV portfolio has variance and portfolio weights given by:

$$\sigma^2(\mu_p) = \frac{A\mu^2 - 2B\mu + C}{D} \text{ and } \omega(\mu_p)_{n\times 1} = g + h\mu_p$$

$$\begin{aligned} \text{where } g_{n\times 1} &= \frac{C\Sigma^{-1}\overrightarrow{1} - B\Sigma^{-1}\overrightarrow{E}}{D},\ h_{n\times 1} = \frac{A\Sigma^{-1}\overrightarrow{E} - B\Sigma^{-1}\overrightarrow{1}}{D}, \\ A &= \overrightarrow{1}^T\Sigma^{-1}\overrightarrow{1},\ B = \overrightarrow{1}^T\Sigma^{-1}\overrightarrow{E},\ C = \overrightarrow{E}^T\Sigma^{-1}\overrightarrow{E},\ D = AC - B^2 \end{aligned}$$

- Prop 2: MV frontier form a convex set. (i.e. if ω_1, ω_2 frontier portfolios with μ_1, μ_2. Then, $\forall \lambda \in \mathbb{R}$, $\omega^* \equiv \lambda\omega_1 + (1-\lambda)\omega_2$ also frontier.)
- Prop 3: Mean, variance, covariance of the MVP (minimum variance portfolio) is given by:

$$\mu_{mvp} = \frac{B}{A}, \sigma^2_{mvp} = \frac{1}{A}, Cov(R_i, R_{mvp}) = \frac{1}{A} = \sigma^2_{mvp}\ \forall i \text{ on frontier}$$

- Prop 4: If p^T is the tangency portfolio, then all securities / portfolios can be priced according to the following beta-representation:

$$E(R_i) - R_f = \frac{Cov(R_i, R^T)}{Var(R^T)}\left[E(R^T) - R_f\right] \quad \forall i$$

- Prop 5: If $\exists$ R.F. asset, then **ANY** portfolio p on the MV frontier (other than the r.f. asset) can be used to price other assets/portfolios according to the following beta-representation:

$$E(R_i) - R_f = \frac{Cov(R_i, R_p)}{Var(R_p)}\left[E(R_p) - R_f\right] \quad \forall i$$

- Prop 6: If no R.F. asset, then $\exists\ \gamma$ s.t. **ANY** portfolio p on the MV frontier (other than the MV portfolio) can be used to price other assets/portfolios according to the following beta-representation:

$$E\left(R_i\right)-\gamma=\frac{Cov\left(R_i,R_p\right)}{Var(R_p)}\left[E\left(R_p\right)-\gamma\right]\quad\forall i$$

and $\gamma=E\left(R_{zcp}\right)$ where zcp is the unique portfolio on the MV frontier that has 0 correlation with p

- Prop 7: If $\exists$ a portfolio p and a γ s.t. $E\left(R_i\right)-\gamma=\frac{Cov(R_i,R_p)}{Var(R_p)}\left[E\left(R_p\right)-\gamma\right]\quad\forall i$, $\Rightarrow$ p must be on the MV frontier.

- Prop 7 + 8: A portfolio p on MV frontier $\iff E\left(R_i\right)-\gamma=\frac{Cov(R_i,R_p)}{Var(R_p)}\left[E\left(R_p\right)-\gamma\right]\quad\forall i$

3 CAPM (The beta-rep model with market portfolio as factor. Or, the model with SDF $m=a+bR_m$) (Notes 6)

- Theo 1: (CAPM with RF) Suppose $\exists$ a R.F. asset, and suppose CAPM assumptions hold (agents optimize, have full information, and no trading frictions) $\Rightarrow$ in equilibrium the market portfolio is the tangency portfolio.

$$\text{So, } E\left(R_i\right)-R_f=\frac{Cov\left(R_i,R_m\right)}{Var(R_m)}\left[E\left(R_m\right)-R_f\right]\quad\forall i$$

- Theo 2: (Zero-Beta CAPM) If no R.F. asset, under CAPM assumptions $\Rightarrow$ in equilibrium the market portfolio is on the MV frontier and

$$E\left(R_i\right)-R_{zc}=\frac{Cov\left(R_i,R_m\right)}{Var(R_m)}\left[E\left(R_m\right)-R_{zc}\right]\quad\forall i$$

