MV, Tangency Portfolios, and What their FOC's Tell Us

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1 Minimum Variance Portfolio

Definition 1 *The MVP is the portfolio w that solves the following problem*

$$\min_{w} w \Sigma w \;\; s.t. \;\; w^T \mathbf{1} = 1$$

1. First Order Condition of the problem

$$
\begin{aligned}
\pounds &= w^T \Sigma w + \lambda \left(w^T \mathbf{1} - 1 \right) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} + \lambda \left(\sum_{i=1}^{N} w_i - 1 \right)
\end{aligned}
$$

$$\text{FOC w.r.t. } w_i: \; \frac{\partial \pounds}{\partial w_i} = 2 \sum_{j=1}^{N} w_j \sigma_{ij} + \lambda = 0 \quad \forall i$$

$$\Rightarrow \forall i, j \;\; \sum_{j=1}^{N} w_j \sigma_{ij} = \sum_{j=1}^{N} w_j \sigma_{kj}$$

$$\text{or } Cov\left(r_p, r_i \right) = Cov\left(r_p, r_j \right)$$

2. Intuition: Recall that the covariance between portfolio return and the return of one of the underlying assets is the marginal contribution to risk of that asset (See Lecture 5).

$$Cov\left(w^T r, r_k \right) = \sum_{i=1}^{N} w_i \sigma_{ik} = \frac{1}{2} \frac{\partial Var\left(w_A^T r \right)}{\partial w_k}$$

If the marginal of contribution to portfolio risk is not equal between all the assets in the MVP, then you can do strictly better by putting a little more weight on an asset with lower marginal risk and a little less weight on an asset with higher marginal risk.

2 Tangency Portfolio

Definition 2 *The tangency portfolio is the portfolio w that solves the following problem*

$$\max_{w} \frac{w^T E^e}{\left(w^T \Sigma w\right)^{1/2}} \quad s.t. \ w^T \mathbf{1} = 1$$

1. A trick: Let's equivalently consider a portfolio as follows

$$r_p = r_T + x r_i - x r_f$$

   Then the objective function can be re-written as (note that I've already substituted the constraint that the weights sum to 0)...

$$\max_{w} \frac{\mathbb{E}\left(r_p\right) - r_f}{\sigma\left(r_p\right)}$$

$$= \max_{w} \frac{\mathbb{E}\left(r_T\right) - r_f + w\mathbb{E}\left(r_i\right) - w r_f}{\left(\sigma^2\left(r_T\right) + w^2\sigma^2\left(r_i\right) + 2w\sigma\left(r_i, r_T\right)\right)^{1/2}}$$

   for some arbitrary i, and where r_T is the tangency portfolio.

2. First order conditions of this re-formulated problem

   FOC wrt w : $0 = \dfrac{\mathbb{E}\left(r_i\right) - r_f}{\left(\sigma^2\left(r_T\right) + w^2\sigma^2\left(r_i\right) + 2w\sigma\left(r_i, r_T\right)\right)^{1/2}}$

   $$-\frac{1}{2}\frac{\mathbb{E}\left(r_T\right) - r_f + w\mathbb{E}\left(r_i\right) - w r_f}{\left(\sigma^2\left(r_T\right) + w^2\sigma^2\left(r_i\right) + 2w\sigma\left(r_i, r_T\right)\right)^{3/2}}\left(2w\sigma^2\left(r_i\right) + 2\sigma\left(r_i, r_T\right)\right)$$

   Note here that we KNOW the optimal w^* is 0, by definition of a tangency portfolio, therefore we can substitute this value in and get...

$$\frac{\mathbb{E}\left(r_T\right) - r_f}{\sigma^2\left(r_T\right)} = \frac{\mathbb{E}\left(r_i\right) - r_f}{\sigma\left(r_i, r_T\right)} \quad \forall i$$

3. This tells us 2 things

   (a) Marginal contribution to reward - to - marginal contribution to risk ratio are the same for all assets

$$\frac{\mathbb{E}\left(r_i\right) - r_f}{\sigma\left(r_i, r_T\right)} = \frac{\mathbb{E}\left(r_j\right) - r_f}{\sigma\left(r_j, r_T\right)} \quad \forall i, j$$

   (b) We obtain a simple model for expected return of an asset in this economy

$$\mathbb{E}\left(r_i\right) - r_f = \underbrace{\frac{\sigma\left(r_i, r_T\right)}{\sigma^2\left(r_T\right)}}_{\beta_{i,T}}\left\{\mathbb{E}\left(r_T\right) - r_f\right\} \quad \forall i \neq T$$

4. Isn't this just CAPM?

   - This is "like" CAPM. But there's one problem - it does NOT help us identify $\mathbb{E}(r_i) - r_f$, but merely states the relationship between $\mathbb{E}(r_i) - r_f$ and $\mathbb{E}(r_T) - r_f$.

   - The problem: To use this formula to find $\mathbb{E}(r_i) - r_f$, we need to know what the tangency portfolio is. But to find the tangency portfolio, we need to maximize the Sharpe ratio, which requires us to know the expected returns of all assets $\rightarrow$ this is circular!

   - CAPM uses economic reasoning and identifies what the tangency portfolio MUST be, and in doing so we get around this circular problem!