Section I. Effective Hamiltonian of accidental Dirac points in Hermitian systems

Here we consider the Hermitian system of a 2D square-lattice PhC tuned to accidental degeneracy between a quadrupole mode and two degenerate dipole modes at the Brillouin zone center (Γ point). Near Γ point where the in-plane wavevectors $k_x$ and $k_y$ are small, the effective Hamiltonian given by first-order degenerate perturbation theory is \[ H_{\text{eff}}^{2D} = \begin{pmatrix} \omega_0 & v_g k_x & v_g k_y \\ v_g k_x & \omega_0 & 0 \\ v_g k_y & 0 & \omega_0 \end{pmatrix}, \] \[(S.1)\]

which can be transformed to

\[ U H_{\text{eff}}^{2D} U^{-1} = \begin{pmatrix} \omega_0 & v_g |k| & 0 \\ v_g |k| & \omega_0 & 0 \\ 0 & 0 & \omega_0 \end{pmatrix} \] \[(S.2)\]

with the orthogonal rotation matrix

\[ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \] \[(S.3)\]

where $\cos \theta = k_x/|k|$, $\sin \theta = k_y/|k|$, and $|k| = \sqrt{k_x^2 + k_y^2}$. After transformation, the $3 \times 3$ matrix becomes two isolated blocks: the upper $2 \times 2$ block gives the conical Dirac dispersion ($\omega = \omega_0 \pm v_g |k|$), while the lower block is the intersecting flat band ($\omega = \omega_0$).

Section II. Effective non-Hermitian Hamiltonian of the exceptional ring

For a 3D PhC slab with finite thickness, the two dipole modes become resonances with finite lifetime due to their radiation, with eigenvalues becoming complex ($\omega_0 - i \gamma_d$). With $C_4$ rotational symmetry, these two dipole modes are related by a $90^\circ$ rotation and therefore share the same complex eigenvalue. Meanwhile, the quadrupole mode does not couple to radiation at the Γ point due to symmetry mismatch [2], and to leading order its eigenvalue remains at $\omega_0$. The effective non-Hermitian Hamiltonian of the 3D PhC slab thus becomes

\[ H_{\text{eff}}^{3D} = \begin{pmatrix} \omega_0 & v_g k_x & v_g k_y \\ v_g k_x & \omega_0 - i \gamma_d & 0 \\ v_g k_y & 0 & \omega_0 - i \gamma_d \end{pmatrix}, \] \[(S.4)\]

which transforms to

\[ U H_{\text{eff}}^{3D} U^{-1} = \begin{pmatrix} \omega_0 & v_g |k| & 0 \\ v_g |k| & \omega_0 - i \gamma_d & 0 \\ 0 & 0 & \omega_0 - i \gamma_d \end{pmatrix} \] \[(S.5)\]

with the same matrix $U$ as in equation (S.3). The upper $2 \times 2$ block is the $H_{\text{eff}}$ in equation (1) of the main text that gives rise to an exceptional ring, while the lower block is the flat band.
Section III, Eigenfunctions in Hermitian and non-Hermitian systems
In this section, we show the numerically calculated eigenfunctions in the Hermitian PhC and the non-Hermitian PhC slab, and compare them to the analytical predictions from first-order perturbation theory.

According to equation S.1, the eigenfunctions of the Hermitian PhC along the $k_x$ axis are $\psi_1 \pm \psi_2$ with eigenvalues $\omega_0 \pm v_g k_x$. Fig. S1a shows that the numerical results agree with this prediction. These two eigenfunctions are orthogonal to each other, as expected for Hermitian systems.

In comparison, for the non-Hermitian PhC slab, equation S.4 predicts the two eigenfunctions to merge into one, $\psi_1 - i \psi_2$, at the exceptional point (EP). This prediction also agrees with our numerical results, shown in Fig. S1b. Both eigenfunctions become very close to the analytical prediction of $\psi_1 - i \psi_2$. Furthermore, this coalesced eigenfunction is self-orthogonal under the unconjugated product. Such coalescence of eigenfunctions can never happen in Hermitian systems and highlights the effect of non-Hermiticity.