4 SDF, Beta Representation, and MV Frontier (Notes 7)

4.1 Beta Representation of Linear Factor Pricing Models

$(E(R_i)=\gamma+\beta^T\lambda_k$ where $\beta: R_i=a_i+\beta^Tf_k+\varepsilon_i)$

- Prop 1: If a factor f_k is a gross return on a portfolio (e.g. R_m in CAPM) and $\exists$ R.F. asset $\Rightarrow\lambda_k$ should always be a risk-premium on that portfolio (i.e. $\lambda_k=E\left(R_m\right)-R_f$)

- Prop 2: If a factor f_k is an excess return (e.g. R_H-R_L in FF3) $\Rightarrow\lambda_k$ is always the expected excess return (i.e. $\lambda_k=E\left(f_k\right)$)

4.2 SDF and Beta Representation $\left(m=a+b^Tf\iff\text{Beta Representation with f as factor}\right)$

- Theo 1: $\exists\ a\in\mathbb{R}\ \&\ b\in\mathbb{R}^k$ s.t. $m=a+b^T\vec{f}$ is a SDF $\iff\exists\ \gamma\in\mathbb{R}\ \&\ \lambda\in\mathbb{R}^k$ s.t. $\forall$ securities/portfolios i, $E\left(R_i\right)=\gamma+\beta_i^T\lambda\ \forall i$

$$\text{where } \vec{\beta}_i\equiv\Sigma_{\vec{f}}^{-1}Cov\left(\bar{f},R_i\right)\ \text{ is the reg coeff of } R_i \text{ on } f \text{ and a constant}$$

- Prop 3: $m=a+b^T\vec{f}\Rightarrow$ in beta representation, $\lambda=E\left(\vec{f}\right)-\gamma E\left(m\vec{f}\right)$ $\left(\text{where }\ \gamma=\frac{1}{E(m)}\right)$

4.3 Single-Factor Special Case: CAPM $(CAPM\iff m=a+bR_m)$

- Prop 4: CAPM $\iff m=a_m+b_mR_m$ for some $\alpha_m,b_m\in\mathbb{R}$

  Moreover,

$$\text{if }\exists\text{ R.F.asset, } a_m=\frac{1}{R_f}+\frac{1}{R_f}\frac{E\left(R_m\right)\left(ER_m-R_f\right)}{Var\left(R_m\right)}\text{ and } b_m=-\frac{1}{R_f}\frac{ER_m-R_f}{Var\left(R_m\right)}$$

$$\text{if no R.F.asset, } a_m=\frac{1}{R_{zc}}+\frac{1}{R_{zc}}\frac{E\left(R_m\right)\left(ER_m-R_{zc}\right)}{Var\left(R_m\right)}\text{ and } b_m=-\frac{1}{R_{zc}}\frac{ER_m-R_{zc}}{Var\left(R_m\right)}$$

- Corr 1: (Complete Mkt + CAPM don't mix!) Suppose market is complete. If CAPM holds and market portfolio return distribution is normal (or has large enough support) $\Rightarrow\exists$ arbitrage opportunities.

4.4 Single-Factor Special Case: SDF as Factor

- Prop 5: Any valid SDF m yields a beta representation with m as factor.

$$i.e.\ E(R_i) = \underbrace{\frac{1}{E(m)}}_{\gamma} + \underbrace{\frac{-Var(m)}{E(m)}}_{\lambda}\underbrace{\frac{Cov(m,R_i)}{Var(m)}}_{\beta}$$

- Prop 6: Given any valid SDF m and under LOOP (so that x^* exists). We can use m-mimicking payoff $x^* \equiv \pi(m \mid \underline{X})$ as factor to yield a beta representation.

$$i.e.\ E(R_i) = \underbrace{\frac{1}{E(x^*)}}_{\gamma} + \underbrace{\frac{-Var(x^*)}{E(x^*)}}_{\lambda}\underbrace{\frac{Cov(x^*,R_i)}{Var(x^*)}}_{\beta}$$

