1. Closed Under Addition: For each x, y in the set, x+y also in the set.
2. Closed Under Scalar Multiplication: For each x in the set, kx is also in the set ($k \in \Re$)
3. Subspace: A subset of a vector space that is also a vector space. **A nonempty subset S of a vector space V is a subspace iff** $\forall x, y \in S, a, b \in \Re, ax + by \in S$ **(MUST SHOW 0 is in the set to verify non-emptiness)**
4. Linear Combination: Let $\{\vec{v}_1, ..., \vec{v}_k\}$ be a subset of vectors in a vector space V. A linear combination is a vector of the form $c_1\vec{v}_1 + ... + c_k\vec{v}_k, c_i \in \Re$. Linear combination of a set of vectors in R^N is **a location the vectors can take us**.
5. Span: The span of a set of vectors contained in a vector space V is the **set of all possible linear combinations** of that set. i.e. the set of all possible locations that a set of vectors in R^N can take us.
6. Linear Transformations: "how vector spaces communicate with each other." Let V and W be vector spaces. A linear transformation is a **Function** ***L*: V→W** that satisfies 2 properties: 1) $L(v_1+v_2)=L(v_1)+L(v_2)$ $\forall$ $v_1,v_2 \in$ V 1) $L(cv)=cL(v)$ $\forall$ v∈ V, c∈ R (V- Domain W- Codomain)
7a. Domain and Co-domain: Given a linear transformation f: $R^N \rightarrow R^M$ represented by A_{mxn}, the domain of f is R^N and codomain is R^M
7b. Image / Column Space of A: The span of the columns of A. It consists of all the values the function takes in its codomain.
$\{\vec{v} : \exists\ \vec{w}$ s.t. $A\vec{w} = \vec{v}\}$ or $\{\vec{y} \in Y : A\vec{x} = \vec{y}$ for some $\vec{x} \in X\}$ → A SET!
7c. Kernel(A) / Null Space: The **set** of vectors x (in the domain X) s.t. Ax = 0 (the transformation returns the 0 vector in the codomain).
8. Linear dependence: A set of vectors in a vector space V is linearly dependent if **at least one** of the vectors can be written as a linear combination of the other vectors (i.e. has redundancy). That is, set of vectors $S = \{\vec{v}_1, ..., \vec{v}_k\} \subseteq V$ is **linearly dependent iff** $c_1\vec{v}_1 + ... + c_k\vec{v}_k = \vec{0}, c_i \in \Re$ **has at least one** $c_i \neq 0$ **(at least one non-trivial solution).**
9. Basis: A subset B of a vector space V is a basis if **1. B is linearly independent and 2. Span B = V** (show Span(B) = $MAT_n(R)$)
9a. $MAT_n(R)$: Set of nxn matrices with real entries.
10. Dimension: Dimension of a vector space V is the number of elements in any basis.
11. Symmetric Matrix: If $A = A^T$, then A is said to be symmetric.
12. Identity Matrix (I_n): There exists an nxn identity matrix for all integer n such that $I_n\vec{x} = \vec{x}$ for all $\vec{x}$ in R^N. (For A_{mxn}, $I_mA = AI_n=A$)
13. 1-1: Let f: x→Y be a function, then f is 1-1 (injective) if for each $y \in Y$ there exists **at most** one $x \in X$ such that f(x) = y.
14. Onto: Let f: x→Y be a function, then f is onto (surjective) if for each $y \in Y$ there exists at least one $x \in X$ such that f(x) = y.
15. Rank-Nullity: For any mxn matrix A, dim(kerA) + dim(imA) = n ⇔ nullity of A + rank(A) = n
⇔ # of dependent vars/columns + # of leading vars/lin indep columns = # of total vars.

Rank: # of leading 1s in rref(A).
Properties of Rank: 1. Rank(A)<=m, Rank(A)<=n for all mxn matrix A.
2. If Rank(A) = m then system is consistent → no 0 row. (But can have either unique solution or infinitely many solutions).
3. If Rank(A) = n then system has **at most** 1 solution. (has 0 solution if inconsistent, i.e. when m>n with incons row).
4. If Rank(A) < n then system has either 0 (if inconsistent) or infinitely many solutions (if consistent, but there's free vars).
5. If Rank(A) = m = n, then rref(A) = I_n (square matrix, invertible).