Section IV, Existence of exceptional points along every direction in momentum space
In this section, we demonstrate that EPs exist in all directions in the $k$ space, not only for a simplified Hamiltonian (equation (1)), but also for realistic structures. To prove their existence, we use the unique topological property of EPs: when the system evolves adiabatically in the parameter space around an EP, the eigenvalues will switch their positions at the end of the loop [3,4]. In our system, the parameter space in which we choose to evolve the eigenfunctions is three-dimensional, consisting of the two in-plane wavevectors ($k_x$, $k_y$) and the radius of the air holes $r$, as shown in Fig. S3a. Here, $r$ can also be other parameters, like the refractive index of the PhC slab ($n$), the periodicity of the square lattice ($a$), or the thickness of the slab ($h$). For simplicity of this demonstration, we choose $r$ as the varying parameter while keeping all other parameters ($n$, $a$, and $h$) fixed throughout.

First, we compare the evolution of the eigenvalues when the system parameters follow (1) a loop that does not enclose an EP, and (2) one that encloses an EP. Following the loop $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ in Fig. S3a,b that does not enclose an EP (point $E_P$), we see that the complex eigenvalues come back to themselves at the end of the loop (Fig. S3c where the red dot and the blue dot return to their initial positions at the end of the loop). However, following the loop $A' \rightarrow B' \rightarrow C' \rightarrow D' \rightarrow A'$ in Fig. S3d, which encloses an EP (point $E_P$), we see that the complex eigenvalues switch their positions in the complex plane (Fig. S3f where the red dot and the blue dot switch their positions). This switching of the eigenvalues shows the existence of an EP along the $\Gamma$ to X direction, at some particular value of $k_x$ and some particular value of radius $r$. This shows the existence of the EP without having to locate the exact parameters of $k_x$ and $r$ at which the EP occurs.

Similarly, we can evolve the parameters along any direction $\theta = \tan(k_y/k_x)$ in the $k$ space and check whether an EP exists along this direction or not. As two examples, we show the evolution of the complex eigenvalues when we evolve the parameters along the $\theta = \pi/8$ direction following the loop $A'' \rightarrow B'' \rightarrow C'' \rightarrow D'' \rightarrow A''$ and along the $\theta = \pi/4$ direction following...
the loop $A'' \rightarrow B'' \rightarrow C'' \rightarrow D'' \rightarrow A''$ in Fig. S3g,h. In both cases, we observe the switching of the eigenvalues, showing the existence of an EP along these two directions. The same should hold for every direction in $k$ space.

The above calculations show that for every direction $\theta$ we examined in the $k$ space, there is always a particular combination of $k_c$ and $r_c$, which supports an EP. However, we note that in general, different directions can have different $k_c$ and different $r_c$, so the exceptional ring for the realistic PhC slab structure is parameterized by $k_c(\theta)$ and $r_c(\theta)$. This angular variation of $k_c(\theta)$ and $r_c(\theta)$ can be described by introducing higher order corrections in the effective Hamiltonian, which we examine in the next section.

Section V, Generalization of the effective Hamiltonian

Here, we generalize the effective Hamiltonian in equations (1) and (S.5). First, the radiation of the quadrapole mode is zero only at the $\Gamma$ point; away from the $\Gamma$ point, the quadrapole mode has a $\sim k$-dependent radiation that is small but non-zero, which we denote with $q$. Second, we consider possible deviation from accidental degeneracy in the Hermitian part, with a frequency walk-off $\delta$. With these two additional ingredients, the effective Hamiltonian becomes

$$
\left( \begin{array}{cc} \omega_0 + \delta & v_g k \\ v_g k & \omega_0 \end{array} \right) - i \left( \begin{array}{cc} \gamma_q & \sqrt{\gamma_q \gamma_d} \\ \sqrt{\gamma_q \gamma_d} & \gamma_d \end{array} \right),
$$

with complex eigenvalues of

$$
\omega_{\pm} = \omega_0 + \frac{\delta}{2} - i \frac{\gamma_q + \gamma_d}{2} \pm \sqrt{\left(v_g k - i \sqrt{\gamma_q \gamma_d}\right)^2 - \left(\frac{\gamma_d - \gamma_q}{2} - i \frac{\delta}{2}\right)^2},
$$

which are generalized forms of equations (1) and (2), respectively. We note that the off-diagonal term $\sqrt{\gamma_q \gamma_d}$ in equation (S.6) is required by energy conservation and time-reversal symmetry [5], as we will discuss more in the last section. Equation (S.7) shows that EP occurs when the two conditions

$$
\begin{cases}
  k = (\gamma_d - \gamma_q)/(2 v_g) \approx \gamma_d/2 v_g, \\
  \delta = 2 \sqrt{\gamma_d \gamma_q},
\end{cases}
$$

are satisfied. In the region of momentum space of interest, $\gamma_q$ is much smaller than $\gamma_0$ (this can be seen, for example, from the imaginary parts of Fig. S2a,c), so the first condition becomes $k_c \approx \gamma_d/2 v_g$, same as in the simplified model. For a given direction $\theta$ in the $k$ space (as discussed in the previous section), we can vary the magnitude $k$ and the radius $r$ to find the $k_c(\theta)$ and $r_c(\theta)$ where these two conditions are met simultaneously.