- Prop 7: Given SDF m and under LOOP (so that x^* exists). Then, $\exists$ beta representation with $R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$ as factor.

$$i.e.\ E(R_i) = \underbrace{\frac{E(R^{*2})}{E(R^*)}}_{\gamma} + \underbrace{\frac{-Var(R^*)}{E(R^*)}}_{\lambda}\underbrace{\frac{Cov(R^*,R_i)}{Var(R^*)}}_{\beta}$$

4.5 MV and an Orthogonal Characterization of the MV Frontier (Under LOOP so that x* exists)

- Prop 8: (R*on MV frontier) $R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$ is on the MV frontier $\left(\text{bc } E(R_i) - \gamma = \beta_{i,R^*}[ER^* - \gamma]\right)$
- Prop 9: (Any frontier portf can be a factor) $\exists\ a, b \in \mathbb{R}$ and a return R_p s.t. $m = a + bR_p$ is a SDF $\Longleftrightarrow$ R_p is on the MV frontier
- Prop10: (Representation of $R^{e*} \equiv \pi(1|\underline{R}^e)$)

$$a) \text{ If } \exists \text{ R}_f,\ \text{then } R^{e*} = 1 - \frac{R^*}{R_f}$$

$$b) \text{ If no R}_f,\ \text{then } R^{e*} = \pi(1|\underline{X}) - \frac{E(R^*)}{E(R^{*2})}R^* \text{ where } \pi(1|\underline{X}) \equiv E\left(\vec{x}^T\right)E\left(\vec{x}\vec{x}^T\right)^{-1}\vec{x}$$

- Prop11: Every R_i can be expressed as

$$R_i = R^* + \omega_i R^{e*} + \eta_i$$

$$\text{where } \omega \in \mathbb{R}, \eta_i \text{ an excess return with } E(\eta_i) = 0,\ \text{and } E(R^*R^{e*}) = E(R^*\eta_i) = E(R^{e*}\eta_i) = 0$$

- Corr 2: R^{mv} is on the MV frontier $\Longleftrightarrow$ $R^{mv} = R^* + \omega R^{e*}$ for some ω
- Prop12: Under LOOP (so that x^* exists), $R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$ is the MV frontier portfolio with the smallest second moment.

5 Arbitrage Pricing Theory (APT) (Notes 8)

- Prop 1 (APT): Suppose all N security returns are described by the following statistical factor decomposition

$$R_i = a_i + \vec{f}^T\beta_i + \varepsilon_i \quad \forall i \qquad (1)$$

$$\text{where } E(\varepsilon_i) = E\left(\varepsilon_i\vec{f}_k\right) = Cov\left(\varepsilon_i, \vec{f}_k\right) = 0 \quad \forall i, \forall k \qquad (2)$$

$$\text{and } E(\varepsilon_i\varepsilon_j) = \sigma_i^2 \le \bar{\sigma}^2 \text{ if i=j and 0 o.w.} \qquad (3)$$

$$\text{or in a system form... } R_{N\times1} = A_{N\times1} + B_{N\times K}f_{K\times1} + E_{N\times1} \qquad (1')$$

$$E(E) = E(Ef) = 0 \qquad (2')$$

$$\Sigma_{N\times N} \equiv E\left(EE^T\right) \text{ is diagonal with + and bdd entries} \quad (3')$$

Then, "arbitrage considerations" alone $\Rightarrow$ expected returns of securities "approximately" satisfy beta representation with factors $\vec{f}$

$$i.e.E(R_i) \cong \gamma + \lambda^T \beta_i \quad (4)$$

Moreover, if $\exists$ R.F. asset and $\vec{f}$ are returns, then

$$E(R_i) \cong R_f + \beta_i^T \left[E\left(\vec{f}\right) - R_f\right] \quad \forall i \quad (5)$$

