

Chapter 15

The Hamiltonian method

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This chapter is to be read in conjunction with *Introduction to Classical Mechanics, With Problems and Solutions* © 2007, by David Morin, Cambridge University Press.

The text in this version is the same as in Version 1, but some new problems and exercises have been added.

More information on the book can be found at:

<http://www.people.fas.harvard.edu/~djmorin/book.html>

At present, we have at our disposal two basic ways of solving mechanics problems. In Chapter 3 we discussed the familiar method involving Newton's laws, in particular the second law, $F = ma$. And in Chapter 6 we learned about the Lagrangian method. These two strategies always yield the same results for a given problem, of course, but they are based on vastly different principles. Depending on the specifics of the problem at hand, one method might lead to a simpler solution than the other.

In this chapter, we'll learn about a third way of solving problems, the *Hamiltonian* method. This method is quite similar to the Lagrangian method, so it's debatable as to whether it should actually count as a third one. Like the Lagrangian method, it contains the principle of stationary action as an ingredient. But it also contains many additional features that are extremely useful in other branches of physics, in particular statistical mechanics and quantum mechanics. Although the Hamiltonian method generally has no advantage over (and in fact is invariably much more cumbersome than) the Lagrangian method when it comes to standard mechanics problems involving a small number of particles, its superiority becomes evident when dealing with systems at the opposite ends of the spectrum compared with "a small number of particles," namely systems with an intractably large number of particles (as in a statistical-mechanics system involving a gas), or systems with no particles at all (as in quantum mechanics, where everything is a wave).

We won't be getting into these topics here, so you'll have to take it on faith how useful the Hamiltonian formalism is. Furthermore, since much of this book is based on problem solving, this chapter probably won't be the most rewarding one, because there is rarely any benefit from using a Hamiltonian instead of a Lagrangian to solve a standard mechanics problem. Indeed, many of the examples and problems in this chapter might seem a bit silly, considering that they can be solved much more quickly using the Lagrangian method. But rest assured, this silliness has a purpose; the techniques you learn here will be very valuable in your future physics studies.

The outline of this chapter is as follows. In Section 15.1 we'll look at the sim-

ilarities between the Hamiltonian and the energy, and then in Section 15.2 we'll rigorously define the Hamiltonian and derive Hamilton's equations, which are the equations that take the place of Newton's laws and the Euler-Lagrange equations. In Section 15.3 we'll discuss the Legendre transform, which is what connects the Hamiltonian to the Lagrangian. In Section 15.4 we'll give three more derivations of Hamilton's equations, just for the fun of it. Finally, in Section 15.5 we'll introduce the concept of phase space and then derive Liouville's theorem, which has countless applications in statistical mechanics, chaos, and other fields.

15.1 Energy

In Eq. (6.52) in Chapter 6 we defined the quantity,

$$E \equiv \left(\sum_{i=1}^N \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - L, \quad (15.1)$$

which under many circumstances is the energy of the system, as we will see below. We then showed in Claim 6.3 that $dE/dt = -\partial L/\partial t$. This implies that if $\partial L/\partial t = 0$ (that is, if t doesn't explicitly appear in L), then E is constant in time. In the present chapter, we will examine many other properties of this quantity E , or more precisely, the quantity H (the *Hamiltonian*) that arises when E is rewritten in a certain way explained in Section 15.2.1.

But before getting into a detailed discussion of the actual Hamiltonian, let's first look at the relation between E and the energy of the system. We chose the letter E in Eq. (6.52/15.1) because the quantity on the right-hand side often turns out to be the total energy of the system. For example, consider a particle undergoing 1-D motion under the influence of a potential $V(x)$, where x is a standard Cartesian coordinate. Then $L \equiv T - V = m\dot{x}^2/2 - V(x)$, which yields

$$E \equiv \frac{\partial L}{\partial \dot{x}} \dot{x} - L = (m\dot{x})\dot{x} - L = 2T - (T - V) = T + V, \quad (15.2)$$

which is simply the total energy. By performing the analogous calculation, it likewise follows that E is the total energy in the case of Cartesian coordinates in N dimensions:

$$\begin{aligned} L &= \left(\frac{1}{2}m\dot{x}_1^2 + \cdots + \frac{1}{2}m\dot{x}_N^2 \right) - V(x_1, \dots, x_N) \\ \implies E &= \left((m\dot{x}_1)\dot{x}_1 + \cdots + (m\dot{x}_N)\dot{x}_N \right) - L \\ &= 2T - (T - V) \\ &= T + V. \end{aligned} \quad (15.3)$$

In view of this, a reasonable question to ask is: Does E always turn out to be the total energy, no matter what coordinates are used to describe the system? Alas, the answer is no. However, when the coordinates satisfy a certain condition, E is indeed the total energy. Let's see what this condition is.

Consider a slight modification to the above 1-D setup. We'll change variables from the nice Cartesian coordinate x to another coordinate q defined by, say, $x(q) = Kq^5$, or equivalently $q(x) = (x/K)^{1/5}$. Since $\dot{x} = 5Kq^4\dot{q}$, we can rewrite the Lagrangian $L(x, \dot{x}) = m\dot{x}^2/2 - V(x)$ in terms of q and \dot{q} as

$$L(q, \dot{q}) = \left(\frac{25K^2mq^8}{2} \right) \dot{q}^2 - V(x(q)) \equiv F(q)\dot{q}^2 - V_u(q), \quad (15.4)$$

where $F(q) \equiv 25K^2mq^8/2$ (so the kinetic energy is $T = F(q)\dot{q}^2$). The quantity E is then

$$E \equiv \frac{\partial L}{\partial \dot{q}} \dot{q} - L = (2F(q)\dot{q})\dot{q} - L = 2T - (T - V) = T + V, \quad (15.5)$$

which again is the total energy. So apparently it is possible for (at least some) non-Cartesian coordinates to yield an E equaling the total energy.

We can easily demonstrate that in 1-D, E equals the total energy if the new coordinate q is related to the old Cartesian coordinate x by any general functional dependence of the form, $x = x(q)$. The reason is that since $\dot{x} = (dx/dq)\dot{q}$ by the chain rule, the kinetic energy always takes the form of \dot{q}^2 times some function of q . That is, $T = F(q)\dot{q}^2$, where $F(q)$ happens to be $(m/2)(dx/dq)^2$. This function $F(q)$ just goes along for the ride in the calculation of E , so the result of $T + V$ arises in exactly the same way as in Eq. (15.5).

What if instead of the simple relation $x = x(q)$ (or equivalently $q = q(x)$) we also have time dependence? That is, $x = x(q, t)$ (or equivalently $q = q(x, t)$)? The task of Problem 15.1 is to show that $L(q, \dot{q}, t)$ yields an E that takes the form,

$$E = T + V - m \left(\left(\frac{\partial x}{\partial q} \right) \left(\frac{\partial x}{\partial t} \right) \dot{q} + \left(\frac{\partial x}{\partial t} \right)^2 \right), \quad (15.6)$$

which is *not* the total energy, $T + V$, due to the $\partial x(q, t)/\partial t \neq 0$ assumption. So a necessary condition for E to be the total energy is that there is no time dependence when the Cartesian coordinates are written in terms of the new coordinates (or vice versa).

Likewise, you can show that if there is \dot{q} dependence, so that $x = x(q, \dot{q})$, the resulting E turns out to be a very large mess that doesn't equal $T + V$. However, this point is moot, because as we did in Chapter 6, we will assume that the transformation between two sets of coordinates never involves the time derivatives of the coordinates.

So far we've dealt with only one variable. What about two? In terms of Cartesian coordinates, the Lagrangian is $L = (m/2)(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2)$. If these coordinates are related to new ones (call them q_1 and q_2) by $x_1 = x_1(q_1, q_2)$ and $x_2 = x_2(q_1, q_2)$, then we have $\dot{x}_1 = (\partial x_1/\partial q_1)\dot{q}_1 + (\partial x_1/\partial q_2)\dot{q}_2$, and similarly for \dot{x}_2 . Therefore, when written in terms of the q 's, the kinetic energy takes the form,

$$T = \frac{m}{2}(A\dot{q}_1^2 + B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2), \quad (15.7)$$

where A , B , and C are various functions of the q 's (but not the \dot{q} 's), the exact forms of which won't be necessary here. So in terms of the new coordinates, we have

$$\begin{aligned} E &= \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 - L \\ &= m \left(A\dot{q}_1 + (B/2)\dot{q}_2 \right) \dot{q}_1 + m \left((B/2)\dot{q}_1 + C\dot{q}_2 \right) \dot{q}_2 - L \\ &= m(A\dot{q}_1^2 + B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2) - L \\ &= 2T - (T - V) \\ &= T + V, \end{aligned} \quad (15.8)$$

which is the total energy. This reasoning quickly generalizes to N coordinates, q_i . The kinetic energy has only two types of terms: ones that involve \dot{q}_i^2 and ones that

involve $\dot{q}_i \dot{q}_j$. These both pick up a factor of 2 (as either a 2 or a 1 + 1, as we just saw in the 2-D case) in the sum $\sum (\partial L / \partial \dot{q}_i) \dot{q}_i$, thereby yielding $2T$.

As in the 1-D case, time dependence in the relation between the Cartesian coordinates and the new coordinates will cause E to not be the total energy, as we saw in Eq. (15.6) for the 1-D case. And again, \dot{q}_i dependence will also have this effect, but we are excluding such dependence. We can sum up all of the above results by saying:

Theorem 15.1 *A necessary and sufficient condition for the quantity E to be the total energy of a system whose Lagrangian is written in terms of a set of coordinates q_i is that these q_i are related to a Cartesian set of coordinates x_i via expressions of the form,*

$$\begin{aligned} x_1 &= x_1(q_1, q_2, \dots), \\ &\vdots \\ x_N &= x_N(q_1, q_2, \dots). \end{aligned} \quad (15.9)$$

That is, there is no t or \dot{q}_i dependence.

In theory, these relations can be inverted to write the q_i as functions of the x_i .

REMARK: It is quite permissible for the number of q_i 's to be smaller than the number of Cartesian x_i 's (N in Eq. (15.9)). Such is the case when there are constraints in the system. For example, if a particle is constrained to move on a plane inclined at a given angle θ , then (assuming that the origin is chosen to be on the plane) the Cartesian coordinates (x, y) are related to the distance along the plane, r , by $x = r \cos \theta$ and $y = r \sin \theta$. Because θ is given, we therefore have only one q_i , namely $q_1 \equiv r$.¹ The point is that even if there are fewer than N q_i 's, the kinetic energy still takes the form of $(m/2) \sum \dot{x}_i^2$ in terms of Cartesian coordinates, and so it still takes the form (in the case of two q_i 's) given in Eq. (15.7) once the constraints have been invoked and the number of coordinates reduced (so that the Lagrangian can be expressed in terms of independent coordinates, which is a requirement in the Lagrangian formalism). So E still ends up being the energy (assuming there is no t or \dot{q}_i dependence in the transformations).

Note that if the system is describable in terms of Cartesian coordinates (which means that either there are no constraints, or the constraints are sufficiently simple), and if we do in fact use these coordinates, then as we showed in Eq. (15.3), E is always the energy.

♣

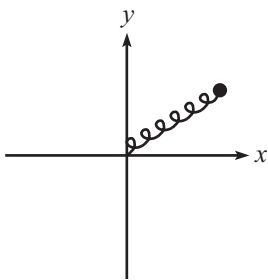


Figure 15.1

Example 1 (Particle in a plane): A particle of mass m moves in a horizontal plane. It is connected to the origin by a spring with spring constant k and relaxed length zero (so the potential energy is $kr^2/2 = k(x^2 + y^2)/2$), as shown in Fig. 15.1. Find L and E in terms of Cartesian coordinates, and then also in terms of polar coordinates. Verify that in both cases, E is the energy and it is conserved.

Solution: In Cartesian coordinates, we have

$$L = T - V = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2), \quad (15.10)$$

¹A more trivial example is a particle constrained to move in the x - y plane. In this case, the Cartesian coordinates (x, y, z) are related to the “new” coordinates (q_1, q_2) in the plane (which we will take to be equal to x and y) by the relations: $x = q_1$, $y = q_2$, and $z = 0$.

and so

$$E = \frac{\partial L}{\partial \dot{x}} \dot{x} + \frac{\partial L}{\partial \dot{y}} \dot{y} - L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k}{2}(x^2 + y^2), \quad (15.11)$$

which is indeed the energy.

In polar coordinates, we have

$$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{kr^2}{2}, \quad (15.12)$$

and so

$$E = \frac{\partial L}{\partial \dot{r}} \dot{r} + \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{kr^2}{2}, \quad (15.13)$$

which is again the energy. As mentioned above, the Cartesian-coordinate E is always the energy. The fact that the polar-coordinate E is also the energy is consistent with Eq. (15.9), because the Cartesian coordinates are functions of the polar coordinates: $x = r \cos \theta$ and $y = r \sin \theta$. In both cases, E is conserved because L has no explicit t dependence (see Claim 6.3 in Chapter 6).

Example 2 (Bead on a rod): A bead of mass m is constrained to move along a massless rod that is pivoted at the origin and arranged (via an external torque) to rotate with constant angular speed ω in a horizontal plane. A spring with spring constant k and relaxed length zero lies along the rod and connects the mass to the origin, as shown in Fig. 15.2. Find L and E in terms of the polar coordinate r , and show that E is conserved but it is not the energy.

Solution: The kinetic energy comes from both radial and tangential motion, so the Lagrangian is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) - \frac{kr^2}{2}. \quad (15.14)$$

E is then

$$E = \frac{\partial L}{\partial \dot{r}} \dot{r} - L = \frac{m}{2}(\dot{r}^2 - r^2\omega^2) + \frac{kr^2}{2}. \quad (15.15)$$

Since L has no explicit t dependence, Claim 6.3 tells us that E is conserved. However, E is not the energy, due to the minus sign in front of the $r^2\omega^2$ term. This is consistent with the above discussion, because the relations between the Cartesian coordinates and the coordinate r (namely $x = r \cos \omega t$ and $y = r \sin \omega t$) involve t and are therefore *not* of the form of Eq. (15.9).

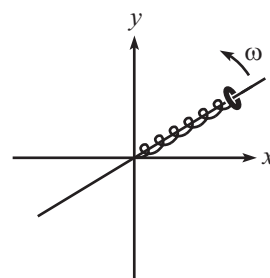


Figure 15.2

Example 3 (Accelerating rod): A bead of mass m is constrained to move on a horizontal rod that is accelerated vertically with constant acceleration a , as shown in Fig. 15.3. Find L and E in terms of the Cartesian coordinate x . Is E conserved? Is E the energy?

Solution: Since $\dot{y} = at$ and $y = at^2/2$, the Lagrangian is

$$L = \frac{m}{2}(\dot{x}^2 + (at)^2) - mg\left(\frac{at^2}{2}\right). \quad (15.16)$$

E is then

$$E = \frac{\partial L}{\partial \dot{x}} \dot{x} - L = \frac{m}{2}(\dot{x}^2 - (at)^2) + mg\left(\frac{at^2}{2}\right). \quad (15.17)$$

E is not conserved, due to the explicit t dependence in L . Also, E is not the energy, due to the minus sign in front of the $(at)^2$ term. This is consistent with the fact that the transformations from the single coordinate along the rod, x , to the two Cartesian coordinates (x, y) are $x = x$ and $y = 0 \cdot x + at$, and the latter involves t .

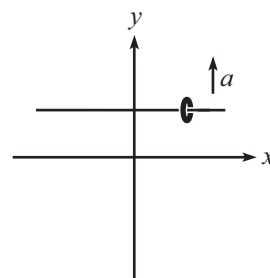


Figure 15.3

REMARKS: In this third example, the t dependence in the transformation between the two sets of coordinates (which caused E to not be the energy) in turn brought about the t dependence in L (which caused E to not be conserved, by Claim 6.3). However, this “bringing about” isn’t a logical necessity, as we saw in the second example above. There, the t dependence in the transformation didn’t show up in L , because all the t ’s canceled out in the calculation of $\dot{x}^2 + \dot{y}^2$, leaving only $\dot{r}^2 + r^2\omega^2$.