Solutions of Linear Systems: A linear system is said to be…
1. Consistent if there exists **at least** 1 solution (a. exactly 1 if all var's are leading and rank = m = n. b. infinite if there is at least 1 free var.
2. Inconsistent (i.e. no solutions) iff the rref form of A in augmented form contains a row $[0 \quad 0 \quad 0 \quad 0 \vdots 1]$

Types of Matrices: (NOTE: **L is a line that passes through the origin**)

1. Scaling (by k): $\begin{matrix} k & 0 \\ 0 & k \end{matrix}, k \in \Re$ **2. Projection** (of $\vec{x}$ onto L in R^2): $\begin{matrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{matrix}$, $\vec{u} = \begin{matrix} u_1 \\ u_2 \end{matrix}$ is a unit vector parallel to L.

3. Reflection (of $\vec{x}$ about L in R^2): $\begin{matrix} a & b \\ b & -a \end{matrix}$, $a^2 + b^2 = 1$ Note: $A(\vec{x}) = ref_L(\vec{x}) = proj_L(\vec{x}) - (\vec{x} - proj_L(\vec{x})) = 2proj_L(\vec{x}) - \vec{x}$

Note2: $Ref_L = 2Proj_L - I$ (These refer to matrices. Once you have the projection matrix we can get a reflection matrix)

4. Rotation (of $\vec{x}$ in R^2): $\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}$ or $\begin{matrix} a & -b \\ b & a \end{matrix}$ where $a^2 + b^2 = 1$

5. Rotation w/ Scaling (of $\vec{x}$ in R^2): $\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}$ or $\begin{matrix} a & -b \\ b & a \end{matrix}$ where $a^2 + b^2 \neq 1$, and $a = r\cos\theta$, $b = r\sin\theta$

6. Sheer: a) Horizontal Sheer: $\begin{matrix} 1 & k \\ 0 & 1 \end{matrix}$ b) Vertical Sheer: $\begin{matrix} 1 & 0 \\ k & 1 \end{matrix}$, $k \in \Re$

Span: 1. Span of a nonzero vector is a line. 2. Span of 2 nonzero, linearly independent vectors is a plane. (all in R^N)
Image: $A\vec{x} = \vec{b}$ has a solution as long as $\vec{b} \in \text{Im}(A)$. i.e. $\vec{b}$ must be contained in the set of all possible lin. combinations of the columns of A.

Image and Kernel: Are both **linear subspaces** → closed under addition, closed under scalar multiplication, contains $\vec{0}$ vector (in R^N/ R^M).
For a linear transformation T from $R^M \rightarrow R^N$, represented by A_{nxm}, **Im(A) is a subset of the codomain R^N and Ker(T) is a subset of the domain R^M**.
Dimension of Im and Ker: Dim(KerA) = # of vectors in the Kernel/Null space (or # of lin. dependent columns in A).
Dim(Ker A) = # of lin. ind. columns of A = rank(A).

Basis of ImA and KerA: Simply row-reduce A (in *implied* augmented matrix with $\vec{b} = \vec{0}$) to find the linearly dep. and independent columns.
Rank(BA)≤Rank(B) and Nullity(BA)≥Nullity(B) → Bc Im(BA) is contained in Im(B) and Ker(A) is contained in Ker(BA).

When is Ker(A) = {0}? a) for a nxn square matrix A, ker(A) = {0} iff A is invertible (i.e. columns are lin ind, etc… see below)
b) Generally, for mxn matrix A, Ker(A) = {0} iff rank(A) = n. (implies $n \le m$, since $n = \text{rank}(A) \le n$)

Linear Transformation: Every linear transf. from a finite dim. VS to a finite dim. VS can be represented by a matrix
Let *L*: V→W be a linear transformation between the vector spaces V and W. The matrix representing *L* is given by
A= $\left[L(\vec{e}_1), L(\vec{e}_2), \ldots, L(\vec{e}_n)\right]$ in standard basis → A$\underline{e}_1$ = first column of A, A$\underline{e}_2$ = second column of A etc…
Or A= $\left[L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_n)\right]$ where all the vectors are in standard coordinates.