We can now analyze the angular dependence of $k_c(\theta)$ and $r_c(\theta)$ without having to find their exact values. The first condition of equation (S.8) says that the angular dependence of $k_c(\theta)$ comes from $\gamma_d$ and $v_g$; in the PhC slab structure here, we find that $\gamma_d$ varies by about 20% as the angle $\theta = \tan(k_y/k_x)$ is varied, while $v_g$ remains almost the same; therefore, $k_c(\theta)$ potentially varies by around 10% along the exceptional ring. For the second condition of equation (S.8), we have $\gamma_d \approx 5 \times 10^{-3} \omega_0$ and $\gamma_q \approx 5 \times 10^{-5} \omega_0$ for our PhC slab structure, so
\[ \delta_c = 2\sqrt{\gamma_d\gamma_q} \approx 1 \times 10^{-3}\omega_0. \] Again, \( \gamma_d \) and \( \gamma_q \) vary by about 20% as the angle \( \theta \) is changed, so \( \delta_c \) can change by around \( 2 \times 10^{-4}\omega_0 \). Empirically, we find that a change of \( \delta \) by \( 2 \times 10^{-4}\omega_0 \) corresponds to a change in the radius \( r \) of around 0.06 nm, which is the estimated range of variation for \( r_c(\theta) \) of all \( \theta \in [0, 2\pi) \). This angular variation is much smaller than our structure can resolve in practice, since the radii of different holes within one fabricated PhC slab will already differ by more than 0.06 nm. So, in practice a given fabricated structure can be close to EP along all different directions \( \theta \), but is unlikely to be an exact EP for any direction.

Section VI, Sample fabrication and experimental details
The Si\(_3\)N\(_4\) layer was grown with the low-pressure chemical vapor deposition method on a 6\( \mu \)m-thick cladding of SiO\(_2\) on the backbone of a silicon wafer (LioniX). Before exposure, the wafer was coated with a layer of polymer as anti-reflection coating, a thin layer of SiO\(_2\) as an intermediate layer for etching, and a layer of negative photoresist for exposure. The square lattice pattern was created with Mach–Zehnder interference lithography using a 325-nm He/Cd laser. The angle between the two arms of the laser beam was chosen for a periodicity of 336 nm. After exposures, the pattern in the photoresist was transferred to Si\(_3\)N\(_4\) by reactive-ion etching.

The source was a supercontinuum laser from NKT Photonics (SuperKCompact). A polarizer selected \( s\)- or \( p\)-polarized light. The sample was immersed in a colorless liquid with tunable refractive indices (Cargille Labs). The sample was mounted on two perpendicular motorized rotation stages (Newport): one to orient the PhC to the \( \Gamma\)-X or \( \Gamma\)-M direction, and the other to determine the incident angle \( \theta \). The reflectivity spectra were measured with a spectrometer with spectral resolution of 0.02 nm (HR4000; Ocean Optics).

Section VII, Model of reflection from two resonances
In Section IX, we will derive the temporal-coupled coupled theory (TCMT) [5–10] equations describing the full experimental system. Here, we first consider a simpler scenario to illustrate the qualitative features of the reflectivity spectrum. Consider two resonances in a photonic crystal slab, governed by the Hamiltonian in equation (3) of the main text,

\[
H = \begin{pmatrix} \omega_1 & \kappa \\ \kappa & \omega_2 \end{pmatrix} - i \begin{pmatrix} \gamma_1 & \gamma_{12} \\ \gamma_{12} & \gamma_2 \end{pmatrix},
\]  

where the first term is the Hermitian part describing a lossless system, and the second term is the anti-Hermitian part describing radiation of the two resonances. We ignore non-radiative loss here for simplicity. Also, we ignore the direct Fresnel reflection between the dielectric layers. We consider the photonic crystal slab to be mirror symmetric in \( z \) direction, with the two resonances having the same mirror symmetry in \( z \). This simple system describes the basic elements of resonant reflection. TCMT predicts the reflectivity to be (see Section IX for the full details)

\[
R(\omega) = \frac{1}{1 + f^2(\omega)}, \quad f(\omega) = \frac{(\omega - \omega_1)(\omega - \omega_2) - \kappa^2}{\gamma_1(\omega - \omega_2) + \gamma_2(\omega - \omega_1) + 2\gamma_{12}\kappa},
\]  