The above three examples cover three of the four possible permutations of E equalling or not equalling the energy, and E being conserved or not conserved. An example of the fourth permutation (where E is the energy, but it isn’t conserved) is the Lagrangian $L = m\dot{x}^2/2 - V(x, t)$. This yields $E = m\dot{x}^2/2 + V(x, t)$, which is the energy. But E isn’t conserved, due to the t dependence in $V(x, t)$. ♣

15.2 Hamilton’s equations

15.2.1 Defining the Hamiltonian

Our goal in this section is to rewrite E in a particular way that will lead to some very useful results, in particular Hamilton’s equations in Section 15.2.2 and Liouville’s theorem in Section 15.5. To start, note that L (and hence E) is a function of q and \dot{q} . We’ll ignore the possibility of t dependence here, since it is irrelevant for the present purposes. Also, we’ll work in just 1-D for now, to concentrate on the main points. Let’s be explicit about the q and \dot{q} dependence and write E as

$$E(q, \dot{q}) \equiv \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \dot{q} - L(q, \dot{q}). \quad (15.18)$$

Our strategy for rewriting E will be to exchange the variable \dot{q} for the variable p , defined by

$$p \equiv \frac{\partial L}{\partial \dot{q}}. \quad (15.19)$$

We already introduced p in Section 6.5.1; it is called the *generalized momentum* or the *conjugate momentum* associated with the coordinate q . It need not, however, have the units of standard linear momentum, as we saw in the examples in Section 6.5.1. Once we exchange \dot{q} for p , the variables in E will be q and p , instead of q and \dot{q} . Unlike q and \dot{q} (where one is simply the time derivative of the other), the variables q and p are truly independent; we’ll discuss this below at the end of Section 15.4.1.

In order to make this exchange, we need to be able to write \dot{q} in terms of q and p . We can do this (at least in theory) by inverting the definition of p , namely $p \equiv \partial L / \partial \dot{q}$, to solve for \dot{q} in terms of q and p . In many cases this inversion is simple. For example, the linear momentum associated with the Lagrangian $L = m\dot{x}^2/2 - V(x)$ is $p = \partial L / \partial \dot{x} = m\dot{x}$, which can be inverted to yield $\dot{x} = p/m$ (which happens to involve p but not x), as we well know. And the angular momentum associated with the central-force Lagrangian $L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$ is $p_\theta = \partial L / \partial \dot{\theta} = mr^2\dot{\theta}$ (we’re including the subscript θ just so that this angular momentum isn’t mistaken for a standard linear momentum), which can be inverted to yield $\dot{\theta} = p_\theta / mr^2$ (which involves both p_θ and r). However, in more involved setups this inversion can be complicated, or even impossible.

Having written \dot{q} in terms of q and p (that is, having produced the function $\dot{q}(q, p)$), we can now replace all the \dot{q} ’s in E with q ’s and p ’s, thereby yielding a

function of only q 's and p 's. When written in this way, the accepted practice is to use the letter H (for *Hamiltonian*) instead of the letter E . So it is understood that H is a function of only q and p , with no \dot{q} 's. Written explicitly, we have turned the E in Eq. (15.18) into the Hamiltonian given by

$$H(q, p) \equiv p\dot{q}(q, p) - L(q, \dot{q}(q, p)). \quad (15.20)$$

This point should be stressed: a Lagrangian is a function of q and \dot{q} , whereas a Hamiltonian is a function of q and p . This switch from \dot{q} to p isn't something we've done on a whim simply because we like one letter more than another. Rather, there are definite motivations (and rewards) for using (q, p) instead of (q, \dot{q}) , as we'll see at the end of Section 15.3.3.

If we have N coordinates q_i instead of just one, then from Eq. (15.1) the Hamiltonian is

$$H(q, p) \equiv \left(\sum_{i=1}^N p_i \dot{q}_i(q, p) \right) - L(q, \dot{q}(q, p)), \quad (15.21)$$

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \quad (15.22)$$

The arguments (q, p) above are shorthand for $(q_1, \dots, q_N, p_1, \dots, p_N)$. (And likewise for the \dot{q} in the last term.) So H is a function of (in general) $2N$ coordinates. And possibly the time t also (if L is a function of t), in which case the arguments (q, p) simply become (q, p, t)

Example (Harmonic oscillator): Consider a 1-D harmonic oscillator described by the Lagrangian $L = m\dot{x}^2/2 - kx^2/2$. The conjugate momentum is $p \equiv \partial L/\partial \dot{x} = m\dot{x}$, which yields $\dot{x} = p/m$. So the Hamiltonian is

$$\begin{aligned} H = p\dot{x} - L &= p\dot{x} - \left(\frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 \right) \\ &= p \left(\frac{p}{m} \right) - \frac{m}{2} \left(\frac{p}{m} \right)^2 + \frac{k}{2}x^2 \\ &= \frac{p^2}{2m} + \frac{kx^2}{2}, \end{aligned} \quad (15.23)$$

where we now have H in terms of x and p , with no \dot{x} 's. H is simply the energy, expressed in terms of x and p .

Example (Central force): Consider a 2-D central-force setup described by the Lagrangian $L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$. The two conjugate momenta are

$$\begin{aligned} p_r &\equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r} \implies \dot{r} = \frac{p_r}{m}, \\ p_\theta &\equiv \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \implies \dot{\theta} = \frac{p_\theta}{mr^2}. \end{aligned} \quad (15.24)$$

So the Hamiltonian is

$$\begin{aligned} H &= \left(\sum p_i \dot{q}_i \right) - L \\ &= p_r \dot{r} + p_\theta \dot{\theta} - \left(\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \right) \\ &= p_r \left(\frac{p_r}{m} \right) + p_\theta \left(\frac{p_\theta}{mr^2} \right) - \frac{m}{2} \left(\frac{p_r}{m} \right)^2 - \frac{mr^2}{2} \left(\frac{p_\theta}{mr^2} \right)^2 + V(r) \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r). \end{aligned} \quad (15.25)$$

H expresses the energy (it is indeed the energy, by Theorem 15.1, because the transformation from Cartesian to polar coordinates doesn't involve t) in terms of r , p_r , and p_θ . It happens to be independent of θ , and this will have consequences, as we'll see below in Section 15.2.3.

Note that in both of these examples, the actual form of the potential energy is irrelevant. It simply gets carried through the calculation, with only a change in sign in going from L to H .

15.2.2 Derivation of Hamilton's equations

Having constructed H as a function of the variables q and p , we can now derive two results that are collectively known as *Hamilton's equations*. And then we'll present three more derivations in Section 15.4, in the interest of going overboard. As in the previous section, we'll ignore the possibility of t dependence (because it wouldn't affect the discussion), and we'll start by working in just 1-D (to keep things from getting too messy).

Since H is a function of q and p , a reasonable thing to do is determine how H depends on these variables, that is, calculate the partial derivatives $\partial H/\partial q$ and $\partial H/\partial p$. Let's look at the latter derivative first. From the definition $H \equiv p\dot{q} - L$, we have

$$\frac{\partial H(q, p)}{\partial p} = \frac{\partial(p\dot{q}(q, p))}{\partial p} - \frac{\partial L(q, \dot{q}(q, p))}{\partial p}, \quad (15.26)$$

where we have explicitly indicated the q and p dependences. In the first term on the right-hand side, $p\dot{q}(q, p)$ depends on p partly because of the factor of p , and partly because \dot{q} depends on p . So we have $\partial(p\dot{q}(q, p))/\partial p = \dot{q} + p(\partial\dot{q}/\partial p)$. In the second term, $L(q, \dot{q}(q, p))$ depends on p only through its dependence on \dot{q} , so we have

$$\frac{\partial L(q, \dot{q}(q, p))}{\partial p} = \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \frac{\partial \dot{q}(q, p)}{\partial p}. \quad (15.27)$$

But $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$. If we substitute these results into Eq. (15.26), we obtain (dropping the (q, p) arguments)

$$\begin{aligned} \frac{\partial H}{\partial p} &= \left(\dot{q} + p \frac{\partial \dot{q}}{\partial p} \right) - p \frac{\partial \dot{q}}{\partial p} \\ &= \dot{q}. \end{aligned} \quad (15.28)$$

This nice result is no coincidence. It is a consequence of the properties of the Legendre transform (as we'll see in Section 15.3), which explains the symmetry between the relations $p \equiv \partial L/\partial \dot{q}$ and $\dot{q} = \partial H/\partial p$.

Let's now calculate $\partial H/\partial q$. From the definition $H \equiv p\dot{q} - L$, we have

$$\frac{\partial H(q, p)}{\partial q} = \frac{\partial(p\dot{q}(q, p))}{\partial q} - \frac{\partial L(q, \dot{q}(q, p))}{\partial q}. \quad (15.29)$$

In the first term on the right-hand side, $p\dot{q}(q, p)$ depends on q only through its dependence on \dot{q} , so we have $\partial(p\dot{q}(q, p))/\partial q = p(\partial\dot{q}/\partial q)$. In the second term, $L(q, \dot{q}(q, p))$ depends on q partly because q is the first argument, and partly because \dot{q} depends on q . So we have

$$\frac{\partial L(q, \dot{q}(q, p))}{\partial q} = \frac{\partial L(q, \dot{q})}{\partial q} + \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \frac{\partial \dot{q}(q, p)}{\partial q}. \quad (15.30)$$

As above, we have $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$. But also, the Euler-Lagrange equation (which holds, since we're looking at the actual classical motion of the particle) tells us that

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) = \frac{\partial L(q, \dot{q})}{\partial q} \implies \dot{p} = \frac{\partial L(q, \dot{q})}{\partial q}. \quad (15.31)$$

If we substitute these results into Eq. (15.29), we obtain (dropping the (q, p) arguments)

$$\begin{aligned} \frac{\partial H}{\partial q} &= p \frac{\partial \dot{q}}{\partial q} - \left(\dot{p} + p \frac{\partial \dot{q}}{\partial q} \right) \\ &= -\dot{p}. \end{aligned} \quad (15.32)$$

Note that we needed to use the E-L equation to derive Eq. (15.32) but not Eq. (15.28). Putting the two results together, we have Hamilton's equations (bringing back in the (q, p) arguments):

$$\dot{q} = \frac{\partial H(q, p)}{\partial p}, \quad \text{and} \quad \dot{p} = -\frac{\partial H(q, p)}{\partial q}. \quad (15.33)$$

In the event that L (and hence H) is a function of t , the arguments (q, p) simply become (q, p, t) .

REMARKS:

1. Some of the above equations look a bit cluttered due to all the arguments of the functions. But things can get confusing if the arguments aren't written out explicitly. For example, the expression $\partial L/\partial q$ is ambiguous, as can be seen by looking at Eq. (15.30). Does $\partial L/\partial q$ refer to the $\partial L(q, \dot{q}(q, p))/\partial q$ on the left-hand side, or the $\partial L(q, \dot{q})/\partial q$ on the right-hand side? These two partial derivatives aren't equal; they differ by the second term on the right-hand side. So there is no way to tell what $\partial L/\partial q$ means without explicitly including the arguments.
2. Hamilton's equations in Eq. (15.33) are two first-order differential equations in the variables q and p . But you might claim that the first one actually isn't a differential equation, since we already know what \dot{q} is in terms of q and p (because we previously needed to invert the expression $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$ to solve for the function $\dot{q}(q, p)$), which means that we can simply write $\dot{q}(q, p) = \partial H(q, p)/\partial p$ for the first equation, which has only q 's and p 's, with no time derivatives. It therefore seems like we can use this to quickly solve for q in terms of p , or vice versa.

However, this actually doesn't work, because the equation $\dot{q}(q, p) = \partial H(q, p)/\partial p$ is *identically* true (equivalent to $0 = 0$) and thus contains no information. The reason for this is that the function $\dot{q}(q, p)$ is derived from the definitional equation $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$. And as we'll see in Section 15.3, this equation contains the same information as the equation $\dot{q}(q, p) = \partial H(q, p)/\partial p$, due to the properties of the Legendre transform. So no new information is obtained by combining them. This will be evident in the examples below, where the first of Hamilton's equations always simply reproduces the definition of p .

Similarly, if you want to use the E-L equation, Eq. (15.31), to substitute $\partial L(q, \dot{q})/\partial \dot{q}$ for \dot{p} in the second of Hamilton's equations (and then plug in $\dot{q}(q, p)$ for \dot{q} *after* taking this partial derivative, because you're interested in $\partial L(q, \dot{q})/\partial q$ and not $\partial L(q, \dot{q}(q, p))/\partial q$), then you will find that you end up with $0 = 0$ after simplifying. This happens because the equation $\partial L(q, \dot{q})/\partial \dot{q} = -\partial H(q, p)/\partial q$ is identically true (see Eq. (15.43) below). ♣

Example (Harmonic oscillator): From the example in Section 15.2.1, the Hamiltonian for a 1-D harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}. \quad (15.34)$$

Hamilton's equations are then

$$\begin{aligned} \dot{x} = \frac{\partial H}{\partial p} &\implies \dot{x} = \frac{p}{m}, & \text{and} \\ \dot{p} = -\frac{\partial H}{\partial x} &\implies \dot{p} = -kx. \end{aligned} \quad (15.35)$$

The first of these simply reproduces the definition of p (namely $p \equiv \partial L / \partial \dot{x} = m\dot{x}$). The second one, when $m\dot{x}$ is substituted for p , yields the $F = ma$ equation,

$$\frac{d(m\dot{x})}{dt} = -kx \implies m\ddot{x} = -kx. \quad (15.36)$$

Many variables

As with the Lagrangian method, the nice thing about the Hamiltonian method is that it quickly generalizes to more than one variable. That is, Hamilton's equations in Eq. (15.33) hold for each coordinate q_i and its conjugate momentum p_i . So if there are N coordinates q_i (and hence N momenta p_i), then we end up with $2N$ equations in all.

The derivation of Hamilton's equations for each q_i and p_i proceeds in exactly the same manner as above, with the only differences being that the index i is tacked on to q and p , and certain terms become sums (but these cancel out in the end). We'll simply write down the equations here, and you can work out the details in Problem 15.4. The $2N$ equations for the Hamiltonian in Eq. (15.21) are:

$$\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H(q, p)}{\partial q_i}, \quad \text{for } 1 \leq i \leq N. \quad (15.37)$$

The arguments (q, p) above are shorthand for $(q_1, \dots, q_N, p_1, \dots, p_N)$. And as in the 1-D case, if L (and hence H) is a function of t , then a t is tacked on to these $2N$ variables.

Example (Central force): From the example in Section 15.2.1, the Hamiltonian for a 2-D central force setup is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r). \quad (15.38)$$

The four Hamilton's equations are then

$$\begin{aligned} \dot{r} = \frac{\partial H}{\partial p_r} &\implies \dot{r} = \frac{p_r}{m}, \\ \dot{p}_r = -\frac{\partial H}{\partial r} &\implies \dot{p}_r = \frac{p_\theta^2}{mr^3} - V'(r), \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} &\implies \dot{\theta} = \frac{p_\theta}{mr^2}, \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} &\implies \dot{p}_\theta = 0. \end{aligned} \quad (15.39)$$

The first and third of these simply reproduce the definitions of p_r and p_θ . The second equation, when $m\dot{r}$ is substituted for p_r , yields the equation of motion for r , namely

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - V'(r). \quad (15.40)$$

This agrees with Eq. (7.8) with $p_\theta \leftrightarrow L$. And the fourth equation is the statement that angular momentum is conserved. If $mr^2\dot{\theta}$ is substituted for p_θ , it says that

$$\frac{d(mr^2\dot{\theta})}{dt} = 0. \quad (15.41)$$

Conservation of angular momentum arises because θ is a cyclic coordinate, so let's talk a little about these...

15.2.3 Cyclic coordinates

In the Lagrangian formalism, the E-L equation tells us that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \implies \dot{p} = \frac{\partial L}{\partial q}. \quad (15.42)$$

Therefore, if a given coordinate q is cyclic (that is, it doesn't appear) in $L(q, \dot{q})$, then $\partial L(q, \dot{q})/\partial q = 0$, and so $\dot{p} = 0$. That is, p is conserved. In the Hamiltonian formalism, this conservation comes about from the fact that q is also cyclic in $H(q, p)$, which means that $\partial H(q, p)/\partial q = 0$, and so the Hamilton equation $\dot{p} = -\partial H(q, p)/\partial q$ yields $\dot{p} = 0$.

To show that everything is consistent here, we should check that a coordinate q is cyclic in $H(q, p)$ if and only if it is cyclic in $L(q, \dot{q})$. This must be the case, of course, in view of the fact that both the Lagrangian and Hamiltonian formalisms are logically sound descriptions of classical mechanics. But let's verify it, just to get more practice throwing partial derivatives around.