- Ex 1: (Exact pricing with $\varepsilon_i = 0$) When $\varepsilon_i = 0$, under APT / no arbitrage, $E(R_i) - R_f = \beta_i^T \left(E(f) - R_f \vec{1}\right)$
- Ex 2: (Approx pricing with $\varepsilon_i \neq 0$) When $\varepsilon_i \neq 0$, under APT / no arbitrage, $E(R_i) - R_f = \beta_i^T \left(E(f) - R_f \vec{1}\right) - R_f Cov(m, \varepsilon_i) \cong \beta_i^T \left(E(f) - R_f \vec{1}\right)$
- Prop 2: (Exact beta-rep at the limit) For a fixed SDF m that prices the factors $\vec{f}$, as $Var(\varepsilon_i) \to 0$ for a security/portfolio $i \Rightarrow E(R_i) \to R_f + \beta_i^T \left(E(f) - R_f \ p\left(\vec{f}\right)\right)$
- Corr 1: (Exact beta-rep at the limit for well-div portfolio) For a fixed SDF m that prices the factors $\vec{f}$, consider a portfolio p with equilibrium weights in n securities. As $n \to \infty$, $E(R_p) \to R_f + \beta_i^T \left(E(f) - R_f \ p\left(\vec{f}\right)\right)$
- Prop 3: (Approx APT Pricing for all but finite securities) For a fixed SDF m that prices the factors $\vec{f}$, $\forall \ \delta > 0$ small, $\exists$ only finitely many sucurities s.t.

$$\left|E(R_i) - R_f - \beta_i^T \left[E\left(\vec{f}\right) - R_f\right]\right| > \delta$$

6 Option Pricing (Notes 11)

6.1 13 Important Properties of Options

We can derive the following with the following 2 no-arbitrage principles: 1) Portfolios with non-neg payoff has non-neg cost. 2) If one portfolio's payoff dominates another then so must its cost.

1. $c, C, p, P \geq 0$ for all parameter values
2. $C(S, t; E, T) \geq S(t) - E$ and $P(S, t; E, T) \geq E - S(t)$
3. For $T_2 > T_1, C(S, t; E, T_2) \geq C(S, t; E, T_1)$ and $P(S, t; E, T_2) \geq P(S, t; E, T_1)$
4. $C(.) \geq c(.)$ and $P(.) \geq p(.)$ given same input values
5. For $E_1 > E_2$,all else equal,
$$C(.; E_1) \leq C(.; E_2)$$
$$c(.; E_1) \leq c(.; E_2)$$
$$P(.; E_1) \geq P(.; E_2)$$
$$p(.; E_1) \geq p(.; E_2)$$
6. $S(t) \equiv C(S, t; 0, \infty) \geq C(S, t; E, T) \geq c(S, t; E, T)$
7. $C(0, t; E, T) = c(0, t; 0, \infty) = 0$
8. If the stock pays no dividends between t and T, then $c(S, t; E, T) \geq S - E \cdot B(t, T)$ where B(t,T) is the time t PV of receiving \$1 at T.
9. European call options are convex functions of E: i.e. $\alpha c(.; E_1) + (1 - \alpha) c(.; E_2) \geq c(.; \alpha E_1 + (1 - \alpha) E_2)$
10. "A portfolio of options is more valuable than an option on a portfolio" (i.e. call option on a portfolio of securities w/ an exercise price of X is less valuable than a portfolio of call options on the underlying with equal exercise)
11. Put-Call Parity (0 dividend stock): $p(S, t; E, T) = c(S, t; E, T) - S(t) + E \cdot B(t, T)$
12. European put option price is convex in E: i.e. $\alpha p(.; E_1) + (1 - \alpha) p(.; E_2) \geq p(.; \alpha E_1 + (1 - \alpha) E_2)$
13. For a stock paying no dividends over (t,T), an American call will never be exercised early. So, $C = c$ when S pays no dividends.

7 Binomial Pricing

- We can replicate the derivatives by holding a portfolio of the underlying stock and cash bond (N, B).

$$N = \frac{f_u - f_d}{S_u - S_d} \text{ and } B = f_{now} - N \cdot S_{now}$$

- We can also use risk-neutral pricing

$$\begin{aligned} f_{now} &= E_{\mathbb{Q}}\left[\frac{1}{R} f_{tomor}\right] \\ &= \frac{1}{R}\{q f_u + (1-q) f_d\} \end{aligned}$$

$$\text{where } q \equiv \frac{R S_{now} - S_d}{S_u - S_d}$$