S.10
with the transmission being $T(\omega) = 1 - R(\omega)$ since there is no non-radiative loss. We immediately see that the reflectivity reaches its maximal value of 1 when the numerator of $f(\omega)$ vanishes, which happens at two frequencies $\omega = \Omega_{1,2}$ with

$$\Omega_{1,2} = \frac{1}{2} \left[ \omega_1 + \omega_2 \pm \sqrt{(\omega_1 - \omega_2)^2 + 4\kappa^2} \right] \quad (S.11)$$

being the eigenvalues for the Hermitian part of the Hamiltonian. Note that the anti-Hermitian part determines the linewidth but has no effect on the location of the reflectivity peaks. This is the first main conclusion of this section.

Secondly, we also observe that the reflectivity reaches its minimal value of 0 when the denominator of $f(\omega)$ vanishes, which happens at one frequency $\omega = (\gamma_1\omega_2 + \gamma_2\omega_1 - 2\gamma_1\kappa)/(\gamma_1 + \gamma_2)$ that is in between $\Omega_1$ and $\Omega_2$. This is called coupled-resonator-induced transparency (CRIT) [11, 12].

We emphasize that the reflectivity peaks, following eigenvalues of the Hermitian part of the Hamiltonian, are different from the real part of the complex eigenvalues of the Hamiltonian $H$. Consider a simple example with $\kappa = 0$ (so that $\Omega_{1,2} = \omega_{1,2}$), $\Omega_{1,2} = \omega_0 \pm b$, and $\gamma_1 = \gamma_2 = \gamma_{12} = b$. The reflection peaks at $\Omega_{1,2} = \omega_0 \pm b$, while the two complex eigenvalues are degenerate at $\omega_+ = \omega_- = \omega_0 - ib$, whose real part is in the middle of the two reflection peaks. This explains the reflectivity from the PhC slabs at 0.3° shown in Fig. 2d and Fig. 3a, where the degenerate complex eigenvalues of the system are in between the two reflection peaks.

In Fig. S6, we use some examples to illustrate the difference between the reflectivity peaks and the real part of the complex eigenvalues. Fig. S6a shows the case when there is only one resonance with complex eigenvalue $\omega_0 - i\gamma$; in this case, the reflection peak (red arrow) is at the same position as the real part of the complex eigenvalue. In contrast, Fig. S6b shows the case when there are two resonances [equation (S.10)] with $\kappa = 0$ and $\Omega_{1,2}$ fixed at $\omega_0 \pm b$; as we vary $\gamma_{1,2}$, the complex eigenvalues (circles) vary accordingly, whereas the reflectivity peaks (red arrows) always show up at $\Omega_{1,2}$.

**Section VIII, Effects of material loss and gain**

The conclusion that reflectivity peaks at the eigenfrequencies $\Omega_{1,2}$ of the Hermitian part is valid when radiation is the only source of loss and when there is no gain in the system. It no longer holds when substantial material loss and/or gain is introduced, as we show in this section.

To account for material loss and gain, we extend the Hamiltonian in Section VII to

$$H = \begin{pmatrix} \omega_1 & \kappa \\ \kappa & \omega_2 \end{pmatrix} - i \begin{pmatrix} \gamma_1 & \gamma_{12} \\ \gamma_{12} & \gamma_2 \end{pmatrix} - i \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$$

with positive (negative) values of $\eta_{1,2}$ describing material loss (gain). Same derivation using
TCMT yields reflection and transmission as

\[ R(\omega) = |g(\omega)|^2, \quad T(\omega) = |1 + g(\omega)|^2 \]

\[ g(\omega) = i \frac{\gamma_1(\omega_2 - i\eta_2 - \omega) + \gamma_2(\omega_1 - i\eta_1 - \omega) - 2\gamma_{12}\kappa}{(\omega_1 - i\gamma_1 - i\eta_1 - \omega)(\omega_2 - i\gamma_2 - i\eta_2 - \omega) - (\kappa - i\gamma_{12})^2}. \] (S.13)

When \( \eta_{1,2} \) is small, the expression reduces back to equation (S.10). When \( \eta_{1,2} \) is substantial, neither the reflectivity peaks nor the transmission dips follow the eigenvalues \( \Omega_{1,2} \).