We'll calculate $\partial H(q, p)/\partial q$ in terms of $\partial L(q, \dot{q})/\partial q$. This calculation is basically a repetition of the one leading up to Eq. (15.32). We'll stick to 1-D (the calculation in the case of many variables is essentially done in Problem 15.4), and we'll ignore any possible t dependence (since it would simply involve tacking a t on to the arguments below). From the definition $H(q, p) \equiv p\dot{q}(q, p) - L(q, \dot{q}(q, p))$, we have

$$\begin{aligned} \frac{\partial H(q, p)}{\partial q} &= \frac{\partial(p\dot{q}(q, p))}{\partial q} - \frac{\partial L(q, \dot{q}(q, p))}{\partial q} \\ &= p \frac{\partial \dot{q}(q, p)}{\partial q} - \left(\frac{\partial L(q, \dot{q})}{\partial q} + \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \frac{\partial \dot{q}(q, p)}{\partial q} \right) \\ &\equiv p \frac{\partial \dot{q}(q, p)}{\partial q} - \left(\frac{\partial L(q, \dot{q})}{\partial q} + p \frac{\partial \dot{q}(q, p)}{\partial q} \right) \\ &= -\frac{\partial L(q, \dot{q})}{\partial q}. \end{aligned} \quad (15.43)$$

Therefore, if one of the partial derivatives is zero, then the other is also, as we wanted to show.

REMARK: Beware of the following incorrect derivation of the $\partial H/\partial q = -\partial L/\partial q$ result: From the definition $H \equiv p\dot{q} - L$, we have $\partial H/\partial q = \partial(p\dot{q})/\partial q - \partial L/\partial q$. Since neither p nor

\dot{q} depends on q , the first term on the right-hand side is zero, and we arrive at the desired result, $\partial H/\partial q = -\partial L/\partial q$.

The error in this reasoning is that \dot{q} *does* depend on q , because it is understood that when writing down H , \dot{q} must be eliminated and written in terms of q and p . So the first term on the right-hand side is in fact *not* equal to zero. Of course, now you might ask how we ended up with the correct result of $\partial H/\partial q = -\partial L/\partial q$ if we made an error in the process. The answer is that we actually *didn't* end up with the correct result, because if we include the arguments of the functions, what we really derived was $\partial H(q, p)/\partial q = -\partial L(q, \dot{q}(q, p))/\partial q$, which is *not* correct. The correct statement is $\partial H(q, p)/\partial q = -\partial L(q, \dot{q})/\partial q$. These two $\partial L/\partial q$ partial derivatives are not equal, as mentioned in the first remark following Eq. (15.33). When the calculation is done correctly, the “extra” term in $\partial L(q, \dot{q}(q, p))/\partial q$ (the last term in the third line of Eq. (15.43)) cancels the $\partial(p\dot{q})/\partial q$ contribution that we missed, and we end up with only the $\partial L(q, \dot{q})/\partial q$ term, as desired. Failure to explicitly write down the arguments of the various functions, especially L , often leads to missteps like this, so the arguments should always be included in calculations of this sort. ♣

15.2.4 Solving Hamilton’s equations

Assuming we’ve written down Hamilton’s equations for a given problem, we still need to *solve* them, of course, to obtain the various coordinates as functions of time. But we need to solve the equations of motion in the Lagrangian and Newtonian formalisms, too. The difference is that for n coordinates, we now have $2n$ *first-order* differential equations in $2n$ variables (n coordinates q_i , and n momenta p_i), instead of n *second-order* differential equations in the n coordinates q_i (namely, the n E-L equations or the n $F = ma$ equations).

In practice, the general procedure for solving the $2n$ Hamilton’s equations is to eliminate the n momenta p_i in favor of the n coordinates q_i and their derivatives \dot{q}_i . This produces n second-order differential equations in the coordinates q_i (as we found in the harmonic-oscillator and central-force examples above), which means that we’re basically in the same place we would be if we simply calculated the n E-L equations or wrote down the n $F = ma$ equations. So certainly nothing is gained, at least in the above two examples, by using the Hamiltonian formalism over the Lagrangian method or $F = ma$.

As mentioned in the introduction to this chapter, this is invariably the case with systems involving a few particles; nothing at all is gained (and usually much is lost, at least with regard to speed and simplicity) when using the Hamiltonian instead of the Lagrangian formalism. In the latter, you write down the Lagrangian and then calculate the E-L equations. In the former, you write down the Lagrangian, and then go through a series of other steps, and then finally arrive back at the E-L (or equivalent) equations. However, in other branches in physics, most notably statistical mechanics and quantum mechanics, the Hamiltonian formalism ranges from being extremely helpful for understanding what is going on, to being absolutely necessary for calculating anything useful.

When solving ordinary mechanics problems with the Hamiltonian formalism, you should keep in mind that the purpose is not to gain anything in efficiency, but rather to become familiar with a branch of physics that has numerous indispensable applications to other branches. Having said all this, we’ll list here the concrete steps that you need to follow when using the Hamiltonian method:

1. Calculate T and V , and write down the Lagrangian, $L \equiv T - V$, in terms of

whatever coordinates q_i (and their derivatives \dot{q}_i) you find convenient.

2. Calculate $p_i \equiv \partial L / \partial \dot{q}_i$ for each of the N coordinates.
3. Invert the expressions for the N p_i to solve for the N \dot{q}_i in terms of the q_i and p_i .²
4. Write down the Hamiltonian, $H \equiv (\sum p_i \dot{q}_i) - L$, and then eliminate all the \dot{q}_i in favor of the q_i and p_i , as indicated in Eq. (15.21).
5. Write down Hamilton's equations, Eq. (15.37); two of them for each of the N coordinates.
6. Solve Hamilton's equations; the usual goal is to obtain the N functions $q_i(t)$. This generally involves eliminating the p_i in favor of the \dot{q}_i . This will turn the $2N$ first-order differential Hamilton's equations into N second-order differential equations. These will be equivalent, in one way or another, to what you would have obtained if you had simply written down the E-L equations after step (1) above.

15.3 Legendre transforms

15.3.1 Formulation

The definition of E in Eq. (15.1), or equivalently H in Eq. (15.20), may seem a bit arbitrary to you. However, there is in fact a definite motivation for it, and this motivation comes from the theory of Legendre transforms. The H defined in Eq. (15.20) is called the *Legendre transform* of L , and we'll now give a review of this topic.

To isolate the essentials, let's forget about H and L for now and consider a function of just one variable, $F(x)$. The derivative of $F(x)$ is $dF(x)/dx \equiv F'(x)$, but for notational convenience let's relabel $F'(x)$ as $s(x)$ (s for "slope"). So we have $dF(x)/dx \equiv s(x)$. Note that we can (at least in principle; and at least locally, assuming s isn't constant) invert the function $s(x)$ to solve for x as a function of s , yielding $x(s)$. For example, if $F(x) = x^3$, then $s(x) \equiv F'(x) = 3x^2$. Inverting this gives $x(s) = (s/3)^{1/2}$.

The purpose of the Legendre transform is to construct another function (call it G) that reverses the roles of x and s . That is, the goal is to construct a function $G(s)$ with the property that

$$\frac{dG(s)}{ds} = x(s). \quad (15.44)$$

$G(s)$ is then called the Legendre transform of $F(x)$. Now, the most obvious way to find $G(s)$ is to simply integrate this equation. In the case of $F(x) = x^3$, we're looking for a function $G(s)$ whose derivative is $(s/3)^{1/2}$. Integrating this, we quickly see that the desired function is $G(s) = 2(s/3)^{3/2}$. So this is the Legendre transform of $F(x) = x^3$. However, it turns out that there is another method that doesn't involve the task of integrating. The derivation of this method is as follows.

If such a G exists, then we can add the two equations, $dF = s dx$ and $dG = x ds$, to obtain $d(F + G) = s dx + x ds \implies d(F + G) = d(sx)$. This implies that

²In some cases the expressions for the p_i may be "coupled" (that is, a given \dot{q}_i may appear in more than one of them), thereby making the inversion difficult (or impossible). But in most cases the equations are uncoupled and easily invertible, as they were in the harmonic-oscillator and central-force examples above.

$F + G = sx$, up to an additive constant which is taken to be zero by convention. The Legendre transform of F is therefore

$$G = sx - F \equiv F'x - F. \quad (15.45)$$

This function G can be written as a function of either x or s (since each of these can be written as a function of the other). But when we talk about Legendre transforms, it is understood that G is a function of s . So to be precise, we should write

$$G(s) = sx(s) - F(x(s)). \quad (15.46)$$

Let's see what this gives in the $F(x) = x^3$ example. We have

$$\begin{aligned} G(s) = sx(s) - F(x(s)) &= s(s/3)^{1/2} - F((s/3)^{1/2}) \\ &= (3 \cdot s/3) \cdot (s/3)^{1/2} - (s/3)^{3/2} \\ &= (3 - 1)(s/3)^{3/2} \\ &= 2(s/3)^{3/2}, \end{aligned} \quad (15.47)$$

in agreement with the above result obtained by direct integration. The advantage of this new method is that it involves the straightforward calculation in Eq. (15.46), whereas the process of integration can often be tricky.

As a double check on Eq. (15.46), let's verify that the $G(s)$ given there does indeed have the property that $dG(s)/ds$ equals x . Using the product and chain rules, along with the definition of s , we have

$$\frac{dG(s)}{ds} = \left(1 \cdot x + s \cdot \frac{dx}{ds} \right) - \frac{dF}{dx} \cdot \frac{dx}{ds} = x + s \cdot \frac{dx}{ds} - s \cdot \frac{dx}{ds} = x, \quad (15.48)$$

as desired.

What is the Legendre transform of $G(s)$? From Eq. (15.45) we see that the rule for calculating the transform is to subtract the function from the product of the slope and the independent variable. In the case of $G(s)$, the slope is x and the independent variable is s . So the Legendre transform of $G(s)$ is

$$G's - G = xs - G = F, \quad (15.49)$$

where the second equality follows from Eq. (15.45). Since F is a function of x , we should make this clear and write $xs(x) - G(s(x)) = F(x)$. We therefore see that two applications of the Legendre transform bring us back to the function we started with. In other words, the Legendre transform is the inverse of itself.

Let's calculate the Legendre transform of a few functions.

Example 1: $F(x) = x^2$:

We have $s \equiv F'(x) = 2x$, so $x(s) = s/2$. The Legendre transform of $F(x)$ is then

$$G(s) = sx(s) - F(x(s)) = s \cdot (s/2) - (s/2)^2 = s^2/4. \quad (15.50)$$

Therefore, $dG(s)/ds = s/2$, which equals x , as expected.

Example 2: $F(x) = \ln x$:

We have $s \equiv F'(x) = 1/x$, so $x(s) = 1/s$. The Legendre transform of $F(x)$ is then

$$G(s) = sx(s) - F(x(s)) = s \cdot (1/s) - \ln(1/s) = 1 + \ln s. \quad (15.51)$$

Therefore, $dG(s)/ds = 1/s$, which equals x , as expected.

Example 3: $F(x) = e^x$:

We have $s \equiv F'(x) = e^x$, so $x(s) = \ln s$. The Legendre transform of $F(x)$ is then

$$G(s) = sx(s) - F(x(s)) = s \cdot \ln s - s. \tag{15.52}$$

Therefore, $dG(s)/ds = (s \cdot (1/s) + 1 \cdot \ln s) - 1 = \ln s$, which equals x , as expected.

15.3.2 Geometric meaning

It is easy to demonstrate geometrically the meaning of the Legendre transform given in Eq. (15.45). And a byproduct of this geometrical explanation is a quick way of seeing why $dG/ds = x$, as we verified algebraically in Eq. (15.48). Consider the function $F(x)$ shown in Fig. 15.4, and draw the tangent line at the point $(x, F(x))$. Let the intersection of this line with the y axis be labeled as point A . And label point B as shown. From the definition of the slope s of the tangent line, the distance AB is sx . Therefore, since the distance OB is $F(x)$, the distance AO is $sx - F(x)$. Note that this distance is defined to be positive if point A lies below the origin. In other words, $sx - F(x)$ equals the negative of the y value of point A . In view of Eq. (15.45), we therefore have our geometric interpretation: For a given x , the value of the Legendre transform of $F(x)$ is simply the negative of the y value of the y intercept of the tangent line.

However, recall that the Legendre transform is understood to be a function of the slope s . So the step-by-step procedure for geometrically obtaining the Legendre transform is the following. Given a function $F(x)$ and a particular value of x , we can draw the tangent line at the point $(x, F(x))$. Let the slope be $s(x)$. We can then find the y intercept of this line. A given coordinate x produces a value of this y intercept, so the y intercept may be thought of as a function of x . However, we can (generally) invert $s(x)$ to obtain the function $x(s)$, so the y intercept (the Legendre transform) may also be considered to be a function of s . In other words,

$$G(s) = (y \text{ intercept}) = sx(s) - F(x(s)). \tag{15.53}$$

As an example, Fig. 15.5 shows a tangent line to the function $F(x) = x^2$. The slope is $s(x) = F'(x) = 2x$, so the distance AO is $G = sx - F(x) = (2x)x - x^2 = x^2$. Writing this as a function of s , we have $G(s) = x(s)^2 = (s/2)^2 = s^2/4$, as in the first example above.

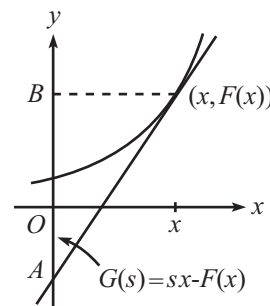


Figure 15.4

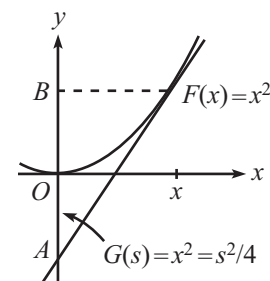


Figure 15.5

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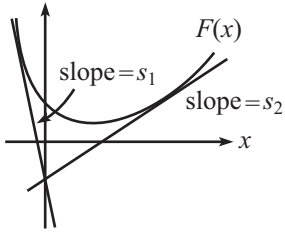


Figure 15.6

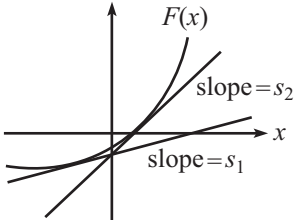


Figure 15.7

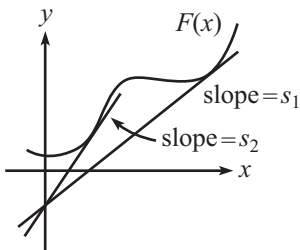


Figure 15.8

REMARK: Can two different values of s yield the same value of $G(s)$? Yes indeed, as shown in Fig. 15.6. The tangent lines with slopes s_1 and s_2 both yield the same y intercept, so $G(s_1) = G(s_2)$. Note that in the special case where $F(x)$ is an even function of x , both s and $-s$ yield the same y intercept, so $G(s)$ is also an even function of s (assuming it is well defined; see the next paragraph). Can two different values of s with the same sign yield the same y intercept? Yes again, as shown in Fig. 15.7. Can two different values of s (of any signs) yield the same y intercept if the two associated values of x have the same sign? Still yes, as shown in Fig. 15.8. The equality of the two $G(s)$ values is shown schematically in Fig. 15.9. The task of Problem 15.14 is to show that $F(x)$ must have an inflection point (a point where the second derivative is zero) for this last scenario to be possible.

Can a given value of s yield two different values of $G(s)$ (which would make the function $G(s)$ not well defined)? Yes, as shown in Fig. 15.10. The same slope yields two different y intercepts. The double-valued nature of $G(s)$ is shown schematically in Fig. 15.11. The task of Problem 15.15 is to show that $F(x)$ must have an inflection point (a point where the second derivative is zero) for this scenario to be possible. Therefore, if there is no inflection point (that is, if $F(x)$ is concave upward or downward), then $G(s)$ is well defined. ♣

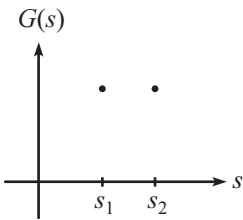


Figure 15.9

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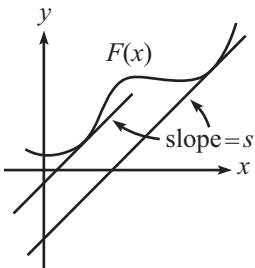


Figure 15.10

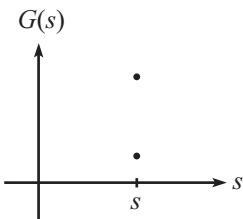


Figure 15.11

Armed with the above y -intercept interpretation of the Legendre transform, we can now see geometrically why $dG(s)/ds = x$. First, note that if we are given the function $G(s)$, we can reconstruct the function $F(x)$ in the following way. For a given value of s , we can draw the line with slope s and y intercept $G(s)$, as shown in Fig. 15.12. We can then draw the line for another value of s , and so on. The envelope of all these lines is the function $F(x)$. This follows from the geometric construction of $G(s)$ we discussed above. Fig. 15.13 shows the parabolic $F(x) = x^2$ function that results from the function $G(s) = s^2/4$.

