MV Fromtier with N Risky Assets

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Summer 2009

1 Mean Variance Frontier with N Risky Assets

Definition 1 *Given N risky assets with non-degenerate security returns (so that $\sum^{-1}$) exists. The mean variance frontier is characterized as the portfolios $w(\mu)$ such that for each level of expected return (μ), $w(\mu)$ minimizes risk, i.e.*

$$\min_w w^T \Sigma w \quad \text{s.t. } w^T 1 = 1 \text{ and } w^T E = \mu$$

1. First Order Condition of the problem

$$£ = w^T \Sigma w + \lambda \left(w^T \mathbf{1} - 1 \right) + \gamma \left(w^T E - \mu \right)$$

$$\text{FOC w.r.t. } w^T\text{: } \frac{\partial £}{\partial w^T} = 2\Sigma w + \lambda \mathbf{1} + \gamma E = 0$$

$$\Rightarrow w = -\frac{1}{2}\Sigma^{-1}(\lambda \mathbf{1} + \gamma E) \ = \Sigma^{-1}(\lambda^* \mathbf{1} + \gamma^* E) \tag{1}$$

where $\lambda^* = -\frac{1}{2}\lambda$ and $\gamma^* = -\frac{1}{2}\gamma$

2. Deriving the frontier

$$\text{Pre-multiply (1) by } \mathbf{1}^T\text{: } 1 = \lambda^* \underbrace{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}_{A} + \gamma^* \underbrace{\mathbf{1}^T \Sigma^{-1} E}_{B} \tag{2}$$

$$\text{Pre-multiply (1) by } \mathbf{E}^T\text{: } \mu(w) = \lambda^* \underbrace{E^T \Sigma^{-1} \mathbf{1}}_{B} + \gamma^* \underbrace{E^T \Sigma^{-1} E}_{C} \tag{3}$$

Let...

(a) $A = \mathbf{1}^T \Sigma^{-1} \mathbf{1} > 0$ since quad form and Σ^{-1} p.d.

(b) $B = E^T \Sigma^{-1} \mathbf{1}$

(c) $C = E^T \Sigma^{-1} E > 0$ since quad form and Σ^{-1} p.d.

(d) $AC - B^2$ (you can also show that this is >0)

From (2) & (3)

$$\left.\begin{array}{l}\lambda^* A+\gamma^* B=1 \\ \lambda^* B+\gamma^* C=\mu\end{array}\right\} \Rightarrow \lambda^*=\frac{C-\mu B}{D}, \quad \gamma^*=\frac{\mu A-B}{D}$$

Thus,

$$\begin{aligned} w^*(\mu; E, \Sigma) &= \Sigma^{-1}\left(\lambda^* \mathbf{1}+\gamma^* E\right) \qquad (4)\\ &= \frac{C-\mu B}{D} \Sigma^{-1} \mathbf{1}+\frac{\mu A-B}{D} \Sigma^{-1} E \\ &= \mu \underbrace{\left(\frac{A}{D} \Sigma^{-1} E-\frac{B}{D} \Sigma^{-1} \mathbf{1}\right)}_{h}+\underbrace{\left(\frac{C}{D} \Sigma^{-1} \mathbf{1}+\frac{B}{D} \Sigma^{-1} E\right)}_{g} \end{aligned}$$

and

$$\begin{aligned} \sigma^{*2}(\mu; E, \Sigma) &= w^{*T} \Sigma^{-1} w^* \qquad (5)\\ &= \ldots \\ &= \frac{A}{D} \mu^2-\frac{2B}{D} \mu+\frac{C}{D} \end{aligned}$$

3. Observe...
   (a) Minimum variance portfolio (given an expected return) has portfolio weights that are linear in μ
   (b) and variance that is quadratic in μ.

Proposition 2 *For a mean porfolio return of μ, the minimum variance portfolio with expected return of μ has variance*

$$\sigma^{*2}(\mu; E, \Sigma)=\frac{A}{D} \mu^2-\frac{2B}{D} \mu+\frac{C}{D}$$

and portfolio weights

$$w^*(\mu; E, \Sigma)=\mu h+g$$

Proof. We showed this above... ■

Proposition 3 *MV portfolios form a convex set (that is, if w_1, w_2 are mv frontier portfolios with μ_1, μ_2. Then, $\forall \lambda \in \mathbb{R}$, $\lambda w_1+(1-\lambda) w_2$ also a frontier portfolio.*

Proof.

$$\left.\begin{array}{l} w(\mu_1)=g+h \mu_1 \\ w(\mu_2)=g+h \mu_2 \end{array}\right\} \Rightarrow \underbrace{\lambda w(\mu_1)+(1-\lambda) w(\mu_2)}_{w_\lambda}=g+h\left(\underbrace{\lambda \mu_1+(1-\lambda) \mu_2}_{\mu_\lambda}\right)$$

By prop above, we see that the weighted average portfolio is a min-var portfolio with mean μ_λ. ■

Proposition 4 *Mean, variance and covariance of the (overall) minimum variance portfolio (MVP) is given by...*

$$\mu_{mvp} = \frac{B}{A}$$

$$\sigma^2_{mvp} = \frac{1}{A}$$

$$Cov\left(r_i, r_{mvp}\right) = \frac{1}{A} = \sigma^2_{mvp} \quad \forall i \neq mvp$$

Proof. MVP is given by the following problem

$$\min_{\mu} \sigma^2\left(\mu\right) = \min_{\mu} \frac{A}{D}\mu^2 - \frac{2B}{D}\mu + \frac{C}{D}$$

FOC wrt μ yields

$$2\frac{A}{D}\mu = 2\frac{B}{D} \Rightarrow \mu_{mvp} = \frac{B}{A}$$

Plugging back into the variance formula yields

$$\begin{aligned} \sigma^2\left(\mu_{mvp}\right) &= \frac{A}{D}\left(\frac{B}{A}\right)^2 - \frac{2B}{D}\left(\frac{B}{A}\right) + \frac{C}{D} \\ &= \frac{1}{A} \end{aligned}$$

Covariance is given by

$$\begin{aligned} Cov\left(r_i, r_{mvp}\right) &= Cov\left(w_p r, w_{mvp} r\right) \\ &= w_p^T \Sigma w_{mvp} \\ &= w_p^T \Sigma \left(g + h\frac{B}{A}\right) \\ &= w_p^T \Sigma \left[\frac{C}{D}\Sigma^{-1}\mathbf{1} + \frac{B}{D}\Sigma^{-1}E + \left(\frac{A}{D}\Sigma^{-1}E - \frac{B}{D}\Sigma^{-1}\mathbf{1}\right)\frac{B}{A}\right] \\ &= w_p^T \Sigma \left[\frac{1}{A}\Sigma^{-1}\mathbf{1}\right] \\ &= \frac{1}{A} \end{aligned}$$

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