To illustrate this point, we consider a system tuned to accidental degeneracy (\( \omega_1 = \omega_2 \equiv \omega_0 \)) with only one resonance radiating (\( \gamma_1 = 1, \gamma_2 = \gamma_{12} = 0 \)) and with equal gain/loss for the two resonances (\( \eta_1 = \eta_2 \equiv \eta_0 \)). This system has linear Dirac dispersion \( \Gamma_1 = \Gamma_2 = \omega_0 \pm \kappa \) for the eigenfrequency of the Hermitian part, while the complex eigenvalues exhibit an exceptional point at \( \kappa = 1/2 \). In Fig. S7, we plot the reflection and transmission spectrum for a system with substantial non-radiative loss (\( \eta_0 = 0.45 \), Fig. S7a), a purely passive system (\( \eta_0 = 0 \), Fig. S7b), and a system with substantial gain (\( \eta_0 = -0.25 \), Fig. S7c). On top of the plots, we indicate the reflection peaks (blue solid lines) and transmission dips (red solid lines), as well as the eigen-frequency of the Hermitian part (green dashed lines) and the real part of the eigenvalues (green solid lines). As can be seen, in the passive case the reflection peaks and transmission dips follow the linear dispersion of \( \Omega_{1,2} \), whereas in the cases with substantial non-radiative loss or material gain, the reflection peaks and transmission dips show branching behavior instead.

**Section IX, Full derivation using temporal coupled-mode theory (TCMT)**

Here, we provide the derivation of reflectivity using temporal coupled-mode theory (TCMT) [5–10] in a more general setup than the previous two sections. We will account for the direct (non-resonant) reflection process in between layers of dielectric, include arbitrary number of \( n \) resonances (as there are other nearby resonances that modify the spectrum) with arbitrary symmetries, and account for the non-radiative loss. We use the resulting expression to fit the experimentally measured reflection data and extract the Hamiltonian matrix elements and eigenvalues.

The time evolution of these \( n \) resonances, whose complex amplitudes are denoted by an \( n \times 1 \) column vector \( A \), is described by the Hamiltonian \( H \) and a driving term,

\[ \frac{dA}{dt} = -iHA + K^T s_+, \] (S.14)

where the Hamiltonian is an \( n \times n \) non-Hermitian matrix

\[ H = \Omega - i\Gamma - i\gamma_{\text{nr}}, \] (S.15)

with \( \Omega \) denoting its Hermitian part that describes a lossless system, \( -i\Gamma \) denoting its anti-Hermitian part from radiation loss, and \( -i\gamma_{\text{nr}} \) its anti-Hermitian part from non-radiative decays. The non-radiative decay includes both material absorption and scattering from surface roughness (in our system, surface roughness is the primary contribution [2]), and here we consider it to be the same for all resonances, so \( \gamma_{\text{nr}} \) is a real number instead of a matrix.
Reflectivity measurements couple the \( n \) resonances to the incoming and outgoing planewaves, whose complex amplitudes we denote by two \( 2 \times 1 \) column vectors, \( s_+ \) and \( s_- \). The direct reflection and transmission of the planewaves through the slab (in the absence of resonances) are described by a \( 2 \times 2 \) complex symmetric matrix \( C \), and

\[
s_- = Cs_+ + DA, \tag{S.16}
\]

where \( D \) and \( K \) in equation (S.14) are \( 2 \times n \) complex matrices denoting coupling between the resonances and the planewaves. We approximate the direct scattering matrix \( C \) by that of a homogeneous slab whose permittivity is equal to the spatial average of the PhC slab [8–10]. Lastly, outgoing planewaves into the silica substrate are reflected at the silica-silicon interface, so

\[
s_{2+} = e^{2i\beta h_s}r_{23}s_{2-}, \tag{S.17}
\]

where \( h_s \) is the thickness of the silica substrate with refractive index \( n_s = 1.46 \), \( \beta = \sqrt{\frac{n_s^2 \omega^2}{c^2} - |k|^2} \) is the propagation constant in silica, and \( r_{23} \) is the Fresnel reflection coefficient between silica and the underlying silicon. The formalism described above is the same as Ref [9] except that here we account for the coupling between the \( n \) resonances (off-diagonal terms of \( H \)).