Matrix Multiplication – Properties $\forall$ nxn square matrix A

1. Associative: $A(BC) = (AB)C, (kA)B = k(AB)$ **2. Distributive**: $(A+B)C = AC + BC$ **3. Rarely Commutative**: $AB \neq BA$ (AI=IA)

3. Identity: Given invertible matrix nxn A there exists A^{-1} s.t. $A^{-1}A = I_n$ **4. Invertibility**: $(BA)^{-1} = B^{-1}A^{-1}$ exists when A, B both invertible.

5. $B_{nxn}A_{nxn} = I_n \Rightarrow A = B^{-1}, B = A^{-1}, AB = B^{-1}A^{-1} = I_n \Rightarrow A, B$ invertible by 4.

6. Linearity: Matrix product is linear. A(C+D) = AC+AD, (A+B)C = AC+BC, (kA)B = k(AB) = A(kB) given k scalar.

7. Matrix in Summation Form: Each entry in a matrix product is a dot product, so $B_{mxn}A_{nxp} = C_{mxp}, c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$

8. Matrices Can be Partitioned (then multiplied): Partitioned matrices can be multiplied as though the submatrices (on the lhs) are scalars.

9. Matrix Multiplication is Function Composition: If $T(\vec{x}) = A\vec{x} = \vec{y}, S(\vec{y}) = B\vec{y} = \vec{z}$, then $C = B_{mxn}A_{qxp} = S(T(\vec{x})) = \vec{z}$
(and matrix-vector multiplication is a linear transformation) (co-domain of A, *q*, must be same as domain of B, *n*. That is, q = n.)

Subspace and Independent Vectors and Spanning Vectors: A subspace V of R^N with dim(V) = m …
1. We can find at most m linearly independent vectors in V. 0 Tcind12.02c9 068 25.361 ( )6( )6( )6( )6( )66( )62 Tm[(1. 06(W(, qeed indlea2(o)-6

Orthogonality/Orthonormality

Orthonormal Vectors: A set of vectors are orthonormal if they are all unit vectors and orthogonal to one another

For $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ in R^N

1. $\|\underline{u}_1\| = \|\underline{u}_2\| = \ldots = \|\underline{u}_n\| = 1$ [Length/Magnitude/Norm $\|\underline{v}\| = \sqrt{\vec{v} \bullet \vec{v}}$]
2. $\underline{u}_i \cdot \underline{u}_j = 0$ (if $i \neq j$) [$\underline{u}$, $\underline{v}$ perpendicular/orthogonal iff $\underline{u} \cdot \underline{v} = 0$]
   [$\underline{u}$, $\underline{v}$ orthogonal iff $\|\underline{u}\|^2 + \|\underline{v}\|^2 = \|\underline{u}^2 + \underline{v}^2\|$]
3. $\underline{u}_i \cdot \underline{u}_j = 1$ (if i = j) [$\underline{u}$ is a unit vector if its length is 1, i.e. $\|\underline{v}\| = \underline{v} \cdot \underline{v} = 1$]
4. Orthonomal Vectors are linearly independent [Prove this w. successive dot products]
5. Orthonormal vectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ form a basis of R^N [Any n linearly independent vectors in R^N forms a basis]

ANY VECTOR $\underline{X}$ IN R^N IS A LINEAR COMBINATION OF $P_V(\underline{X})$ AND $X^{\|}$ FOR SOME SUBSPACE V OF R^N

Given any vector $\underline{x}$ in R^N and any (**arbitrary**) subspace V of R^N, we can express $\underline{x}$ as a sum of perp and projection on to V:

$\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp}$ [$\vec{x}^{\|}$ is the orth. Proj. of $\underline{x}$ onto V and $\vec{x}^{\perp}$ is perpendicular/orthogonal complement to V]