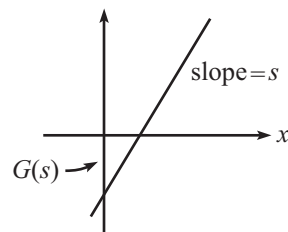


Figure 15.12

In Fig. 15.14 consider two nearby values of s (call them s_1 and s_2), and draw the lines with slopes s_i and y intercepts $G(s_i)$. Due to the above fact that the function $F(x)$ is the envelope of the tangent lines, these two lines will meet at the point $(x, F(x))$, where x is the coordinate that generates the coordinate s via the Legendre transformation (where $s = s_1 = s_2$ in the limit where s_1 and s_2 are infinitesimally close to each other).

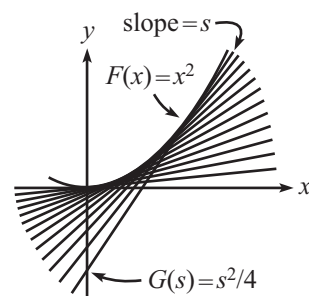


Figure 15.13

Define the points P , B , A_1 , and A_2 as shown. Using triangles PBA_1 and PBA_2 , along with the definition of the slope s , we see that the distances BA_1 and BA_2 are simply the slopes multiplied by x . So they are sx_1 and sx_2 . But from our geometric interpretation of the Legendre transform, these distances are also $BA_1 = F(x) + G(s_1)$ and $BA_2 = F(x) + G(s_2)$, where this is the *same* $F(x)$ appearing in both of these equations. We therefore have

$$s_1x = F(x) + G(s_1), \quad \text{and} \quad s_2x = F(x) + G(s_2). \quad (15.54)$$

Subtracting these gives

$$(s_2 - s_1)x = G(s_2) - G(s_1) \implies x = \frac{G(s_2) - G(s_1)}{s_2 - s_1}. \quad (15.55)$$

In the limit $s_1 \rightarrow s_2$, this becomes $x = dG(s)/ds$, as desired. Basically, the point is that multiplying x by the difference in slopes gives the difference in the y intercepts (the $G(s)$ values). That is, $x\Delta s = \Delta G \implies x = dG/ds$. This is the “mirror” statement of the standard fact that multiplying the slope of a function by the difference in x values gives the difference in the $F(x)$ values (to first order, at least). That is, $s\Delta x = \Delta F \implies s = dF/dx$.

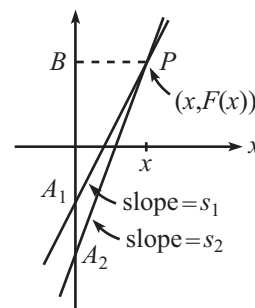


Figure 15.14

15.3.3 Application to the Hamiltonian

Let’s now look at how the Legendre transform relates to the Hamiltonian, H . The Lagrangian L is a function of q and \dot{q} (and possibly t , but we’ll ignore this dependence, since it is irrelevant for the present discussion). We will choose to form the Legendre transform of L with respect to the variable \dot{q} . That is, we will ignore the q dependence and consider L to be a function of only \dot{q} . (The terminology is then that \dot{q} is the “active” variable and q is the “passive” variable.) So \dot{q} is the analog of the variable x in the preceding discussion, and $p \equiv \partial L/\partial \dot{q}$ is the analog of $s \equiv dF/dx$. Following Eq. (15.46), the Legendre transform of L (with \dot{q} as the active variable) is defined to be the Hamiltonian,

$$H(q, p) \equiv p\dot{q}(q, p) - L(q, \dot{q}(q, p)) \quad (15.56)$$

This is simply Eq. (15.46) with G, F, x, s replaced by H, L, \dot{q}, p , respectively, and with q (the passive variable) tacked on all the arguments.

Having defined H as the Legendre transform of L in this way, the statement analogous to the $x = dG/ds$ result in Eq. (15.44) is (with G, x, s replaced by H, \dot{q}, p ,

respectively, and with q tacked on all the arguments)

$$\dot{q}(q, p) = \frac{\partial H(q, p)}{\partial p}. \quad (15.57)$$

This is the first of Hamilton's equations in Eq. (15.33). As mentioned in the second remark following Eq. (15.33), this equation comes about simply by the definition of the Legendre transform. There is no physics involved.

REMARK: In practice, when solving for q as a function of t in a given problem by combining Eq. (15.57) with the second Hamilton's equation, the usual strategy is not to think of \dot{q} as a function of q and p as indicated in Eq. (15.57), but rather to eliminate p by writing it in terms of q and \dot{q} (see the examples in Section 15.2.2). ♣

We've thus far ignored the variable q . But since it is indeed an argument of H , it can't hurt to calculate the partial derivative $\partial H/\partial q$. But we already did this in Eq. (15.43), and the result is

$$\frac{\partial H(q, p)}{\partial q} = -\frac{\partial L(q, \dot{q})}{\partial q}. \quad (15.58)$$

(Note that any t dependence in L and H wouldn't affect this relation.) So far we've used only the definition of the Legendre transform; no mention has been made of any actual physics. But if we now combine Eq. (15.58) with the E-L equation, $d(\partial L/\partial \dot{q})/dt = \partial L/\partial q \implies \dot{p} = \partial L/\partial q$, we obtain the second Hamilton's equation,

$$\dot{p} = -\frac{\partial H}{\partial q}. \quad (15.59)$$

This has basically been a repeat of the derivation of Eq. (15.32).

Many variables

How do we form a Legendre transform if there is more than one active variable? To answer this, consider a general function of two variables, $F(x_1, x_2)$. The differential of F is

$$dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 \equiv s_1 dx_1 + s_2 dx_2, \quad (15.60)$$

where $s_1 \equiv \partial F/\partial x_1$ and $s_2 \equiv \partial F/\partial x_2$ are functions of x_1 and x_2 .

Let's assume we want to construct a function $G(s_1, s_2)$ with the properties that $\partial G/\partial s_1 = x_1$ and $\partial G/\partial s_2 = x_2$ (so x_1 and x_2 are functions of s_1 and s_2). If these relations hold, then the differential of G is

$$dG = \frac{\partial G}{\partial s_1} ds_1 + \frac{\partial G}{\partial s_2} ds_2 = x_1 ds_1 + x_2 ds_2. \quad (15.61)$$

Adding the two previous equations gives

$$\begin{aligned} d(F + G) &= (s_1 dx_1 + s_2 dx_2) + (x_1 ds_1 + x_2 ds_2) \\ &= (s_1 dx_1 + x_1 ds_1) + (s_2 dx_2 + x_2 ds_2) \\ &= d(s_1 x_1 + s_2 x_2). \end{aligned} \quad (15.62)$$

The Legendre transform of $F(x_1, x_2)$, with both x_1 and x_2 as active variables, is therefore $G = s_1 x_1 + s_2 x_2 - F$. But since it is understood that G is a function of s_1 and s_2 , we should write

$$G(s_1, s_2) = s_1 \cdot x_1(s_1, s_2) + s_2 \cdot x_2(s_1, s_2) - F(x_1(s_1, s_2), x_2(s_1, s_2)). \quad (15.63)$$

You can quickly check that this procedure generalizes to N variables. So if we have a function $F(x)$, where x stands for the N variables x_i , then the Legendre transform, with all the x_i as active variables, is

$$G(s) = \left(\sum s_i x_i \right) - F(x(s)), \quad (15.64)$$

where $s_i \equiv \partial F / \partial x_i$, and s stands for the N variables s_i . This is exactly the form of the Hamiltonian in Eq. (15.21), where H, L, p, \dot{q} take the place of G, F, s, x , respectively. So in the case of many variables, H is indeed (if you had any doubts) the Legendre transform of L , with the \dot{q}_i as the active variables.

Motivation

Let's now take a step back and ask why we should consider applying a Legendre transform to the Lagrangian in the first place. There are some deeper reasons for considering a Legendre transform, but a practical motivation is the following. In the Lagrangian formalism, we have two basic equations at our disposal. First, there is the simple definitional equation,

$$p \equiv \frac{\partial L}{\partial \dot{q}}. \quad (15.65)$$

And second, there is the E-L equation,

$$\dot{p} = \frac{\partial L}{\partial q}, \quad (15.66)$$

which involves some actual physics, namely the principle of stationary action. These two equations look a bit similar; they both have a p, q, L , and a "dot." However, one dot is in the numerator, and the other is in the denominator. So a reasonable goal is to try to make the equations more symmetrical. It would be nice if we could somehow switch the locations of the p and \dot{q} in the first equation and end up with something along the lines of $\dot{q} = \partial L / \partial p$. Now, we of course can't just go around switching the placements of variables in equations. But what we can do is recognize that the basic effect of a Legendre transform is precisely this switching. That is, it takes the equation $s = dF/dx$ and produces the equation $x = dG/ds$. In the case of the Legendre transform from the Lagrangian to the Hamiltonian, we can turn Eq. (15.65) into Eq. (15.57),

$$\dot{q} = \frac{\partial H}{\partial p}. \quad (15.67)$$

We now have the problem that there is an H here instead of an L , so this equation has lost its symmetry with Eq. (15.66). But this is easy to remedy by exchanging the L in Eq. (15.66) for an H . Again, we can't just go switching letters around, but Eq. (15.58) shows us how to do it legally. The only "penalty" is a minus sign, and we end up with

$$\dot{p} = -\frac{\partial H}{\partial q}. \quad (15.68)$$

The two preceding equations now have a remarkable amount of symmetry. The variables q and p are (nearly) interchangeable. The only nuisance is the minus sign in the second equation. However, two points: First, this minus sign is simply the way it is, and there's no getting around it. And second, the minus sign is actually much more of a blessing than a nuisance, for various reasons, one of which we'll see in Section 15.5 when we derive Liouville's theorem.

Our goal of (near) symmetry has therefore been achieved. Of course, you might ask why this should be our goal. But we'll just take this as an *a priori* fact: When a physical principle (which for the subject at hand is the principle of stationary action, because that is the critical ingredient in both the E-L and Hamilton's equations) is written in a form where a symmetry is evident, interesting and useful results are invariably much easier to see.

15.4 Three more derivations

We now present three more derivations of Hamilton's equations, in addition to the one we gave in Section 15.2.2. So we'll call them the 2nd, 3rd, and 4th derivations. The first of these shows in detail how the equations follow from the principle of stationary action, so it is well worth your attention, as is the "Discussion of coordinate independence, symmetry" that follows. However, the other two derivations aren't critical, so feel free to skip them for now and return later. As you'll see, all four of the derivations essentially come down to the same two ingredients (the principle of stationary action, plus the definition of H via the Legendre transform), so it's debatable whether they should be counted as distinct derivations. But since each one is instructive in its own right, we'll include them all.

In the following three derivations, we'll deal only with the 1-D case, lest the main points get buried in a swarm of indices. You can easily extend the reasoning to many dimensions; see the discussion at the end of Section 15.2.2. Also, time dependence is irrelevant in the 2nd and 3rd derivations, so we won't bother including t in the arguments. But it is relevant in the 4th derivation, so we'll include it there.

15.4.1 Second derivation

The derivation of Eq. (15.32), $\dot{p} = -\partial H/\partial q$, made use of the Euler-Lagrange equation. Therefore (as we've noted before), just as the E-L equation is a consequence of the principle of stationary action, so are Hamilton's equations (or at least the $\dot{p} = -\partial H/\partial q$ one). However, the issue of stationarity got a bit buried in the earlier derivation, so let's start from scratch here and derive Hamilton's equations directly from the principle of stationary action. The interesting thing we'll find is that *both* of the equations arise from this principle.

From the definition $H \equiv p\dot{q} - L$, we have $L = p\dot{q} - H$. So the action S may be written as

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (p\dot{q} - H) dt = \int_{t_1}^{t_2} (p dq - H dt). \quad (15.69)$$

If we want to be explicit with all the arguments, then the integrand should be written as $p(t)dq(t) - H(q(t), p(t)) dt$. In the end, everything is a function of t . As t marches along, the integral is obtained by adding up the tiny changes due to the dq and dt terms, until we end up with the total integral from a given t_1 to a given t_2 .

The integrand depends on the functions $p(t)$ and $q(t)$. We will now show that Hamilton's equations are a consequence of demanding that the action be stationary with respect to variations in *both* p and q . It turns out that we will need the variation in q to vanish at the endpoints of the path (as was the case in Chapter 6), but there will be no such restriction on the variation in p .

Let the variations in $p(t)$ and $q(t)$ yield the functions $p(t) + \delta p(t)$ and $q(t) + \delta q(t)$.³ Using the product rule, the S in Eq. (15.69) acquires the following first-order change due to these variations (dropping all the arguments, for ease of notation):

$$\delta S = \int_{t_1}^{t_2} \left(\delta p dq + p \delta(dq) - \left(\frac{\partial H}{\partial q} \delta q dt + \frac{\partial H}{\partial p} \delta p dt \right) \right). \quad (15.70)$$

In the second term here, we can rewrite $p \delta(dq)$ as $p d(\delta q)$. That is, the variation in the change (between two nearby times, due to the traversing of the path) in q equals the change (due to the traversing) in the variation in q .

REMARK: This is probably better explained in plain math than plain English. Let's change the notation so that $f(t)$ is the original function $q(t)$, and $g(t)$ is the modified function $q(t) + \delta q(t)$. Then $\delta q(t) = g(t) - f(t)$. (To repeat, so there isn't any confusion: the δ symbol is associated with the change from the function $f(t)$ to the function $g(t)$, and the d symbol is associated with changes due to time marching along.) If we consider two nearby times t_1 and t_2 , then what is $\delta(dq)$? Well, $df = f(t_2) - f(t_1)$ and $dg = g(t_2) - g(t_1)$. Therefore, $\delta(dq)$ (which is the change in dq in going from f to g) is

$$\delta(dq) = (g(t_2) - g(t_1)) - (f(t_2) - f(t_1)). \quad (15.71)$$

And what is $d(\delta q)$? Well, $\delta q(t) = g(t) - f(t)$. Therefore, $d(\delta q)$ (which is the change in $g(t) - f(t)$ in going from t_1 to t_2) is

$$d(\delta q) = (g(t_2) - f(t_2)) - (g(t_1) - f(t_1)). \quad (15.72)$$

The two preceding results are equal, as promised. ♣

If we now integrate the $p d(\delta q)$ term by parts, we obtain $\int p d(\delta q) = p \delta q - \int dp \delta q$. The first of these terms vanishes, assuming that we are requiring δq to be zero at the endpoints (which we are). So Eq. (15.70) becomes

$$\delta S = \int_{t_1}^{t_2} \left(\delta p \left(dq - \frac{\partial H}{\partial p} dt \right) - \delta q \left(dp + \frac{\partial H}{\partial q} dt \right) \right). \quad (15.73)$$

If we assume that the variations δp and δq are independent (see the following discussion for more on this assumption), then the terms in parentheses must independently vanish if we are to have $\delta S = 0$. Dividing each term by dt , we obtain Hamilton's equations,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad (15.74)$$

as desired.

Discussion of coordinate independence, symmetry

In view of the fact that two equations popped out of this variational argument, while only one (the E-L equation) popped out of the variational argument in Chapter 6, you might wonder if we performed some sort of cheat. Or said in another way: to obtain the second Hamilton's equation, we needed to integrate by parts and demand that the variation in q vanish at the endpoints. But we didn't need to do anything at all to obtain the first Hamilton's equation. This seems a little fishy. Or said in yet another way: there was no restriction on the variation in p (as long as it was

³The $\delta q(t)$ variation here is the same as the $a\beta(t)$ variation we used in Section 6.2. The present notation makes things look a little nicer.

infinitesimal, since we kept only the first-order changes in Eq. (15.70)); the variation didn't need to vanish at the endpoints. So we seem to have gotten something for nothing.