For steady state with \( e^{-i\omega t} \) time dependence, we solve for vector \( A \) from equation (S.14) to get the scattering matrix of the whole system that includes both direct and resonant processes,

\[
s_- = (C + C_{\text{res}})s_+, \tag{S.18}
\]

where the effect of the \( n \) resonances is captured in a \( 2 \times 2 \) matrix

\[
C_{\text{res}} = iD(\omega - H)^{-1}K^T. \tag{S.19}
\]

We can solve equation (S.17) and equation (S.18) to obtain the reflectivity

\[
R(\omega) = \left| \frac{s_{1-}}{s_{1+}} \right|^2. \tag{S.20}
\]

In this expression, the only unknown is \( C_{\text{res}} \). Therefore, by comparing the experimentally measured reflectivity spectrum and the one given by TCMT in equation (S.20), we can extract the unknown parameters in the resonant scattering matrix \( C_{\text{res}} \) and obtain the eigenvalues of the Hamiltonian \( H \).

The remaining task is to write \( C_{\text{res}} \) using as few unknowns as possible so that the eigenvalues of \( H \) can be extracted unambiguously. In equation (S.19), there are a large number of unknowns in the matrix elements of \( H \), \( D \), and \( K \), but there is much redundancy because the matrix elements are not independent variables and because \( C_{\text{res}} \) is independent of the basis choice. Below, we show that we can express \( C_{\text{res}} \) with only \( 2n + 1 \) unknown real numbers, and these \( 2n + 1 \) real numbers are enough to determine the \( n \) complex eigenvalues of \( H \).

First, we normalize the amplitudes of \( A \) and \( s_\pm \) such that their magnitudes squared are the energy of the resonances per unit cell and the power of the incoming/outgoing planewaves per
unit cell, respectively. Then, energy conservation, time-reversal symmetry, and $C_2$ rotational symmetry of the PhC slab [5,13] require the direct scattering matrix to satisfy $C^\dagger = C^* = C^{-1}$ and the coupling matrices to satisfy $D^\dagger D = 2\Gamma$, $K = D$, and $CD^* = -D$. It follows that the matrix $\Gamma$ is real and symmetric. Next, using the Woodbury matrix identity and these constraints, we can rewrite equation (S.19) as

$$C_{\text{res}} = -2W (2 + W)^{-1} C,$$

(S.21)

where $W \equiv iD(\omega - \Omega + i\gamma_0)^{-1}D^\dagger$ is a 2-by-2 matrix.

We note that the matrix $W$, and therefore the matrix $C_{\text{res}}$, is invariant under a change of basis for the resonances through any orthogonal matrix $U$ (where $\Omega$ is transformed to $U\Omega U^{-1}$, and $D$ is transformed to $DU^{-1}$). The eigenvalues of $H$ is also invariant under a change of basis. Therefore, we are free to choose any basis. We kept the basis choice general in the two-resonance system described in the previous two sections, but here for the purpose of fitting, we need as few unknowns as possible, and it is most convenient to choose the basis where $\Omega$ is diagonal. So we let $\Omega_{ij} = \Omega_j \delta_{ij}$, with $\{\Omega_j\}_{j=1}^{n}$ being the eigenvalues of $\Omega$.

To proceed further, we note that the PhC slab sits on a silica substrate with $n_s = 1.46$ and is immersed in a liquid with $n = 1.48$, so the structure is nearly symmetric in the $z$ direction. The mirror symmetry requires the coupling to the two sides to be symmetric or anti-symmetric [7],

$$\frac{D_{1j}}{D_{2j}} \equiv \sigma_j = \pm 1, \quad j = 1, \ldots, n,$$

(S.22)

where $\sigma_j = 1$ for TE-like resonances and $\sigma_j = -1$ for TM-like resonances, in the convention where $(E_x, E_y)$ determines the phase of $A_j$ and $s_{\pm}$. Then, the diagonal elements of $\Gamma$ are related to $D$ by $\Gamma_{jj} \equiv \gamma_j = |D_{1j}|^2$, and in this basis we have

$$W = \sum_{j=1}^{n} \frac{i\gamma_j}{\omega - \Omega_j + i\gamma_{\text{nr}}} \begin{pmatrix} 1 & \sigma_j \\ \sigma_j & 1 \end{pmatrix}.$$

(S.23)

This completes our derivation. Equations (S.21) and (S.23) provide an expression for $C_{\text{res}}$ that depends only on $2n + 1$ unknown non-negative real numbers: the $n$ eigenvalues $\{\Omega_j\}_{j=1}^{n}$ of the Hermitian matrix $\Omega$, the $n$ diagonal elements $\{\gamma_j\}_{j=1}^{n}$ of the real-symmetric radiation matrix $\Gamma$ in the basis where $\Omega$ is diagonal, and the non-radiative decay rate $\gamma_{\text{nr}}$.