Orthogonal Projections: $\underline{x}^{\|}$ (Note: Orthogonal Projections are **linear** and are **NOT orthogonal** transformations)

1. Given a subspace V of R^N with an orthonormal basis $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m$: $\vec{x}^{\|} = P_V(\underline{x}) = (\underline{u}_1.\underline{x})\underline{u}_1 + (\underline{u}_2.\underline{x})\underline{u}_2 + \ldots + (\underline{u}_m.\underline{x})\underline{u}_m$
   Note: Given any orthonormal basis $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ of R^N, $\underline{x} = (\underline{u}_1.\underline{x})\underline{u}_1 + (\underline{u}_2.\underline{x})\underline{u}_2 + \ldots + (\underline{u}_n.\underline{x})\underline{u}_n$
   [Remember that the coordinates of a vector in any basis is just the constants of the linear combination of the basis elements.]
2. Given a $\underline{x}$ in R^N and a subspace V of R^N, $\|\underline{x}\| \geq \|Proj_V(\underline{x})\|$ or $\vec{x}^{\|}$ [$\|\underline{x}\|^2 = \|\underline{x}^{\|}\|^2 + \|\vec{x}^{\perp}\|^2 \rightarrow \|\underline{x}\| = \|\underline{x}^{\|}\|$ **iff** $\|\vec{x}^{\perp}\| = 0$]
   [Cauchy Inequality. **The statement is an equality iff $\underline{x}$ is in V**]
3. Given a $\underline{x}$ and $\underline{y}$ in R^N and a subspace V of R^N, $\|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\|\,\|\underline{y}\|$
   [Cauchy-Scwartz Inequality. **The statement is an equality iff $\underline{x}$ and $\underline{y}$ are parallel**]

   Derivation: $\|\vec{x}\| \geq \|(\vec{x} \bullet \vec{u})\vec{u}\| = |\vec{x} \bullet \vec{u}| = \left|\vec{x} \bullet \frac{\vec{y}}{\|\vec{y}\|}\right| = \frac{1}{\|\vec{y}\|}\left|\vec{x} \bullet \vec{y}\right|$ ($\underline{u}$ is a unit vector on a line spanned by $\underline{y}$)

   Note $\|k\underline{v}\| = |k|\,\|v\|$, so $\|(\underline{x}.\underline{u})\underline{u}\| = |\underline{x}.\underline{u}|$
4. Matrix of an orthogonal projection of $\underline{x}$ in R^N onto a subspace V of R^N can be constructed as…
   $Proj_V(\underline{x}) = QQ^T$ where Q is an orthogonal matrix composed of an orthonormal basis of V.
5. Orthogonal projection of $\underline{x}$ in R^N onto a subspace V of in R^N can be thought of as **the vector in V closest to $\underline{x}$**

Orthogonal Complement: $\vec{x}^{\perp}$ (Given a subspace V of R^N, the orthogonal complement $V^{\perp}$ of R^N…)

Def: Orthogonal complement of V is $V^{\perp} = \{\vec{x} \in R^N : \vec{v} \bullet \vec{x} = \vec{0} \forall \vec{v} \in V\}$

1. $V^{\perp} = Ker(T)$ for $T(\underline{x}) = P_V(\underline{x})$ [$V^{\perp}$ = {$\underline{x}$: $\underline{v} \cdot \underline{x} = 0$ for all $\underline{v}$ in V = im(T)} → $\underline{x}$ in Ker(T)]
2. $V^{\perp}$ is a subspace of R^N [$V^{\perp}$ = Ker(T) for T which projects onto V which is a subspace of R^N]
3. $V \cap V^{\perp} = \{\vec{0}\}$ [If $\underline{x}$ in V and $V^{\perp}$, then $\underline{x}$ is orthogonal to itself, i.e. $\underline{x} \cdot \underline{x} = 0 \rightarrow \underline{x} = 0$]
4. $Dim(V) + Dim(V^{\perp}) = n$ [Dim(V) = Rank(T), Dim($V^{\perp}$) = Dim(Ker(T)) = Nullity(T)]
5. $(V^{\perp})^{\perp} = V$ [$V^{\perp}$ = {$\underline{x}$:$\underline{v}.\underline{x}$ = 0 for $\underline{v}$ in V}→ $(V^{\perp})^{\perp}$ = {$\underline{u}$: $\underline{x}.\underline{u}$ = 0 for $\underline{x}$ in $V^{\perp}$ }→$\underline{u}$ in V]
6. $[\text{Im}(A)]^{\perp} = Ker(A^T)$ [$[\text{Im}(A)]^{\perp}$ = {$\underline{x}$: $\underline{v}.\underline{x}$ = 0 for all $\underline{v}$ in V} = {$\underline{x}$: $\underline{v}^T\underline{x}$ = 0 for all $\underline{v}$ in V}]
   for **any** matrix A s.t. im(A) = subspace V [= Ker(A^T)]