The explanation of all this is that we *did* get something for nothing. Or more precisely, we got something *by definition*. As we saw in Section 15.3.3, the first Hamilton's equation, $\dot{q} = \partial H/\partial p$ is true by definition due to the fact that H is defined to be the Legendre transform of L . So it's no surprise that we didn't need to make any effort in deriving this equation, and no surprise that there weren't any restrictions on the variation in p . The first term in parentheses in Eq. (15.73) is simply zero by definition.

Having said this, we should note that there are (at least) two possible points of view concerning the stationary property of S when written in terms of H (and thus written in terms of q and p , instead of the q and \dot{q} in the Lagrangian formalism).

- **FIRST POINT OF VIEW:** The facts we will take are given are (1) the action $\int L$ is stationary with respect to variations in q that vanish at the endpoints, and (2) H is the Legendre transform of L . We then find:

Looking at the second term in Eq. (15.73), fact (1) implies that Hamilton's second equation holds. Looking at the first term in Eq. (15.73), we see that it is zero by definition (that is, Hamilton's first equation holds by definition), because of fact (2).⁴ It then *follows* that $\delta S = 0$, no matter how we choose to vary p . So we conclude that we can vary q and p independently and have S be stationary, as long as Hamilton's equations are satisfied.

- **SECOND POINT OF VIEW:** The facts we will take are given are (1) we start with the Hamiltonian, $H(p, q)$, and pretend that we don't know anything about Legendre transforms or the Lagrangian (and in particular the definition $p \equiv \partial L/\partial \dot{q}$), and (2) the integral⁵ $\int (p \dot{q} - H) dt$ is stationary with respect to independent variations in the variables q and p .⁶ We then find:

Upon deriving Eq. (15.73) above, facts (1) and (2) immediately imply that both of Hamilton's equations must be satisfied.

Note that in both of these points of view, the variables q and p are independent. In the first point of view, this independence is reached as a conclusion (by using the properties of the Legendre transform); while in the second, it is postulated. But in either case the result is the same: q and p are independent variables, unlike the (very dependent) variables q and \dot{q} in the Lagrangian formalism.

However, in the end this independence is fairly irrelevant. The important issue is the *symmetry* between q and p in Hamilton's equations (the minus sign aside).⁷ This fundamental difference between the Hamiltonian formalism (where q and p are symmetric) and the Lagrangian formalism (where q and \dot{q} are not) leads to many added features in the former, such as Liouville's theorem which we'll discuss in Section 15.5 below.

⁴Note that this reasoning makes no mention of δp variations, in particular their independence of δq variations.

⁵Yes, this integral comes out of the blue, but so does the $\int (T - V)$ integral in the Lagrangian formalism.

⁶Note that the \dot{q} in the integral, being the time derivative of q , is very much dependent on q ; the variation in \dot{q} is simply the time derivative of the variation in q . The \dot{q} just stays as \dot{q} here; we are *not* writing it as $\dot{q}(q, p)$, because this functional form is based on the definition $p \equiv \partial L/\partial \dot{q}$, which we're pretending we don't know anything about.

⁷Of course, if you take the second point of view above, you would say that this symmetry between q and p is a consequence of their independence. But in the first point of view, the symmetry is based on the Legendre transform.

15.4.2 Third derivation

The strategy of this derivation will be to calculate the differential of $H(q, p)$ by first calculating the differential of $L(q, \dot{q})$, and to then read off the partial derivatives, $\partial H/\partial q$ and $\partial H/\partial p$, from the result.

The Lagrangian $L(q, \dot{q})$ is a function of q and \dot{q} , so let's consider the change in $L(q, \dot{q})$ brought about by changes in q and \dot{q} . If we label these changes as dq and $d\dot{q}$, then the change in $L(q, \dot{q})$ is, by definition,

$$dL(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial q} dq + \frac{\partial L(q, \dot{q})}{\partial \dot{q}} d\dot{q}. \quad (15.75)$$

REMARK: As noted in Footnote 7 in Chapter 6, we are not assuming that the changes dq and $d\dot{q}$ are independent. They are in fact quite dependent; if q becomes $q + \epsilon$, then \dot{q} becomes $\dot{q} + \dot{\epsilon}$. So we have $d\dot{q} = \dot{\epsilon} \equiv d\epsilon/dt \equiv d(dq)/dt$. The dependence here is clear, in that one change is the derivative of the other. This is perfectly fine, because in writing down the change in $L(q, \dot{q})$ in terms of the partial derivatives in Eq. (15.75), nowhere is it assumed that these partials are independent. Equation (15.75) arises simply by letting $q \rightarrow q + dq$ and $\dot{q} \rightarrow \dot{q} + d\dot{q}$ in $L(q, \dot{q})$, and by then looking at what the first order change is. For example, if $L = Aq + B\dot{q} + Cq^2\dot{q}^3$, then you can quickly use the binomial expansion to show that the substitutions $q \rightarrow q + dq$ and $\dot{q} \rightarrow \dot{q} + d\dot{q}$ yield a first order change in L equal to

$$\begin{aligned} dL &= A dq + B d\dot{q} + C(2q\dot{q}^3 dq + 3q^2\dot{q}^2 d\dot{q}) \\ &= (A + 2Cq\dot{q}^3) dq + (B + 3Cq^2\dot{q}^2) d\dot{q} \\ &= \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}, \end{aligned} \quad (15.76)$$

as desired. Even if we have a function $f(x, y)$ of two variables that are trivially dependent (say, $y = 5x$), the relation $df = (\partial f/\partial x) dx + (\partial f/\partial y) dy$ still holds. ♣

Recalling the definition $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$ in Eq. (15.19), we can rewrite Eq. (15.75) as

$$dL(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial q} dq + p d\dot{q}. \quad (15.77)$$

Our goal here is to say something about the Hamiltonian, which is function of q and p , but not of \dot{q} , so let's get rid of the $d\dot{q}$ term by using

$$d(p\dot{q}) = \dot{q} dp + p d\dot{q} \implies p d\dot{q} = d(p\dot{q}) - \dot{q} dp. \quad (15.78)$$

This is basically the integration-by-parts trick that we've used many times in the past, which in the end is nothing more than the product rule for derivatives. Equation (15.77) can now be written as

$$\begin{aligned} dL(q, \dot{q}) &= \frac{\partial L(q, \dot{q})}{\partial q} dq + d(p\dot{q}) - \dot{q} dp \\ \implies d(p\dot{q} - L) &= -\frac{\partial L(q, \dot{q})}{\partial q} dq + \dot{q} dp. \end{aligned} \quad (15.79)$$

If we now use the definition $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$ to invert and solve for \dot{q} in terms of q and p , we can write the $p\dot{q} - L(q, \dot{q})$ on the left hand side in terms of only q and p , with no \dot{q} 's, as $p\dot{q}(q, p) - L(q, \dot{q}(q, p))$. But this is simply the Hamiltonian $H(q, p)$ defined in Eq. (15.20), so we have

$$dH(q, p) = -\frac{\partial L(q, \dot{q})}{\partial q} dq + \dot{q} dp. \quad (15.80)$$

But by definition, we also have

$$dH(q, p) = \frac{\partial H(q, p)}{\partial q} dq + \frac{\partial H(q, p)}{\partial p} dp. \quad (15.81)$$

Comparing the two previous equations, we therefore find

$$\frac{\partial L(q, \dot{q})}{\partial q} = -\frac{\partial H(q, p)}{\partial q}, \quad \text{and} \quad \dot{q} = \frac{\partial H(q, p)}{\partial p}. \quad (15.82)$$

The inclusion of the arguments of the functions is very important. In particular, the $\partial L/\partial q$ term here is $\partial L(q, \dot{q})/\partial q$ and not $\partial L(q, \dot{q}(q, p))/\partial q$; see the first remark following Eq. (15.33).

Up to this point we have done nothing except make definitions and manipulate mathematical relations. We haven't done any physics. Given the definitions of p and H we have made, the equations in Eq. (15.82) are simply identically true. But we will now invoke some actual physics and use the fact that the E-L equation states that $(d/dt)((\partial L(q, \dot{q})/\partial \dot{q}) = \partial L(q, \dot{q})/\partial q$. Using the definition $p \equiv \partial L(q, \dot{q})/\partial \dot{q}$, this becomes $\dot{p} = \partial L(q, \dot{q})/\partial q$. Substituting this in the first of Eqs. (15.82), we have Hamilton's equations,

$$\dot{p} = -\frac{\partial H(q, p)}{\partial q}, \quad \text{and} \quad \dot{q} = \frac{\partial H(q, p)}{\partial p}. \quad (15.83)$$

The second of these equations is an identically true mathematical statement, given the definitions of p and H in terms of L (with the definition of H being motivated by the Legendre transform); there is no physics involved. But the first equation uses the E-L equation, which arose from the physical statement that the action is stationary.

15.4.3 Fourth derivation

In this derivation, we will use the fact that H is the Legendre transform of L . And we will also use the fact that $dH/dt = -\partial L/\partial t$ (see below), which in turn depends on the Euler-Lagrange equation being satisfied, or equivalently on the action being stationary. The t argument of the functions will be important in this derivation, so we'll include it throughout.

Because H is the Legendre transform of L , the definition $p \equiv \partial L(q, \dot{q}, t)/\partial \dot{q}$ implies the "mirror" statement,

$$\dot{q} = \frac{\partial H(q, p, t)}{\partial p}. \quad (15.84)$$

This is the first of Hamilton's equations. But as we have already noted many times, this equation isn't a physical statement, but rather a mathematical one arising from the definition of H as the Legendre transform of L . Alternatively, you can just derive it from scratch as we did in Eq. (15.28) (although this still relies on the fact that H is the Legendre transform of L). Therefore, as far as the first Hamilton's equation is concerned, this fourth derivation is basically the same as the first one.

Now let's derive the second Hamilton's equation. In Chapter 6, we showed in Eq. (6.54) that the E defined in Eq. (6.52/15.1) satisfies $dE(q, \dot{q}, t)/dt = -\partial L(q, \dot{q}, t)/\partial t$. Therefore, since H and E are the same quantity, we also have

$$\frac{dH(q, p, t)}{dt} = -\frac{\partial L(q, \dot{q}, t)}{\partial t}. \quad (15.85)$$

The fact that H 's arguments (q, p, t) are different from E 's arguments (q, \dot{q}, t) is irrelevant when taking the *total* time derivative, because in the end, H and E are the same function of t when all the coordinates are expressed in term of t . So perhaps we should write $H(q(t), p(t), t)$, etc. here, but that gets to be rather cumbersome.

We'll need one more fact, namely $\partial H(q, p, t)/\partial t = -\partial L(q, \dot{q}, t)/\partial t$. This can be shown as follows. Since $H(q, p, t) \equiv p\dot{q}(q, p, t) - L(q, \dot{q}(q, p, t), t)$, we have

$$\frac{\partial H(q, p, t)}{\partial t} = p \frac{\partial \dot{q}}{\partial t} - \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial t} + \frac{\partial L(q, \dot{q}, t)}{\partial t} \right). \quad (15.86)$$

But $p \equiv \partial L(q, \dot{q}, t)/\partial \dot{q}$, so the first and second terms on the right-hand side cancel, and we are left with

$$\frac{\partial H(q, p, t)}{\partial t} = -\frac{\partial L(q, \dot{q}, t)}{\partial t}, \quad (15.87)$$

as desired. Eqs. (15.85) and (15.87) therefore imply that $dH/dt = \partial H/\partial t$.

The second Hamilton equation now quickly follows. The chain rule yields

$$\frac{dH}{dt} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial t}. \quad (15.88)$$

Using $dH/dt = \partial H/\partial t$, along with $\dot{q} = \partial H/\partial p$ from Eq. (15.84), the preceding equation reduces to

$$0 = \frac{\partial H}{\partial q} \dot{q} + \dot{q} \dot{p} \implies \dot{q} \left(\frac{\partial H}{\partial q} + \dot{p} \right) = 0 \implies \dot{p} = -\frac{\partial H}{\partial q}, \quad (15.89)$$

which is the second of Hamilton's equations, as desired. (We're ignoring the trivial case where \dot{q} is identically zero.)

As in the other three derivations of Hamilton's equations, the first equation is simply a mathematical statement that follows from the properties of the Legendre transform, while the second equation is a physical statement that can be traced to the principle of stationary action.

REMARK: The main ingredient in this derivation of Hamilton's second equation was the $dH/dt = -\partial L/\partial t$ fact in Eq. (15.85). In the special case where $\partial L/\partial t = 0$, this gives $dH/dt = 0$. That is, energy is conserved. So Hamilton's second equation is closely related to conservation of energy (although it should be stressed that the equation holds even in the general case where $\partial L/\partial t \neq 0$). This shouldn't be a surprise, because we already know that Hamilton's equations contain the same information as Newton's $F = ma$ law, and we originally derived conservation of energy in Section 5.1 by integrating the $F = ma$ statement. ♣

15.5 Phase space, Liouville's theorem

15.5.1 Phase space

Consider a particle undergoing motion (let's just deal with 1-D) governed by a given potential (or equivalently a given Lagrangian, or equivalently a given Hamiltonian). At each point in the motion, the particle has a certain value of q and a certain value of p . That is, each point in the particle's motion is associated with a unique point (q, p) in the q - p plane. The space spanned by the q and p axes is called *phase space*. The coordinates q and p are often taken to be a standard Cartesian coordinate x

and the associated linear momentum $m\dot{x}$, or perhaps an angular coordinate θ and the associated angular momentum $mr^2\dot{\theta}$. But it should be stressed that all of the following results hold for a generalized coordinate q and the conjugate momentum $p \equiv \partial L/\partial \dot{q}$.

If a particle is located at a particular point (q_0, p_0) at a given instant, then its motion is completely determined for all time. This is true because given the values of q_0 and p_0 at a given instant, Hamilton's equations, Eqs. (15.33), yield the values of \dot{q} and \dot{p} at this instant. These derivatives in turn determine the values of q and p at a nearby time, which in turn yield the values of \dot{q} and \dot{p} at this nearby time, and so on. We can therefore march through time, successively obtaining values of (q, p) and (\dot{q}, \dot{p}) . So the initial coordinates (q_0, p_0) , together with Hamilton's equations, uniquely determine the path of the particle in phase space.

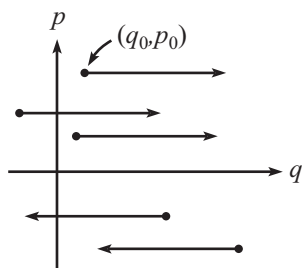


Figure 15.15

Example 1 (Constant velocity): Consider a particle that moves with constant velocity in 1-D. Since there is no force, the potential energy is constant (which we will take to be zero), so the Hamiltonian is $H = p^2/2m$. You can quickly show that Hamilton's equations are $\dot{q} = p/m$ and $\dot{p} = 0$. The second of these is the statement that p doesn't change, as expected. So the possible "curves" in phase space are horizontal lines, as shown in Fig. 15.15. The actual line the particle is on is determined by the initial coordinates (q_0, p_0) . Note that the lines associated with negative p are traced out leftward (that is, in the direction of decreasing q), as should be the case. Note also that although all the lines look basically the same on the page, the particle traverses the ones with larger $|p|$ more quickly, due to the above $\dot{q} = p/m$ equation; the larger the $|p|$, the greater the rate at which q changes, as expected.

Example 2 (Falling balls): With positive q defined to be *downward* (which will reduce the number of minus signs in the expressions below), the Hamiltonian for a vertically thrown ball is $H = p^2/2m - mgq$. You can quickly show that Hamilton's equations are $\dot{q} = p/m$ and $\dot{p} = mg$. Eliminating p gives the familiar equation of motion, $\ddot{q} = g$. Integrating twice then gives the standard result, $q(t) = q_0 + v_0t + gt^2/2$. However, if we want to see what the path looks like in phase space, the best thing to do in general is to solve for q in terms of p (or p in terms of q , whichever is easier). Since $\dot{p} = mg \implies p_0 - p = mgt \implies t = (p - p_0)/mg$, we can eliminate t from the above expression for $q(t)$ to obtain

$$q(p) = q_0 + \left(\frac{p_0}{m}\right) \left(\frac{p - p_0}{mg}\right) + \frac{g}{2} \left(\frac{p - p_0}{mg}\right)^2 = q_0 + \frac{p^2 - p_0^2}{2m^2g}. \quad (15.90)$$

We see that q is a positive quadratic function of p , which means that the curves in phase space are rightward-opening parabolas, as shown in Fig. 15.16. Note that the curves are traced out leftward below the q axis (where p is negative), and rightward above the q axis (where p is positive), as expected.

