Review Materials for Lecture Notes 4 ~ 7

Provides an analytical framework to evaluate attractiveness of assets / Portfolio.

Assume ppl have utility over mean & std of asset returns

- 3 Stage analysis for portfolio / asset allocation
  1. Make forecasts about $E(r_i)$ and $\sigma(r_i)$ $\forall$ i.
  2. Figure out the investment opportunity set i.e.) all feasible
  2. [and in particular the "frontier"]
  3. Figure out, from utility function, the Most "attractive" choice of investment

$\boxed{\text{Insert Portfolio Math}} \rightarrow$

$\boxed{\text{Insert MV Frontier}} \rightarrow$

$U(E, S) = E - \frac{1}{2}\alpha S^2$

Note: When you combine ~~port~~ assets, you get portfolio variance improvement so long as $\rho < 1$.

$\boxed{\text{Insert Separation Prop}} \rightarrow$

$\boxed{\text{eg}}$: slide 15 gives a special case where you have r_f and $r_{S\&P}$ 500 : 2 Asset Allocation problem.

$$w^* = \frac{E_{S\&P} - r_f}{\alpha S^2_{S\&P}}$$

$\boxed{\text{Note}}$: More risk averse $\Rightarrow w^* \downarrow$.

$E_{S\&P} \uparrow \Rightarrow w^* \uparrow$

$S^2_{S\&P} \uparrow \Rightarrow w^* \downarrow$

- Portfolio Math

  - Exp Ret: $E(r_p) = E\left(\sum_i w_i r_i\right) = \underline{\sum_i w_i E(r_i)}$

  - Var : $Var(r_p) = Var\left(\sum_i w_i r_i\right)$

$$= Cov\left(\sum w_i r_i, \sum w_i r_i\right)$$
$$= Cov\left(w^T \vec{r}, w^T \vec{r}\right)$$
$$= w^T Cov(\vec{r}, \vec{r})\, w$$
$$= w^T \Sigma\, w$$

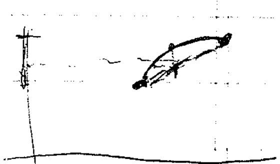
2-asset case:

$$Var(r_p) = Var\left(w_1 r_1 + (1-w_1) r_2\right)$$
$$= w_1^2 Var\, r_1 + (1-w_1)^2 Var\, r_2 + 2 w_1 (1-w_1) Cov(r_1, r_2)$$

delay til later.

$$= w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2 w_1 (1-w_1) \rho_{12} \sigma_1 \sigma_2$$

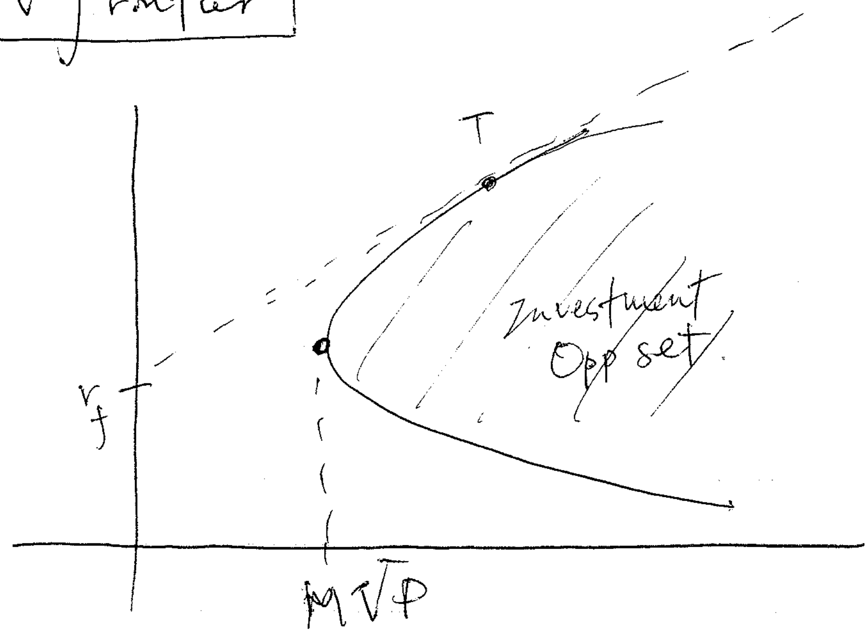
$$\begin{cases} = (w_1 \sigma_1 + (1-w_1)\sigma_2)^2 & \text{if } \rho_{12} = 1 \\ < & \text{if } \rho_{12} < 1 \end{cases}$$



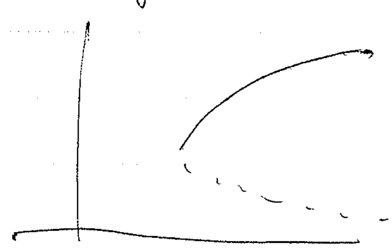
Note: This says that so long as 2 assets are not perfectly correlated ($\rho_{12} < 1$), there are <u>benefits to diversification</u> b/c weighted avg of variances is always <u>larger</u> than variance of portfolio.