At each angle and each polarization, we fit the experimentally measured reflectivity spectrum to the TCMT expression equation (S.20) to determine these $2n + 1$ unknown parameters. Fig. 3a and Fig. S8a show the comparison between the experimental reflectivity spectrum and the fitted TCMT reflectivity spectrum at some representative angles. The near-perfect agreement between the two demonstrates the validity of the TCMT model.

To obtain the eigenvalues of the Hamiltonian $H$, we also need to know the off-diagonal elements of $\Gamma$. From $D^\dagger D = 2\Gamma$ and $D_{1j}/D_{2j} = \sigma_j$, we see that $\Gamma_{ij} = 0$ when resonance $i$ and resonance $j$ have different symmetries in $z$ (i.e. when $\sigma_i\sigma_j \neq 1$), and that $\Gamma_{ij} = \pm \sqrt{\gamma_i\gamma_j}$ when
\( \sigma_i \sigma_j = 1 \). In the latter case, the sign of \( \Gamma_{ij} \) depends on the choice of basis; the eigenvalues of \( H \) are independent of the basis choice, so to calculate the eigenvalues of \( H \), we can simply take the positive roots of all of the non-zero off-diagonal elements of \( \Gamma \).

We note that the model Hamiltonians introduced previously, such as equation (1) in the main text and equation (S.6) above, are all special cases of the general Hamiltonian in equation (S.15) that we consider in the TCMT formalism in this section. The TCMT formalism in this section does not make assumptions on the particular forms of the matrix elements (aside from basic principles such as energy conservation and time-reversal symmetry), so it can be used as an unbiased method for analyzing the experimental data.
References


**Supplementary Figure 1:** Eigenfunctions in the Hermitian PhC and the non-Hermitian PhC slab. 

**a,** Numerically calculated band structure and eigenfunctions for the Hermitian 2D PhC. The basis eigenfunctions at the Γ point (ψ₁ and ψ₂) are shown in the black box. The eigenfunctions away from Γ point (at $k_x a/2\pi = 0.0035$) are shown in the red and blue boxes, agreeing well with the analytical prediction of $\psi_1 \pm \psi_2$. 

**b,** Corresponding results for the non-Hermitian PhC slab, showing the basis eigenfunctions at the Γ point (black box) and the two eigenfunctions near the exceptional point ($k_x a/2\pi = 0.0035$, red and blue boxes). The two eigenfunction near EP almost coalesce and both agree with the analytical prediction of $\psi_1 - i\psi_2$. 

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**Note:** The figures illustrate the eigenfunctions and their behavior in both Hermitian and non-Hermitian PhC slabs. The complex eigenfunctions are shown near the exceptional point, highlighting the unique properties of non-Hermitian systems compared to their Hermitian counterparts.
Supplementary Figure 2: Simulation results of the complex eigenvalues of the PhC slabs with and without accidental degeneracy. The real (upper panels) and imaginary (lower panels) parts of the complex eigenvalues are shown for structures with (b) and without (a, c) accidental degeneracy. The bands with quadrapole modes in the middle of the Brillouin zone are shown in red solid lines, while the bands with the dipole mode are shown in blue solid lines and gray dashed lines. The bands shown in red and blue solid lines couple to each other, while the band in gray dashed lines is decoupled from the other two due to symmetry. When accidental degeneracy happens (at $r = r_c$ as shown in b), the characteristic branching features are observed, demonstrating the existence of EPs. When the accidental degeneracy is lifted, the quadrupole band splits from the dipole bands from different directions: from the bottom when radius of the holes is too small ($r < r_c$ as shown in a), or from the top when the radius is too big ($r > r_c$ as shown in c).
**Supplementary Figure 3:** Existence of EP along every direction in the momentum space for the realistic PhC slab structure.  

**a.** A loop is created in the parameter space of the structure ($A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$), which does not enclose the EP of the system (point $E_p$). Here, $r$ is the radius of the air holes, and $k_{x,y}$ are the in-plane wavevectors.  