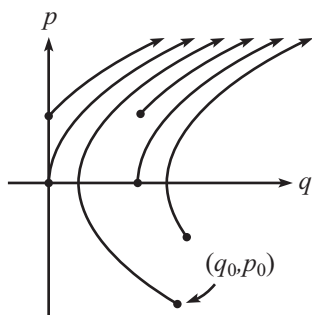


Figure 15.16

REMARK: Another way of obtaining Eq. (15.90) is to use the standard kinematic result for constant acceleration, $v_f^2 = v_i^2 + 2a(x_f - x_i)$. You can verify that this does yield the correct $q(p)$. And yet another way is to simply note that H is constant (because it has no explicit time dependence), so $p^2/2m - mgq = C$, where C is determined by the initial conditions. So C must equal $p_0^2/2m - mgq_0$, and we obtain Eq. (15.90). ♣

The q in Eq. (15.90) takes the form of $q(p) = Ap^2 + B$, where $B = q_0 - p_0^2/2m^2g$ depends on the initial coordinates p_0 and q_0 , but where $A = 1/2m^2g$ doesn't. Because A is the same for every curve, the curves are all parts of parabolas with the same curvature. The only difference between the curves is the q intercept (namely

B), and also the location along the underlying parabola where the motion starts. This is shown in Fig. 15.17.

What is the slope of a curve at a given point (q, p) ? From Eq. (15.90) we have $dq/dp = p/m^2g$, so the slope of the curve in phase space is $dp/dq = m^2g/p$. This has the properties of being infinite when $p = 0$ and being independent of q , both of which are evident in the above figures. This slope can also be obtained by dividing the two Hamilton's equations from above:

$$\frac{dp}{dq} = \frac{dp/dt}{dq/dt} = \frac{\dot{p}}{\dot{q}} = \frac{mg}{p/m} = \frac{m^2g}{p}. \quad (15.91)$$

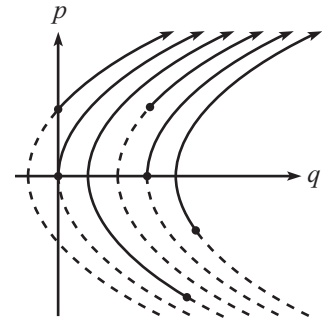


Figure 15.17

Example 3 (Harmonic oscillator): The Hamiltonian for a harmonic oscillator is $H = p^2/2m + kq^2/2$. You can quickly show that Hamilton's equations are $\dot{q} = p/m$ and $\dot{p} = -kq$. To solve for q in terms of p , you could solve for q in terms of t (the result would be a trig function) and then differentiate to obtain p (another trig function) and then eliminate t . You would find that $p^2/2m + kq^2/2$ equals a constant that depends on the initial conditions (namely, $p_0^2/2m + kq_0^2/2$). Or you could do it the easy way and just note that H is constant (because it has no explicit time dependence), which immediately gives the same result. Either way, we have

$$\frac{p^2}{2m} + \frac{kq^2}{2} = \frac{p_0^2}{2m} + \frac{kq_0^2}{2}. \quad (15.92)$$

This is the equation for an ellipse in the q - p plane. The (q, p) coordinates of the particle keep going around and around a particular ellipse as time goes by, as shown in Fig. 15.18.

From Eq. (15.92), the q intercept (that is, where $p = 0$) of a given ellipse is located at $q^2 = p_0^2/km + q_0^2$. And the p intercept (that is, where $q = 0$) is located at $p^2 = p_0^2 + kmq_0^2$. The curves are traced out leftward below the q axis (where p is negative), and rightward above the q axis (where p is positive), as expected. From Hamilton's equations, the slope of a curve at a given point is $dp/dq = \dot{p}/\dot{q} = -kq/(p/m) = -kmq/p$. This is infinite when $p = 0$ and zero when $q = 0$, in agreement with Fig. 15.18.

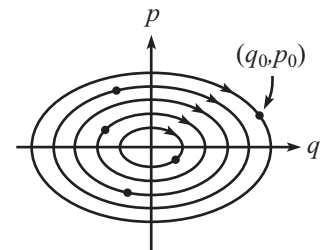


Figure 15.18

Although it isn't obvious from the figure, each ellipse is traced out in the same amount of time (as we know from Chapter 4). But this is at least believable, because the larger ellipses have points with larger values of $|p|$, so the curves are traversed more quickly there, allowing (apparently) the larger total distance to be covered in the same amount of time.

In the above examples, none of the paths cross each other. This is a general result; paths in phase space never cross. The proof is simple: if two paths did cross, then there would be two different velocity vectors (\dot{q}, \dot{p}) at a given point. However, Hamilton's equations uniquely determine the velocity vector $(\dot{q}, \dot{p}) = (\partial H/\partial p, -\partial H/\partial q)$ at a given point (q, p) . So there must in fact be only one curve through any given point, and hence no crossing. Another fact along these lines is that a path in phase space can never branch into two paths, even if the velocities of the two branches are equal at the branching point. The demonstration of this is the task of Problem 15.17.

Note that there is no mention of time in the above phase-space plots. So as we mentioned, you can't just look at a curve and tell how fast the particle is traversing it. But you can easily determine this by using Hamilton's equations (assuming you've been given the Hamiltonian) to write down the particle's velocity vector in

phase space:

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right). \quad (15.93)$$

Note also that there is no mention of absolute time. If the Lagrangian (and hence Hamiltonian) doesn't depend on time, so that the partial derivatives in Eq. (15.93) depend only on q and p , then if a particle has initial coordinates (q_0, p_0) at, say, 3:14 pm, it will trace out exactly the same path in phase space as another particle with the same initial coordinates that started earlier at, say, 1:59 pm.

In the case of N dimensions, there are N coordinates q_i and N momenta p_i , so phase space has $2N$ dimensions. This is clearly a bit difficult to draw for $N > 1$, but the scenario is the same as above: a particle starts at a given location in phase space, and then the $2N$ Hamilton's equations dictate the ensuing path.

If you wanted, you could make a plot of \dot{q} vs. q instead of p vs. q , but it wouldn't be very useful. The advantage of q - p phase space is that the motion is governed by Hamilton's equations which are (nearly) symmetric in q and p , whereas there is no such symmetry with q and \dot{q} . This symmetry leads to many useful results, the most notable of which is Liouville's theorem. . .

15.5.2 Liouville's theorem

Loosely stated, Liouville's theorem states that a given initial region in phase space keeps the same area (even though the shape invariably changes) as it moves through phase space due to the $(\dot{q}, \dot{p}) = (\partial H/\partial p, -\partial H/\partial q)$ velocity vector of each point in the region. Before giving a proof of the theorem, let's look at two examples to get a feel for it.

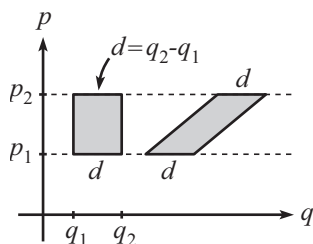


Figure 15.19

Example (Constant velocity): Consider the setup in the first example (“Constant velocity”) from Section 15.5.1, and consider the rectangle in phase space shown in Fig. 15.19. This rectangle (including its interior) may be thought of as representing the (q, p) coordinates of a very large number of particles (so that we may view them as being essentially continuously distributed in phase space) at a given time.⁸ What shape does this rectangle get carried into at a later time? What is the area of this shape?

Solution: Let the initial rectangle span a height from p_1 to p_2 and a width from q_1 to q_2 . As time goes by, the points on the top side of the rectangle move faster to the right than the points on the bottom side, due to the different p (which equals $m\dot{q}$) values. Therefore, at a later time we have a parallelogram as shown. For the sake of drawing the figure we have arbitrarily assumed that $p_2 = 2p_1$, which means that points on the top side move twice as far to the right as points on the bottom side. Note that the left and right sides do indeed remain straight lines, because the speeds with which points move to the right in phase space grow linearly with p . So the desired shape is a parallelogram, and it becomes more and more sheared as time goes by.

What is the area of this parallelogram? The top and bottom sides always have the same lengths as the top and bottom sides of the initial rectangle (namely $q_1 - q_2$), because points with the same value of p move with the same speed (namely p/m) to the right, so the relative distances don't change. And the height of the parallelogram

⁸In particular, the corners of the rectangle represent four objects at a given instant. Two start with momentum p_1 at positions q_1 and q_2 , and two start with momentum p_2 at positions q_1 and q_2 .

is still $p_2 - p_1$, because none of the speeds change which means there is no vertical motion in phase space. Therefore, the area is $(q_2 - q_1)(p_2 - p_1)$ which is the same as the area of the original rectangle.

This “conservation of area” result actually holds for any arbitrarily-shaped initial region in phase space, not just a rectangle. This follows from the fact that we can break up the initial region into a large number of infinitesimal rectangles, and we can then use the reasoning in the previous paragraph for each little rectangle. So we see that the area in phase space is *conserved*. That is, if we look at any region in phase space and then look at the area that this region gets mapped into at any later time, the final area will equal the initial area.

For all we know, this “conservation of phase space area” rule might hold only for the preceding specific scenario involving constant velocity. So let’s look at another example and see what happens to the area of an evolving phase-space region.

Example (Falling balls): Consider the setup in the second example (“Falling balls”) from Section 15.5.1, and consider the rectangle in phase space shown in Fig. 15.20.⁹ What shape does this rectangle get carried into at a later time? What is the area of this shape?

Solution: It turns out that we again end up with a parallelogram, and it again has the same area as the initial rectangle. The parallelogram shape follows from two facts. First, the top and bottom sides always have equal lengths (namely $q_2 - q_1$), because points with the same value of p move with the same speed (namely p/m) to the right in phase space.¹⁰ And second, the left and right sides remain straight lines, because the speeds with which points move to the right in phase space grow linearly with p , and also because all points move vertically with the same speed, namely $\dot{p} = mg$ from the second of Hamilton’s equations. (See Problem 15.16 for the general condition under which straight lines in phase space remain straight.)

What is the area of the parallelogram? Since all points move vertically in phase space with the same speed (namely $\dot{p} = mg$), the height of the parallelogram always remains $p_2 - p_1$. And since the “base” sides always have length $q_2 - q_1$, the area is always $(q_2 - q_1)(p_2 - p_1)$, which is the same as the area of the original rectangle.

Again, as in the preceding example, this result holds for any arbitrarily-shaped initial region in phase space, because we can break up this region into a large number of infinitesimal rectangles for which we know the result holds.

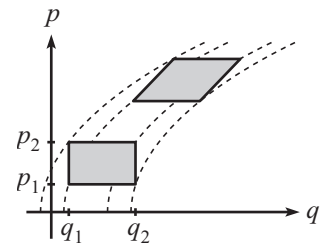


Figure 15.20

Problem 15.19 deals with the third example (“Harmonic oscillator”) from Section 15.5.1, and the result is the same: the area of a given region in phase space doesn’t change as it moves through phase space.¹¹ This “conservation of area” result is a general one; it holds for any system, not just for the simple setups above. It is known as *Liouville’s theorem*, and the proof is as follows.

⁹Physically, the corners of the rectangle represent four balls at a given instant. Two are thrown downward with momentum p_1 from positions q_1 and q_2 , and two are thrown downward with momentum p_2 from positions q_1 and q_2 .

¹⁰If you want to picture balls falling, the two balls associated with, say, the two bottom corners of the rectangle start out a distance $q_2 - q_1$ apart with the same speed. Therefore, since the acceleration is independent of position, they always have the same speed and hence always remain a distance $q_2 - q_1$ apart.

¹¹The parallelogram result is actually the same too, but this doesn’t hold in general for other setups; see Problem 15.16.

Theorem 15.2 (*Liouville’s Theorem*) *Given a system of N coordinates q_i , the $2N$ -dimensional “volume” enclosed by a given $(2N - 1)$ -dimensional “surface” in phase space is conserved (that is, independent of time) as the surface moves through phase space.*

Proof: Let’s restrict ourselves to the $N = 1$ case for simplicity. After proving the theorem for this case, it is fairly straightforward to generalize to higher dimensions (the task of Problem 15.18). For one-dimensional motion, phase space is two-dimensional (one q axis and one p axis), so the “volume” in the statement of the theorem is simply an area, and the “surface” is a closed curve that bounds this area. Our goal is therefore to show that if we look at the area bounded by a given closed curve in the q - p plane at a given time, and if we consider where this curve ends up at an arbitrary later time and look at the area it bounds, then these two areas are equal.

We’ll demonstrate this equality by considering an arbitrary curve and its image an infinitesimal time dt later. If we can show that these two curves enclose the same area,¹² then we can march forward through time in small steps, with the area remaining the same after each step. It then follows that the final region at an arbitrary later time has the same area as the initial region.

The velocity vector of a given point (corresponding to a given object – a ball, atom, or whatever) as it moves through phase space is by definition $\mathbf{v} = (\dot{q}, \dot{p})$. So the displacement vector in the small time dt is $(dq, dp) = \mathbf{v} dt = (\dot{q}, \dot{p}) dt$. Fig. 15.21 shows how these displacement vectors take the points on the initial curve C_0 and carry them into the points on the new curve C_{dt} .¹³ Our goal is to show that the area of C_{dt} equals the area of C_0 .

How might the area change as the curve C moves (which in general involves translation, rotation, and distortion) through phase space? It increases due to the area it picks up in the right part of the figure (at least in the scenario we’ve shown), and it decreases due to the area it loses in the left part. To get a handle on the increase in the right part, consider the motion of two nearby points, A and B , in Fig. 15.21. The corresponding increase in the area bounded by C is the tiny area of the shaded region shown. If A and B are sufficiently close together and if dt is sufficiently small, then this shaded region is essentially a parallelogram (because the two displacement vectors are essentially equal).

A close-up view of the shaded region is shown in Fig. 15.22. Its area equals $d\ell$ times the height dh , where dh is the component of the displacement vector, $\mathbf{v} dt = (\dot{q}, \dot{p}) dt$, that is perpendicular to the curve. Using Eq. (B.2) from Appendix B, we can take advantage of the dot product to write dh as $dh = \mathbf{n} \cdot (\mathbf{v} dt)$, where \mathbf{n} is defined to be the (outward) unit vector perpendicular to the curve. (The $\cos \theta$ in the dot product picks out the component perpendicular to the curve.) The area of the shaded region is therefore $d\ell dh = d\ell (\mathbf{n} \cdot \mathbf{v} dt)$. Note that this automatically gets the sign of the area correct. For example, in the left part of Fig. 15.21, $\mathbf{v} dt$ points inward and \mathbf{n} points outward (by definition), so the area of a tiny parallelogram there is negative. That is, it signifies a decrease in the area bounded by the curve, as it should.

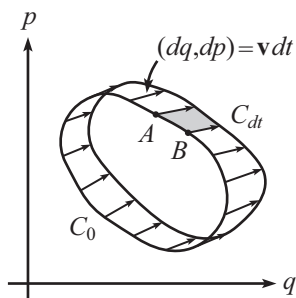


Figure 15.21

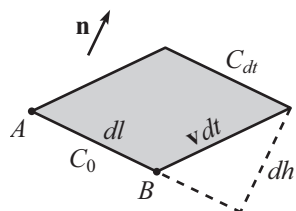


Figure 15.22

¹²Technically, we need to show only that the areas are the same to first order in dt here, but the end result will be that they are equal to all orders.

¹³An infinitesimal dt wouldn’t yield the fairly large difference between C_0 and C_{dt} that is shown in the figure; we’ve exaggerated the difference so that the displacement vectors are discernable. Also, note that since the velocity vector (\dot{q}, \dot{p}) is in general a function of the position (q, p) , the displacement vectors’ lengths and directions can vary depending on the location on the curve, as shown.