Orthogonal Transformations

Properties of Orthogonal Matrix A. A orthogonal **IFF**…

⇔ The trans $T(\underline{x})=A\underline{x}$ preserves length [Definitional ~ T an orthogonal trans if preserves length]

⇔ The trans $T(\underline{x})=A\underline{x}$ preserves orthogonality [if $\underline{x},\underline{y}$ orth, $\|T(\underline{x})+T(\underline{y})\|^2=\|T(\underline{x}+\underline{y})\|^2=\|\underline{x}+\underline{y}\|^2=\|\underline{x}\|^2+\|\underline{y}\|^2=\|T(\underline{x})\|^2+\|T(\underline{y})\|^2$]

⇔ The trans $T(\underline{x})=A\underline{x}$ preserves dot product [if $\underline{x},\underline{y}$ orth, $A\underline{x} \cdot A\underline{y})=(A\underline{x})^T(A\underline{y})=\underline{x}^TA^TA\underline{y}=\underline{x}^T\underline{y}=\underline{x}\cdot\underline{y}$]

⇔ Columns of A form an orthonormal basis of R^N

⇔ $A^TA = I_N$

⇔ $A^T = A^{-1}$

> EXAMPLES: **Rotations** and **Reflections** are transformations that preserve length.
> Properties: If A,B orthogonal and k a constant, then AB, kA, orthogonal,

QR Factorization and Gram-Schmidt

1. Gram-Schmidt algorithm represents a CHANGE OF BASIS from old basis to a new orthonormal basis *U* of V.
2. Given any nxm matrix M with linearly independent columns then there exists an orthogonal matrix Q and diagonal matrix R such that: **M = QR** **(This representation is UNIQUE)**

Least Squares Solution/Approximation

For a system $A\underline{x} = \underline{b}$ that is inconsistent (i.e. $\underline{b}$ is not in the im(A) = subspace V), the solution vector $\underline{x}$ can be approximated by the **vector $\underline{x}^*$ in R^N such that $A\underline{x}^*$ (in V) is closest to b**. (See orthogonal projection above)

$\underline{x}^*$ is the least squares solution of the system $A\underline{x} = \underline{b}$ for any mxn matrix A…

⇔ $\|\underline{b} - A\underline{x}^*\| \leq \|\underline{b} - A\underline{x}\|$ for all $\underline{x}$ in R^N

⇔ $A\underline{x}^* = Proj_V(\underline{b})$, where V = im(A)

⇔ $\underline{b} - A\underline{x}^*$ ($=\vec{b} - b^{\|} = b^{\perp}$) in $V^{\perp} = (im(A))^{\perp} = Ker(A^T)$

⇔ $A^T(\underline{b} - A\underline{x}^*) = \underline{0}$

⇔ $A^T\underline{b} = A^T A\underline{x}^*$ ("normal equation")

→ if A invertible, then A^TA invertible (see below), and we have a **unique least squares solution:**

$$\underline{x}^* = (A^T A)^{-1}A^T\underline{b}$$

> Note: If $A\underline{x} = \underline{b}$ is consistent, then the least squares solution is the EXACT solution (since the error would be 0, and its orthogonal projection onto V is itself).