- $\rho_{12} = \dfrac{Cov(r_1, r_2)}{\sigma_1 \sigma_2}$ (unitless measure).
- $\dfrac{\partial Var(r_p)}{\partial w_i} = Cov(r_i, r_p)$.

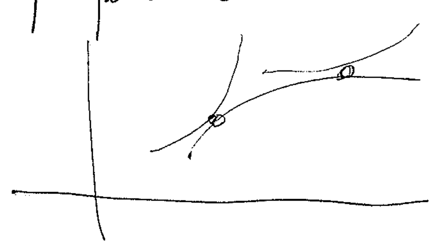
$\boxed{\text{MV Frontier}}$



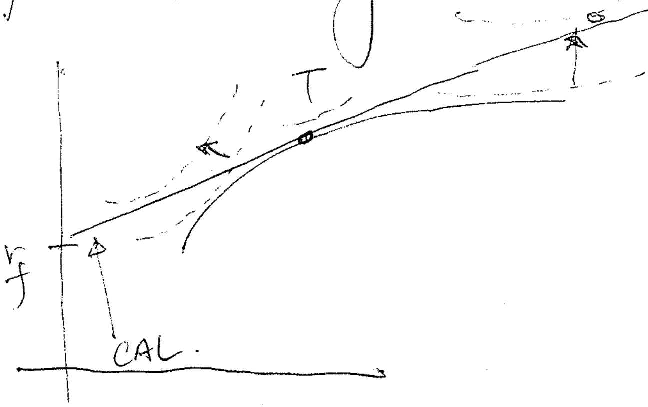
Point 1: Upper frontier better, more efficient



Point 2: Absent r.f. security, can choose any point on the eff frontier, depending on risk preference



Point 3: w/ r.f., we can push out the efficient frontier and make utility improvement for almost everyone (except for the guy who was happiest w/ the ~~rate~~ portfolio to begin w/!)



Tangency Portf. Characteriz.

Point 4: Tangency portfolio is the portfolio of risky assets that pushes the frontier out by the most.

ie) the one that max. slope of C.A.L. or the Sharpe Ratio.

ie) Solution to $\max_w \dfrac{w^T \vec{E} - r_f}{w^T \Sigma w}$ s.t. $w^T 1 = 1$

slide 29 of Note 6 gives a special case w/ 2 risky asset allocation problem to find the tangency portfolio.

Note: It can be show that all assets have the property

$$\frac{E(r_i) - r_f}{Cov(r_i, r_T)} = \frac{E(r_j) - r_f}{Cov(r_j, r_T)} \quad \forall i \quad (*)$$

marg. Contribution to portfolio ret

marg. contr. to portfolio variance.

In this tangency portfolio, we must have that the marginal ~~reward~~ contr. to return given marginal contr. to Var is same, otherwise we can ~~dos~~ increase sharpe ratio

eg) if $\frac{E(r_i)-r_f}{Cov(r_i, r_T)} > \frac{E(r_j)-r_f}{Cov(r_j, r_T)}$

then hold more i and less j

$\Rightarrow$ Sharpe ratio improvement.

We can also get this by the following:

$$\max_{w_i} \frac{w^T E + w E_i - w r_f}{Var(w^T E + w r_i)}$$

Then, we'll have FOC $\Rightarrow$ condition $(*)$.

<u>Point 5</u>: MV portfolio is the solution to

$$\min_{w} \underbrace{w^T \Sigma w}_{Var(r_p)} \quad s.t. \quad w^T 1 = 1.$$

$\rightarrow$ Slide 30 of Lecture 6 has a special case formula for a 2 risky asset case:

$$w^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

<u>Note</u>: MV portfolio satisfy the following property:

$$\boxed{\underbrace{Cov(r_i, r_{mvp})}_{i\text{'s Marg. Contr. to Var}} = \underbrace{Cov(r_j, r_{mvp})}_{j\text{'s Marg. Contr. to Var. of MVP}} \quad \forall i}$$

why? If $Cov(r_i, r_{mvp}) > Cov(r_j, r_{mvp})$
hold more j and less i
$\Rightarrow$ Variance improvement.

Point 6: Benefits to diversification.

Holding a portfolio of investments allows us to make efficiency improvements (i.e. diversify away some idiosyncratic risk).

As long as assets returns are not perfectly correlated.