The total change in area, dA , in going from C_0 to C_{dt} is the sum of the little changes arising from all the little parallelograms. In other words, it is the integral of the parallelogram areas over the whole curve, which is

$$dA = \int_C (\mathbf{n} \cdot \mathbf{v} dt) d\ell \implies \frac{dA}{dt} = \int_C \mathbf{v} \cdot (\mathbf{n} d\ell). \quad (15.94)$$

But the right-hand side of this equation is exactly the type of integral that appears in the divergence theorem (or ‘‘Gauss’ theorem’’). In the case of a 2-D surface bounding a 3-D volume, this theorem is stated in Eq. (B.19) as

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{A}, \quad (15.95)$$

but it holds generally for any $(n - 1)$ -D ‘‘surface’’ bounding an n -D ‘‘volume.’’ In (reasonably) plain English, the divergence theorem says that the integral of the divergence of a vector field over a ‘‘volume’’ equals the flux of the vector field through the bounding ‘‘surface’’ (which is obtained by integrating the component of the vector perpendicular to the surface). The proof in the general n -D case proceeds by exactly the same reasoning we used in the 3-D ‘‘proof’’ in Section B.5.¹⁴

The right-hand side of Eq. (15.94) is the flux of the vector \mathbf{v} through the ‘‘surface’’ (which is just the 1-D curve C here), with $\mathbf{n} d\ell$ being analogous to the $d\mathbf{A}$ in Eq. (15.95).¹⁵ So in the 2-D case of a curve bounding an area, the divergence theorem says

$$\int_A \nabla \cdot \mathbf{v} dA = \int_C \mathbf{v} \cdot (\mathbf{n} d\ell). \quad (15.96)$$

This actually holds for any vector field \mathbf{v} , but we’re concerned only with the velocity vector \mathbf{v} here. Combining Eq. (15.94) with Eq. (15.96) gives

$$\begin{aligned} \frac{dA}{dt} &= \int_A \nabla \cdot \mathbf{v} dA \\ &= \int_A \nabla \cdot (\dot{q}, \dot{p}) dA \\ &= \int_A \left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} \right) dA. \end{aligned} \quad (15.97)$$

This is valid for any general velocity field $\mathbf{v} \equiv (\dot{q}, \dot{p})$. But we will now invoke the fact that we are actually dealing with the motion of points in phase space, which means that the motion is governed by Hamilton’s equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (15.98)$$

¹⁴Basically (in the language of the 3-D case), the divergence measures the flux through an infinitesimal cube, so if we break up the volume V into many little cubes, and if we add up (that is, integrate) the fluxes through all the little cubes throughout V (which equals the left-hand side of Eq. (15.95)), then the fluxes cancel in pairs in the interior of V (because whatever flows out of one cube flows into another), so the only fluxes that don’t get canceled are the ones through the faces of the cubes that lie on the boundary surface C . We therefore end up with the flux through C (which equals the right-hand side of Eq. (15.95)). In the present 2-D case, analogous reasoning holds if we break up the area bounded by C into many little rectangles.

¹⁵Recall that in the 3-D case, $d\mathbf{A}$ is defined to be the vector perpendicular to the surface, with its magnitude equal to the tiny area element. In the present 2-D case, $\mathbf{n} d\ell$ is the vector perpendicular to the curve, with its magnitude equal to the tiny arclength element.

Plugging these into Eq. (15.97) gives

$$\begin{aligned} \frac{dA}{dt} &= \int_A \left(\frac{\partial}{\partial q} \left(\frac{\partial H}{\partial p} \right) + \frac{\partial}{\partial p} \left(-\frac{\partial H}{\partial q} \right) \right) dA \\ &= \int_A \left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} \right) dA \\ &= 0, \end{aligned} \tag{15.99}$$

where we have used the fact that partial derivatives commute. ■

REMARKS:

1. In the case where there are N coordinates q_i , so that phase space has $2N$ dimensions, the proof of Liouville's theorem is basically the same as above. The only difference is that now there are N pairs of the cancelations in Eq. (15.99), as you will see in Problem 15.18.
2. As mentioned earlier, the (near) symmetry in Hamilton's equations is what allows Liouville's theorem to hold. It is this symmetry that leads to the two terms in Eq. (15.99) looking very similar. And it is the "nearness" of the symmetry (with the one minus sign) that leads to the two terms actually canceling.
3. Liouville's theorem is sometimes stated tersely as: the density in phase space remains constant. But you should be careful not to interpret this as the density at a *given fixed point* in phase space remaining constant. This is certainly not always true. For example, in the "Constant velocity" example above, the density at a given point inside the initial rectangle is nonzero. But after the parallelogram flows to the right, the density at the given point becomes zero.

The correct interpretation is that if you *follow a point* as it moves through phase space, then the density in the near vicinity of this point remains constant. That is, if you look at the density of particles (balls, atoms, or whatever) inside a small region around the point, then this density remains constant as the point and surrounding region move through phase space. This is true because the area of the region remains constant due to Liouville's theorem, and also because the number of particles remains constant (they don't magically appear or disappear). ♣

15.6 Problems

Section 15.1: Energy

15.1. Time dependence *

Consider a Cartesian coordinate x and a Lagrangian $L = m\dot{x}^2/2 - V(x)$. Show that if another coordinate q depends on both x and t , or equivalently if x depends on both q and t (that is, $x = x(q, t)$), then $L(q, \dot{q}, t)$ yields an $E \equiv (\partial L/\partial \dot{q})\dot{q} - L$ that takes the form,

$$E = T + V - m \left(\left(\frac{\partial x}{\partial q} \right) \left(\frac{\partial x}{\partial t} \right) \dot{q} + \left(\frac{\partial x}{\partial t} \right)^2 \right). \quad (15.100)$$

15.2. Conservation of E *

Consider the Lagrangian, $L = m\dot{x}^2/2 - V(x)$. Show that the statement that the E in Eq. (15.1) is conserved is essentially the same statement as the Euler-Lagrange equation.

Section 15.2: Hamilton's equations

15.3. Poisson brackets

Consider a function $f(q, p)$ of the coordinates q and p . Use Hamilton's equations to show that the time derivative of f can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}. \quad (15.101)$$

Note: this combination of partial derivatives comes up often enough to warrant a name, so the *Poisson bracket* of two functions, f_1 and f_2 , is defined to be

$$\{f_1, f_2\} \equiv \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial q}. \quad (15.102)$$

With this definition, the time derivative of f takes the nice compact form, $df/dt = \{f, H\}$.

15.4. Hamilton's equations for many variables **

Show that for a system with N degrees of freedom, the $2N$ Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \text{for } 1 \leq i \leq N. \quad (15.103)$$

15.5. Equivalent Lagrangians **

Given a Lagrangian $L(q, \dot{q}, t)$ that describes a certain system, is it possible to find another Lagrangian that describes the same system? That is, can we construct another $L'(q, \dot{q}, t)$ that yields the same equation of motion? The answer is certainly "yes," in the trivial sense that we can add on a constant to L , which of course doesn't change the equation of motion (which involves only derivatives of L). But are there any nontrivial changes to L we can make? The answer is again "yes." Show that if we construct a new Lagrangian L'

by adding to L the time derivative of an arbitrary function $F(q, t)$ depending on q and t (but not \dot{q}),

$$L'(q, \dot{q}, t) \equiv L(q, \dot{q}, t) + \frac{dF(q, t)}{dt}, \quad (15.104)$$

then L' has the same equation of motion.¹⁶

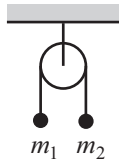


Figure 15.23

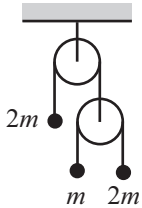


Figure 15.24

15.6. **Atwood's 1** *

Consider the Atwood's machine shown in Fig. 15.23. Let x be the vertical position of the left mass, with upward taken to be positive. Find the Hamiltonian in terms of x and its conjugate momentum, and then write down Hamilton's equations.

15.7. **Atwood's 2** **

Consider the Atwood's machine shown in Fig. 15.24. Let x and y be the vertical positions of the middle mass and right mass, respectively, with upward taken to be positive. Find the Hamiltonian in terms of x and y and their conjugate momenta, and then write down the four Hamilton's equations.

15.8. **Two masses and a spring** **

Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let the position of the left bead be x , and let the stretch of the spring (relative to equilibrium) be z . Find the Hamiltonian in terms of x and z and their conjugate momenta, and then write down the four Hamilton's equations. (See Exercise 15.28 for a slightly easier variation of this problem.)

15.9. **$y = f(x)$ constraint** **

A particle of mass m is constrained to move on a curve whose height is given by the function $y = f(x)$. Find the Hamiltonian in terms of x and its conjugate momentum, and then write down Hamilton's equations. Show that the $\dot{p} = -\partial H/\partial x$ equation reproduces the E-L equation.

15.10. **Spring and moving wall** **

A mass m is connected to a wall by a horizontal spring with spring constant k and relaxed length ℓ_0 . The wall is arranged to move back and forth with position $X_{\text{wall}} = A \sin \omega t$. Let z measure the stretch of the spring. Find the Hamiltonian in terms of z and its conjugate momentum, and then write down Hamilton's equations. Is H the energy? Is H conserved? (See Exercise 15.26 for a slightly easier variation of this problem.)

15.11. **Bead on a rotating hoop** **

A bead is free to slide along a frictionless hoop of radius R . The hoop rotates with constant angular speed ω around a vertical diameter (see Fig. 15.25). Find the Hamiltonian in terms of the angle θ shown and its conjugate momentum, and then write down Hamilton's equations. Is H the energy? Is H conserved?

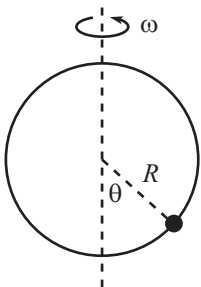


Figure 15.25

Section 15.3: Legendre transforms

¹⁶This problem probably belongs in Chapter 6, since it deals only with Lagrangians. However, the existence of equivalent Lagrangians of the form in Eq. (15.104) is the basis of the so-called *canonical transformations* which lead to the *Hamilton-Jacobi* extension of Hamiltonian mechanics.

15.12. $F(x) = x^n$ *

Find the Legendre transform of $F(x) = x^n$, and verify that $dG(s)/ds = x$.

15.13. **Alternate definition of $G(s)$** *

Given a concave function $F(x)$, an alternate definition of the Legendre transform $G(s)$ is that it equals the maximum value of the quantity $sx - F(x)$, where s is assumed given and x is allowed to vary.¹⁷ Explain why this agrees with the y -intercept definition of $G(s)$ given in Eq. (15.53).

15.14. **Equal $G(s)$ values** **

Let $G(s)$ be the Legendre transform of $F(x)$. Show that if two different values of s yield the same value of $G(s)$, and if additionally the two associated values of x have the same sign, then $F(x)$ must have an inflection point (a point where the second derivative is zero).

15.15. **Double-valued $G(s)$** *

Let $G(s)$ be the Legendre transform of $F(x)$. Show that if $G(s)$ is double valued (that is, if a given value of s yields two different values of $G(s)$), then $F(x)$ must have an inflection point (a point where the second derivative is zero).

*Section 15.5: Phase space, Liouville's theorem*15.16. **Straight lines** **

Under what conditions do straight lines in phase space always remain straight lines as time evolves?

15.17. **No branching** **

Show that a path in phase space can never branch into two paths, even if the velocities of the two branches are equal at the branching point, as shown in Fig. 15.26.

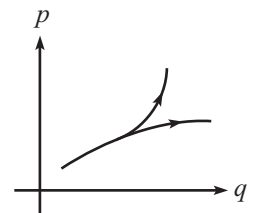


Figure 15.26

15.18. **Liouville for higher N** *

Prove Liouville's theorem for the general case of N coordinates q_i , with phase space being $2N$ -dimensional.

15.19. **Harmonic oscillator** **

Consider a harmonic oscillator governed by the Hamiltonian $H = p^2/2m + kx^2/2$. If x_i and p_i are the initial position and momentum, then from the results of Chapter 4, we know that x and p at a general later time are given by

$$(x, p) = \left(x_i \cos \omega t + \frac{p_i}{m\omega} \sin \omega t, -m\omega x_i \sin \omega t + p_i \cos \omega t \right). \quad (15.105)$$

Consider the rectangle shown in Fig. 15.27, with one corner at the point (x_0, p_0) and sides of lengths Δx and Δp . Show that at any later time, the image of the rectangle is a parallelogram with area $\Delta x \Delta p$.

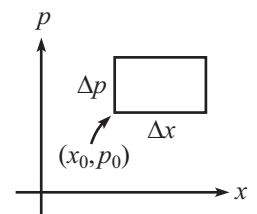


Figure 15.27

¹⁷Note that no mention of the slope is made in this definition; it is not assumed that s represents the slope for the coordinate x (which is allowed to vary, whereas s is fixed).

15.20. **Harmonic oscillator, easier method** *

This problem demonstrates the result of the previous problem in a simpler way. To make Eq. (15.105) more symmetrical, consider the variable $z \equiv m\omega x$. The coordinates (z, p) as functions of time are then

$$(z, p) = (z_i \cos \omega t + p_i \sin \omega t, -z_i \sin \omega t + p_i \cos \omega t). \quad (15.106)$$

Describe the motion of a given initial region in the z - p plane, and then use the result to show that the area of a given initial region in the x - p plane doesn't change.

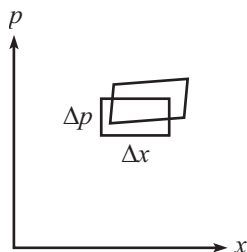


Figure 15.28

15.21. **Liouville's theorem** ***

This problem gives another proof of Liouville's theorem. Consider an infinitesimal rectangle in the x - p plane with side lengths Δx and Δp . In an infinitesimal time dt , this rectangle gets mapped into another region, as shown in Fig. 15.28. Show that this new region is a parallelogram and that its area is (to leading order in Δx , Δp , and dt)

$$A = \Delta x \Delta p \left(1 + \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{p}}{\partial p} \right) dt \right), \quad (15.107)$$

where (\dot{x}, \dot{p}) is the velocity vector field.

This is actually a general mathematical result, independent of how \dot{x} and \dot{p} depend on position. That is, it need not have anything to do with physics and phase space. But if we now restrict ourselves to motion in phase space, which means that it is governed by a Hamiltonian $H(x, p)$, then Hamilton's equations quickly tell us that the two partial derivatives cancel (because $\dot{x} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial x$, and because partial derivatives commute), so we end up with $A = \Delta x \Delta p$. That is, the area remains the same, in agreement with Liouville's theorem.

15.7 Exercises

Section 15.2: Hamilton's equations

15.22. General potential in 1-D

Given the Hamiltonian, $H = p^2/2m + V(x)$, show that Hamilton's equations are equivalent to $F = ma$.

15.23. Atwood's 1 *

Consider the Atwood's machine shown in Fig. 15.29. Let x be the vertical position of the right mass, with upward taken to be positive. Find the Hamiltonian in terms of x and its conjugate momentum, and then write down Hamilton's equations.

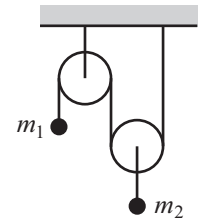


Figure 15.29

15.24. Atwood's 2 **

Consider the Atwood's machine shown in Fig. 15.30. Let x and y be the vertical positions of the left and right masses, respectively, with upward taken to be positive. Find the Hamiltonian in terms of x and y and their conjugate momenta, and then write down the four Hamilton's equations.

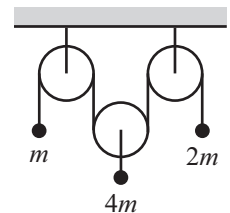


Figure 15.30

15.25. $r = k\theta$ spiral **

A particle of mass m is constrained to move on a spiral described by $r = k\theta$. The spiral lies in a horizontal plane. Find the Hamiltonian in terms of θ and its conjugate momentum, and then write down Hamilton's equations. Show that the $\dot{p} = -\partial H/\partial\theta$ equation reproduces the E-L equation.

15.26. Spring and moving wall *

A mass m is connected to a wall by a horizontal spring with spring constant k and relaxed length ℓ_0 . The wall is arranged to move back and forth with position $X_{\text{wall}} = A\sin\omega t$. Let x be the position of the mass relative to the location of the wall at $t = 0$. Find the Hamiltonian in terms of x and its conjugate momentum, and then write down Hamilton's equations. Is H the energy? Is H conserved? (See Problem 15.10 for a slightly harder variation of this exercise.)

15.27. Spring and rolling wheel *

The axle of a wheel of mass m and radius r is connected to a horizontal spring with spring constant k , and the spring is connected to a wall, as shown in Fig. 15.31. All of the mass of the wheel is on the rim (so its moment of inertia is $I = mr^2$), and it rolls without slipping. Let θ be the angle relative to equilibrium through which the cylinder has rolled. Find the Hamiltonian in terms of θ and its conjugate momentum, and then write down Hamilton's equations.

