Why Geometric Average < Arithmetic Average

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Proposition 1 *The geometric average for a sequence of non-negative numbers* $(R_1, ..., R_n)$ *is always smaller than the arithmetic average*

Proof. Let's use Jensen's Inequality

$$E\left(f\left(X\right)\right) < f\left(E\left(X\right)\right) \text{ for f strictly convex}$$

Geometric average can be written as

$$\left(\prod_{i=1}^{T} R_i\right)^{1/T} = \exp\left(\frac{1}{T}\sum_{i=1}^{T}\log\left(R_i\right)\right) = \exp\left(\mathbb{E}_{\hat{p}}\log\left(R_i\right)\right)$$

Arithmetic average is

$$\frac{1}{T}\sum R_i = \exp\left(\log\left(\frac{1}{T}\sum R_i\right)\right) = \exp\left(\log\left(\mathbb{E}_{\hat{p}}R_i\right)\right)$$

Now, since log is strictly concave, by Jensen's

$$\mathbb{E}_{\hat{p}}\log\left(R_i\right) < \log\mathbb{E}_{\hat{p}}R_i$$

Since exponentiating is a monotonic transformation,

$$\exp\left\{\mathbb{E}_{\hat{p}}\log\left(R_i\right)\right\} < \exp\left\{\log\mathbb{E}_{\hat{p}}R_i\right\}$$

or

$$\left(\prod_{i=1}^{T} R_i\right)^{1/T} < \frac{1}{T}\sum R_i$$

QED ■

Note: Gross returns is always non-negative, since the most you can lose is your whole investment ($R = 0$)!