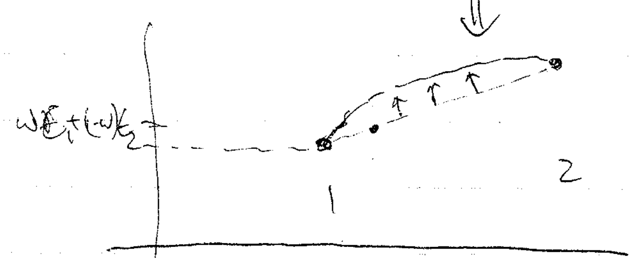
Why? Take 2-risky asset ex...

$$Var(w_0 r_1 + (1-w) r_2)$$

$$= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2$$

$$\begin{cases} = (w\sigma_1 + (1-w)\sigma_2)^2 & \text{if } \rho = 1 \\ < & \text{if } \rho < 1 \end{cases}$$

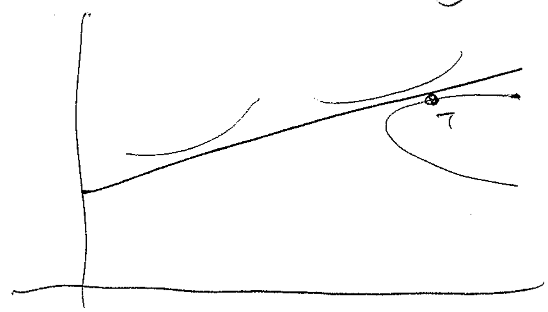
So that $\sigma(r_p) \; \substack{= \\ <} \; w\sigma_1 + (1-w)\sigma_2$ iff $\rho = 1$, iff $\rho < 1$.



The property that weighted avg of the ~~variances~~ std is larger than variance of portfolio IS REFLECTED IN THE GRAPH: WE PUSH OUT BOUNDARY.

• ~~CAPM~~ Separation Princ.

Since CAL is the best frontier, everyone holds the same risky asset : i.e. Tangency portfolio.



but differ based on WHERE on the CAL the lie, based on Indiff. Curve.

$\boxed{\text{CAPM}}$:

From MV, just adds 2 assumptions :

① Demand = Supply

② Everyone agrees on ~~prices~~ return and σ.

$\Rightarrow$ Tangency Portfolio = Market Portfolio

Then,

$$\frac{E(r_i) - r_f}{Cov(r_i, r_m)} = \frac{E(r_j) - r_f}{Cov(r_j, r_m)} = \frac{E(r_m) - r_f}{Var(r_m)} \quad \forall i, j.$$

$$\Rightarrow \boxed{E(r_i) - r_f = \underbrace{\frac{Cov(r_i, r_m)}{Var(r_m)}}_{\text{"}\beta\text{"}} \underbrace{\left[E(r_m) - r_f\right]}_{\text{MKT risk premium}}}$$

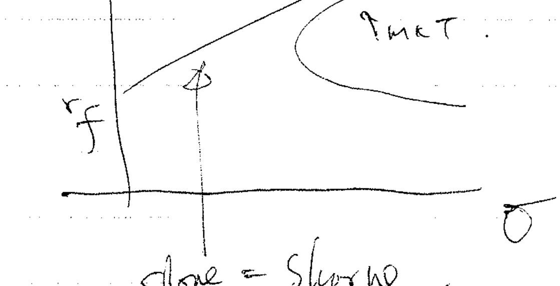
Interpretation: ① β is "sensitivity" to MKT risk and $E(r_m) - r_f$ is premium/reward for MKT risk (systematic & undivers.)

② $$\underbrace{Cov(r_i, r_m)}_{\text{marg. contr. of asset } i \text{ to risk of MKT portfolio}} \cdot \underbrace{\frac{E(r_m) - r_f}{Var(r_m)}}_{\text{risk reward ratio of the market}}$$

Implications:
- MKT Portfolio is MV eff.
- β is sufficient and the only thing we need to explain expected returns.

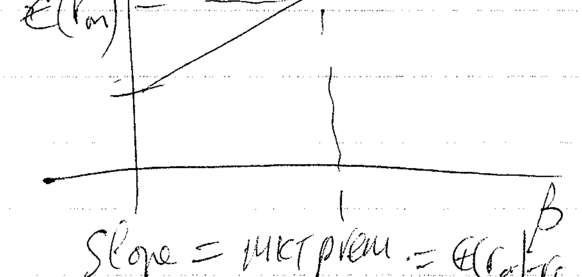
Useful 2 Graphs:

CAL/CML



slope = Sharpe.

SML



Slope = MKT prem = $E(r_m) - r_f$

Idiosyncratic risk is not priced.

~~Var(r)~~ We can always write the following decomp:

$$r_i = \alpha + \beta r_m + \varepsilon_i \qquad \text{s.t. } \mathrm{Cov}(r_m, \varepsilon_i) = 0.$$

$$\Rightarrow \mathrm{Var}(r_i) = \beta^2 \sigma_m^2 + \sigma_\varepsilon^2.$$

Note that everything on the CML has $\sigma_\varepsilon^2 = 0$ since we only hold MKT portfolio as risky asset, so no idiosyncratic risk.