**b.** For each point along the loop, we numerically calculate the eigenvalues of the PhC slab with the corresponding hole radius at the corresponding in-plane wavevector.  

**c.** The complex eigenvalues return to their initial state when not looping around EPs.

**d.** Existence of EP in every direction.
-tial positions at the end of the loop (namely, the blue dot and the red dot come back to themselves) when the system parameters come back to point A. d,e,f. Another loop is created (A′ → B′ → C′ → D′ → A′), which encloses an EP of the system (the same point E_p as in a). Following this new loop, the two eigenvalues switch their positions at the end of the loop (namely, the blue dot and the red dot switch their positions) when the system parameters come back to point A′. g,h. The two complex eigenvalues always switch their positions when we choose the right loops along other directions in the momentum space (Γ to N in g and Γ to M in h).

Supplementary Figure 4: Experimental results of reflectivity showing an accidental Dirac cone. Light with different polarizations (s and p) is selected to excite different resonances of the PhC slab along different directions in the $k$ space. Depending on the choice of polarization, the two bands forming the conical dispersion are excited (s-polarized along $\Gamma$-X and p-polarized along $\Gamma$-M, shown in the left panel), or the flat band in the middle is excited (p-polarized along $\Gamma$-X and s-polarized along $\Gamma$-M, shown in the right panel).
Supplementary Figure 5: Experimental results of reflectivity from PhC slabs with and without accidental degeneracy. Angle-resolved reflectivity along the Γ to X direction, measured for three different PhC slabs: a, with bigger hole radius than the structure with accidental degeneracy ($r > r'$); b, with accidental degeneracy ($r = r'$); c, with smaller hole radius than the structure with accidental degeneracy ($r < r'$). The reflectivity peaks of the structures without accidental degeneracy (a,c) follow quadratic dispersions; while the reflectivity peaks of the structure with accidental degeneracy (b) follow linear Dirac dispersion. Data shown in b is the same data as in Fig. 2c and the left panel of Fig. S3.
Supplementary Figure 6: Illustrative reflectivity spectrum from one resonance and from two coupled resonances. a, When a single resonance dominates, the reflectivity peak is at the same position as the real part of the complex eigenvalue. b, With two coupled resonances, the reflectivity peaks (red arrows) no longer follow the eigenvalues of the system (blue circles). As we vary the radiation loss $\gamma_{1,2}$ of the two resonances while fixing the eigenvalues $\Omega_{1,2}$ of the Hermitian matrix, we see the complex eigenvalues vary, whereas the reflectivity peaks are fixed. The middle panel of (b) shows a situation when the two complex eigenvalues coalesce into an EP.
**Supplementary Figure 7:** Effect of material loss and gain on the reflection and transmission spectra. Reflection spectra (upper panels) and transmission spectra (lower panels) are plotted for three systems with the same radiation losses, but different material gain/loss: with material loss (a), purely passive with no gain or loss (b), and with material gain (c). For the purely passive system (b), the reflection peak (blue solid line) and the transmission dip (red solid line) correspond to the eigenvalue of the Hermitian part of the Hamiltonian (green dashed line), and show the linear Dirac dispersion. When material loss (a) or gain (c) is added to the system, the reflection peaks and transmission dips deviate from the linear Dirac dispersion and show branching behavior instead. The system parameters are given in Section VIII.
Supplementary Figure 8: TCMT fitting and visualization of accidental Dirac cone. a, Examples of reflection spectrum measured at five different incident angles (0°, 0.14°, 0.24°, 0.5° and 1°) along the Γ-X direction for s polarization, with comparison to the TCMT expression in equation (S.20) after fitting. Dotted lines indicate the resonances, with the two relevant resonances marked in red and blue. b, Parameters obtained from the TCMT fitting. The eigenvalues for the Hermitian part of the Hamiltonian, \( \Omega_{1,2} \), are shown in the left panel and reveal the Dirac dispersion arising from accidental degeneracy. The diagonal terms for the anti-Hermitian part of the Hamiltonian, \( \gamma_{1,2} \), are shown in the right panel. Note that the anti-Hermitian part of the Hamiltonian also has off-diagonal terms, so \( \Omega_{1,2} \) and \( \gamma_{1,2} \) are not the eigenvalues of the Hamiltonian. The eigenvalues of the Hamiltonian are shown in Fig. 3 of the main text.