Transpose: A^T

1. $(A+B)^T = A^T+B^T$

1. $(AB)^T = B^TA^T$

2. $(A^T)^{-1} = (A^{-1})^T$ if A invertible [$AA^{-1}=I_n$ → $(AA^{-1})^T=(I_n)^T$ → $(A^{-1})^TA^T=I_n$ → $(A^T)^{-1} = (A^{-1})^T$]

3. rank(A) = rank(A^T) for any A

4. Ker(A) = Ker(A^TA) for any nxm matrix A. [$Ker(A) \subseteq Ker(A^TA)$, $Ker(A^TA) \subseteq Ker(A)$]

5. If Ker(A) = {$\underline{0}$} then A^TA is invertible for any nxm matrix A [$Ker(A^TA) = Ker(A) = \{\underline{0}\}$]

6. Det(A) = Det(A^T) for square matrix A

7. Dot Product: $\vec{v} \bullet \vec{u} = \vec{v}^T\vec{u}$

8. For Orthogonal Matrices: $A^TA = I_n$ ⇔ $A^{-1} = A^T$

9. For Matrix of Orthogonal Projection (of x onto subspace V)**:** $P_V(x) = QQ^T$ [Columns of Q = orthonormal basis of V]

10. Quadratic Forms: $q(\vec{x}) = \vec{x} \bullet A\vec{x} = \vec{x}^T A\vec{x}$

Determinant

Calculating Determinant

1. For 2x2 matrix $A = \begin{matrix} a & b \\ c & d \end{matrix}$ $\det A = ad - bc$

2. For 3x3 matrix $A = \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$

3. Laplace Expansion: We can f9ind det. of nxn matrix A by Laplace expansion down any column or across any row of A

Expansion across ith row: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$ for fixed row i

Expansion across jth column: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$ for fixed column j

4. Gauss Elimination: Using the antisymmetric and multilinear properties, we can reduce A into a simpler matrix B such that Det(B) is easy to compute, then multiply by the appropriate constants and sign changes to back out **Det(A) = $(-1)^S k_1 k_2 \ldots k_r$Det(B)** [Swap rows **s** times and divide rows by scalars **$k_1 \ldots k_r$** to go from A to

5. Geometric / Columns of A: For invertible nxn matrix A = [$\underline{v}_1 \ldots \underline{v}_n$]…

|Det(A)| = |Det(QR)| = |Det(Q)||Det(R)| = |Det(R)| = $\| \vec{v}_1 \| \| \vec{v}_2^{\perp} \| \| \vec{v}_3^{\perp} \| \ldots \| \vec{v}_n^{\perp} \|$ → from Gram-Schmidt

= Volume of m-dimensional paralellopepid formed by $\underline{v}_1 \ldots \underline{v}_n$ → columns of A

[see below for general case]

6. Product of Eigenvalues: For nxn matrix A (diagonalizeable) with eigenvalues, listed with their algebraic multiplicities $\lambda_1, \lambda_2, \ldots \lambda_n$, then Det(A) = $\lambda_1 \lambda_2 \ldots \lambda_n$

Properties

1. Det(A^T) = Det (A) [So everything that is true for rows is true for columns]
2. If A has a row/column of 0's then Det (A) = 0 [Rows/Columns are linear dependence]
3. If A has 2 identical rows/columns then Det(A) = 0 [Rows/Columns are linear dependence]
4. Antisymmetric: If we switch 2 columns of A, Det(A) changes sign. [As a result of the sign map in the det. expansion]
5. Multilinear:
   a. Scalars can be factored out of col/row, so if we factored out k from a column, then det(A) = (1/k)det(A')
   b. We can break up a row/column to compute A
6. Det (I_n) = 1
7. Adding a multiple of one col/row to another does not change the Det (A) [We can row reduce A w/o changing the det]
8. Det (AB) = Det(A) Det(B)
9. If A invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$ [A invertible → $AA^{-1} = I_n$ →det(A)det(A^{-1})=1]
10. For **upper/lower** triangular A, Det(A) = product of A's diagonal entries. [Expand on row/col that has all 0's]
11. For a **partitioned matrix** (if we can partition A into 4 square matrices, not necessarily of the same size), then

$$\det(M) = \det \begin{matrix} A & B \\ 0 & D \end{matrix} = \det(A)\det(D)$$

[Det(A)Det(D)-Det(B)Det(C) does not generally hold for sq matrices A,B,C,D]

12. For **similar matrices** A and B, Det(A) = Det(B) [B=S^{-1}AS, Det(B) = Det(S^{-1})Det(A)Det(S) = Det(A)]
13. For **orthogonal matrix** A, Det(A) = 1 or -1 [$A^TA = I_n$, 1 = Det(A^TA) = Det(A^T)Det(A) = Det(A)^2]
14. For nxn matrix A (diagonalizeable) with eigenvalues, listed with their algebraic multiplicities $\lambda_1, \lambda_2, \ldots \lambda_n$, then Det(A) = $\lambda_1 \lambda_2 \ldots \lambda_n$

Geometric Interpretation

1. Parallelepipeds in R^N: Given **linearly independent** vectors $\underline{v}_1 \dots \underline{v}_m$ in R^N, the m-dimensional parallelepipeds defined by these vectors has the volume, $V(\underline{v}_1 \dots \underline{v}_m)$ defined recursively by …

$V(\underline{v}_1) = \|\underline{v}_1\|$ and $V(\underline{v}_1 \dots \underline{v}_m) = V(\underline{v}_1 \dots \underline{v}_{m-1})\|\vec{v}_m^{\perp}\|$ ←this mimics the formula (base)(height)

($\vec{v}_m^{\perp}$ is the orth. Complement of $\underline{v}_m$ onto the subspace V spanned by $\underline{v}_1 \dots \underline{v}_{m-1}$, as defined by Gram-Schmidt algorithm)

2. Volume of a parallelepiped in R^N: For a nxm matrix $A = [\underline{v}_1 \dots \underline{v}_m]$, the volume of the parallelepiped defined by linearly **independent vectors** $\underline{v}_1 \dots \underline{v}_m$:

$$\boxed{V(\vec{v}_1,\dots,\vec{v}_m) = \sqrt{\det(A^T A)}}$$

Pf: $A = QR \rightarrow A^TA = (QR)^T(QR)=R^TQ^TQR=R^TR$

$\text{Det}(A^TA)=\text{Det}(R^TR)=\text{Det}(R^T)\text{Det}(R) = \text{Det}(R)^2 = (r_{11}r_{22}\dots r_{mm})^2 = \left(\|\vec{v}_1\|\|\vec{v}_2^{\perp}\|\dots\|\vec{v}_m^{\perp}\|\right)^2 = V(\vec{v}_1,\dots,\vec{v}_m)^2$

In particular, if m = n, then $V(\vec{v}_1,\dots,\vec{v}_n) = \sqrt{\det(A^T A)} = \sqrt{\det(A^T)\det(A)} = \sqrt{\det(A)^2} = |\det(A)|$

3. Expansion Factor:

Diagonal Matrices

1. QR Factorization: Any nxm matrix can be represented

Symmetric Matrices $A = A^T$ (Skew symmetric: $A^T = -A$)

1. $A + A^T$ always symmetric. $[(A+A^T)^T = A^T + A = A + A^T]$

Similar Matrices: A & B similar there exists a change of basis matrix S such that $\mathbf{B = S^{-1}AS}$

1. Same Determinant: $\text{Det } B = \text{Det } S^{-1}AS = \text{Det } A$
2.

Im(A) =…

1. $\text{Im}(A) = [Ker(A^T)]^{\perp}$
2. $\text{Im}(A) = \text{Im}(AA^T)$

Proof: $\text{Im}(A) = [Ker(A^T)]^{\perp} = [Ker(AA^T)]^{\perp} = [Ker(AA^T)^T]^{\perp} = [\text{Im}(AA^T)^{\perp}]^{\perp} = \text{Im}(AA^T)$

3. $\text{Im}(A) \subseteq \text{Im}(AB)$