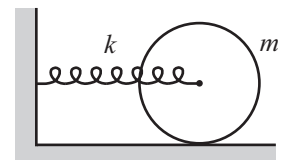


Figure 15.31

15.28. Two masses and a spring *

Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let their positions be x_1 and x_2 . Find the Hamiltonian in terms of x_1 and x_2 and their conjugate momenta, and then write down the four Hamilton's equations. (See Problem 15.8 for a slightly harder variation of this exercise.)

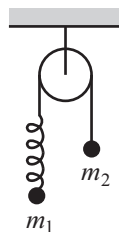


Figure 15.32

15.29. **Atwood with a spring** *

A spring with spring constant k and relaxed length zero is inserted in a standard Atwood's machine to create the setup shown in Fig. 15.32. Let x_1 and x_2 be the positions of the two masses (with downward taken to be positive) relative to an arbitrary configuration where the spring has no stretch. Find the Hamiltonian in terms of x_1 and x_2 and their conjugate momenta, and then write down the four Hamilton's equations. (Assume that the spring is always stretched a positive amount.)

15.30. **Bead on a uniformly moving rod** *

A bead is free to move along a frictionless horizontal rod which initially lies along the y axis in the horizontal plane. The rod is arranged to move with constant speed v in the x direction, and the bead is connected to the origin by a spring with spring constant k and relaxed length zero. Find the Hamiltonian in terms of y and its conjugate momentum, and then write down Hamilton's equations. Is H the energy? Is H conserved?

*Section 15.5: Phase space, Liouville's theorem*15.31. **Harmonic oscillator, quarter cycle** *

Consider a harmonic oscillator governed by the Hamiltonian $H = p^2/2m + kx^2/2$. If x_i and p_i are the initial position and momentum, then from the results of Chapter 4, we know that x and p at a general later time are given by

$$(x, p) = \left(x_i \cos \omega t + \frac{p_i}{m\omega} \sin \omega t, -m\omega x_i \sin \omega t + p_i \cos \omega t \right). \quad (15.108)$$

Consider the rectangle shown in Fig. 15.33 with one corner at the point (x_0, p_0) and sides of lengths Δx and Δp . After a time $t = \pi/2\omega$, what is the image of this rectangle? Show that the area is still $\Delta x \Delta p$.

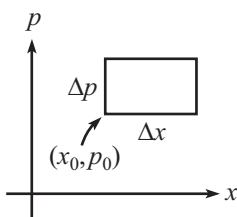


Figure 15.33

15.8 Solutions

15.1. Time dependence

Since $x(q, t)$ depends on both q and t , we have $\dot{x} = (\partial x/\partial q)\dot{q} + \partial x/\partial t$. So the Lagrangian $L = m\dot{x}^2/2 - V(x)$ becomes

$$L = \frac{m}{2} \left(\left(\frac{\partial x}{\partial q} \right)^2 \dot{q}^2 + 2 \left(\frac{\partial x}{\partial q} \right) \left(\frac{\partial x}{\partial t} \right) \dot{q} + \left(\frac{\partial x}{\partial t} \right)^2 \right) - V(x(q, t)). \quad (15.109)$$

Therefore,

$$\begin{aligned} E \equiv \frac{\partial L}{\partial \dot{q}} \dot{q} - L &= m \left(\frac{\partial x}{\partial q} \right)^2 \dot{q}^2 + m \left(\frac{\partial x}{\partial q} \right) \left(\frac{\partial x}{\partial t} \right) \dot{q} - L \\ &= \frac{m}{2} \left(\frac{\partial x}{\partial q} \right)^2 \dot{q}^2 - \frac{m}{2} \left(\frac{\partial x}{\partial t} \right)^2 + V(x(q, t)). \end{aligned} \quad (15.110)$$

In view of the expression for the kinetic energy (the term in parentheses) in Eq. (15.109), we can write E as

$$E = T + V - m \left(\left(\frac{\partial x}{\partial q} \right) \left(\frac{\partial x}{\partial t} \right) \dot{q} + \left(\frac{\partial x}{\partial t} \right)^2 \right), \quad (15.111)$$

as desired.

15.2. Conservation of E

Equation (15.1) gives $E = (\partial L/\partial \dot{x})\dot{x} - L = m\dot{x}^2/2 + V(x)$, and so

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \left(\frac{dV}{dx} \right) \dot{x} = \dot{x} \left(m\ddot{x} + \frac{dV}{dx} \right). \quad (15.112)$$

Ignoring the trivial case where \dot{x} is identically zero,¹⁸ conservation of E is equivalent to the statement that $m\ddot{x} + dV/dx = 0$. But the E-L equation is $(d/dt)(\partial L/\partial \dot{x}) = \partial L/\partial x \implies m\ddot{x} = -dV/dx$, so the two statements are indeed equivalent (and also equivalent to the $F = ma$ statement).

15.3. Poisson brackets

Using Eq. (15.33) and the chain rule, we have

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial p} \dot{q} + \frac{\partial f}{\partial q} \dot{p} \\ &= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}, \end{aligned} \quad (15.113)$$

as desired.

15.4. Hamilton's equations for many variables

Let's look at the $\partial H/\partial p_i$ derivative first. Using $H \equiv (\sum p_k \dot{q}_k) - L$, we have

$$\frac{\partial H(q, p)}{\partial p_i} = \frac{\partial (\sum p_k \dot{q}_k(q, p))}{\partial p_i} - \frac{\partial L(q, \dot{q}(q, p))}{\partial p_i}. \quad (15.114)$$

The arguments (q, p) above are shorthand for $(q_1, \dots, q_N, p_1, \dots, p_N)$. And likewise for the \dot{q} in the last term. We'll ignore any possible t dependence, since it wouldn't affect the discussion.

¹⁸This clearly makes for a constant E , although it has nothing to do with the actual physical motion, except in the case of constant $V(x)$ and an initial velocity of zero. Hence the word "essentially" in the statement of the problem.

In the first term on the right-hand side of Eq. (15.114), $\sum p_k \dot{q}_k(q, p)$ depends on p_i partly because of the factor of p_k when $k = i$, and partly because all the \dot{q}_k (possibly) depend on p_i . So we have $\partial(\sum p_k \dot{q}_k(q, p))/\partial p_i = \dot{q}_i + \sum p_k (\partial \dot{q}_k / \partial p_i)$. In the second term, $L(q, \dot{q}(q, p))$ depends on p_i because all the \dot{q} 's (possibly) depend on p_i , so we have

$$\frac{\partial L(q, \dot{q}(q, p))}{\partial p_i} = \sum_k \frac{\partial L(q, \dot{q})}{\partial \dot{q}_k} \frac{\partial \dot{q}_k(q, p)}{\partial p_i}. \quad (15.115)$$

But $p_k \equiv \partial L(q, \dot{q}) / \partial \dot{q}_k$, so if we substitute these results into Eq. (15.114), we obtain (dropping the (q, p) arguments)

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \left(\dot{q}_i + \sum_k p_k \frac{\partial \dot{q}_k}{\partial p_i} \right) - \sum_k p_k \frac{\partial \dot{q}_k}{\partial p_i} \\ &= \dot{q}_i, \end{aligned} \quad (15.116)$$

as desired.

Let's now calculate $\partial H / \partial q_i$. Using $H \equiv (\sum p_k \dot{q}_k) - L$, we have

$$\frac{\partial H(q, p)}{\partial q_i} = \frac{\partial(\sum p_k \dot{q}_k(q, p))}{\partial q_i} - \frac{\partial L(q, \dot{q}(q, p))}{\partial q_i}. \quad (15.117)$$

In the first term on the right-hand side, $\sum p_k \dot{q}_k(q, p)$ depends on q_i because all the \dot{q} 's (possibly) depend on q_i , so we have $\partial(\sum p_k \dot{q}_k(q, p)) / \partial q_i = \sum p_k \partial \dot{q}_k / \partial q_i$. In the second term, $L(q, \dot{q}(q, p))$ depends on q_i partly because of the (possible) q_i dependence in the first argument, and partly because all the \dot{q} 's (possibly) depend on q_i . So we have

$$\frac{\partial L(q, \dot{q}(q, p))}{\partial q_i} = \frac{\partial L(q, \dot{q})}{\partial q_i} + \sum_k \frac{\partial L(q, \dot{q})}{\partial \dot{q}_k} \frac{\partial \dot{q}_k(q, p)}{\partial q_i}. \quad (15.118)$$

As above, we can use $p_k \equiv \partial L(q, \dot{q}) / \partial \dot{q}_k$ in the second term here. But also, for the first term, the Euler-Lagrange equation for the i th coordinate (which holds, because we're looking at the actual classical motion of the particle) tells us that

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} \right) = \frac{\partial L(q, \dot{q})}{\partial q_i} \implies \dot{p}_i = \frac{\partial L(q, \dot{q})}{\partial q_i}. \quad (15.119)$$

If we substitute these results into Eq. (15.117), we obtain (dropping the (q, p) arguments)

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_i} - \left(\dot{p}_i + \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_i} \right) \\ &= -\dot{p}_i, \end{aligned} \quad (15.120)$$

as desired. Putting everything together, we have the $2N$ Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \text{for } 1 \leq i \leq N. \quad (15.121)$$

15.5. Equivalent Lagrangians

FIRST SOLUTION: With $L' \equiv L + dF/dt$, our goal is to show that the E-L equation,

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}} \right) = \frac{\partial L'}{\partial q}, \quad (15.122)$$

is exactly the same equation as the original E-L equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}. \quad (15.123)$$

We can demonstrate this by simply plugging $L' \equiv L + dF/dt$ into Eq. (15.122). First note that the chain rule gives

$$\frac{dF(q, t)}{dt} = \frac{\partial F(q, t)}{\partial q} \dot{q} + \frac{\partial F(q, t)}{\partial t}. \quad (15.124)$$

(The possible $(\partial F/\partial \dot{q})\dot{q}$ term is missing due to the assumption of no \dot{q} dependence.) So we have $(\partial/\partial \dot{q})(dF/dt) = \partial F/\partial q$. Using this in Eq. (15.122) gives

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial(L + dF/dt)}{\partial \dot{q}} \right) &= \frac{\partial(L + dF/dt)}{\partial q} \\ \implies \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial q} \right) &= \frac{\partial}{\partial q} \left(L + \frac{dF}{dt} \right). \end{aligned} \quad (15.125)$$

This equation is the same as Eq. (15.123) if the terms involving F cancel, that is, if

$$\frac{d}{dt} \left(\frac{\partial F(q, t)}{\partial q} \right) = \frac{\partial}{\partial q} \left(\frac{dF(q, t)}{dt} \right). \quad (15.126)$$

Using the chain rule on the left side and substituting Eq. (15.124) into the parentheses on the right, this is equivalent to

$$\frac{\partial^2 F}{\partial q^2} \dot{q} + \frac{\partial^2 F}{\partial t \partial q} = \frac{\partial^2 F}{\partial q^2} \dot{q} + \frac{\partial^2 F}{\partial q \partial t}, \quad (15.127)$$

which is indeed true, due to the commutativity of partial derivatives.

SECOND SOLUTION: A general function $F(q, t)$ is the sum of terms of the form $f(q, t) = Aq^n t^m$. So it will suffice to show explicitly that the function $f(q, t) = Aq^n t^m$ yields no effect on the equation of motion. This is straightforward but tedious. First, we have $d(Aq^n t^m)/dt = Anq^{n-1} \dot{q} t^m + Aq^n m t^{m-1}$. The new E-L equation,

$$\frac{d}{dt} \left(\frac{\partial(L + df/dt)}{\partial \dot{q}} \right) = \frac{\partial(L + df/dt)}{\partial q}, \quad (15.128)$$

is the same as the old E-L equation, $(d/dt)(\partial L/\partial \dot{q}) = \partial L/\partial q$, if the terms involving f cancel, that is, if

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial(df/dt)}{\partial \dot{q}} \right) &= \frac{\partial(df/dt)}{\partial q} \\ \iff \frac{d}{dt} \left(\frac{\partial(Anq^{n-1} \dot{q} t^m + Aq^n m t^{m-1})}{\partial \dot{q}} \right) &= \frac{\partial(Anq^{n-1} \dot{q} t^m + Aq^n m t^{m-1})}{\partial q} \\ \iff \frac{d(Anq^{n-1} t^m)}{dt} &= An(n-1)q^{n-2} \dot{q} t^m + Anq^{n-1} m t^{m-1}, \end{aligned} \quad (15.129)$$

which is indeed true, as you can quickly verify.

THIRD SOLUTION: The above two mathematical solutions probably weren't too enlightening, so we'll present a solution here that actually gets to the underlying physics of why L' yields the same E-L equation as L . The E-L equation is derived from the principle of stationary action, so it might behoove us to back up and start our reasoning there.

The critical point to realize is that the addition of $dF(q, t)/dt$ doesn't affect the stationary property of the action, because the action for the new Lagrangian is

$$\int_{t_1}^{t_2} L' dt = \int_{t_1}^{t_2} (L + dF/dt) dt = \int_{t_1}^{t_2} L dt + \left(F(q(t_2), t_2) - F(q(t_1), t_1) \right), \quad (15.130)$$

and the last terms here involving F don't change when we vary the path $q(t)$, because it is understood that the variation vanishes at the boundary points.¹⁹ So we have effectively added on a constant (assuming t_1 and t_2 are given), as far as the action is concerned. Therefore, since L and L' differ by a constant, if a given path yields a stationary action for L , then it also yields a stationary action for L' . In other words, L and L' describe the same system. And since the E-L equation is derived from the principle of stationary action, the path that satisfies the E-L equation for L also satisfies the E-L equation for L' .

15.6. Atwood's 1

If the left mass moves up by x , then the right mass moves down by x , so the Lagrangian is

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - (m_1 - m_2)x. \quad (15.131)$$

The momentum conjugate to x is $p \equiv \partial L/\partial \dot{x} = (m_1 + m_2)\dot{x}$, so the Hamiltonian is

$$\begin{aligned} H = p\dot{x} - L &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)x \\ &= \frac{p^2}{2(m_1 + m_2)} + (m_1 - m_2)x. \end{aligned} \quad (15.132)$$

Hamilton's equations are then

$$\begin{aligned} \dot{x} = \frac{\partial H}{\partial p} &\implies \dot{x} = \frac{p}{(m_1 + m_2)}, \\ \dot{p} = -\frac{\partial H}{\partial x} &\implies \dot{p} = -(m_1 - m_2)g. \end{aligned} \quad (15.133)$$

The first of these simply reproduces the definition of p , while the second is identical to the Euler-Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \implies \frac{dp}{dt} = \frac{\partial L}{\partial x} \implies \dot{p} = -(m_1 - m_2)g. \quad (15.134)$$

15.7. Atwood's 2

If the right two masses move up by x and y , then the left mass moves down by $(x + y)/2$. This follows from conservation of string, because the average height of the right two masses always remains the same distance below the center of the right pulley, and the displacement of the left mass is equal and opposite to the displacement of the right pulley. The Lagrangian is therefore

$$\begin{aligned} L &= \frac{1}{2}(2m) \left(\frac{\dot{x} + \dot{y}}{2} \right)^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(2m)\dot{y}^2 + (2m)g \left(\frac{x + y}{2} \right) - mgx - (2m)gy \\ &= \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2 - mgy. \end{aligned} \quad (15.135)$$

The conjugate momenta are

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = \frac{3m\dot{x}}{2} + \frac{m\dot{y}}{2} \quad \text{and} \quad p_y \equiv \frac{\partial L}{\partial \dot{y}} = \frac{m\dot{x}}{2} + \frac{5m\dot{y}}{2}. \quad (15.136)$$

¹⁹These terms depend on t_1 and t_2 , of course, but the point is that they don't depend on the path, because $q(t_1)$ is assumed to be the same for all paths, and likewise for $q(t_2)$.

Inverting these to solve for \dot{x} and \dot{y} gives

$$\dot{x} = \frac{1}{7m}(5p_x - p_y) \quad \text{and} \quad \dot{y} = \frac{1}{7m}(-p_x + 3p_y). \quad (15.137)$$

Using these expressions to eliminate \dot{x} and \dot{y} in favor of p_x and p_y , you can verify that the Hamiltonian is

$$\begin{aligned} H = p_x \dot{x} + p_y \dot{y} - L &= \frac{3}{4}m\dot{x}^2 + \frac{1}{2}m\dot{x}\dot{y} + \frac{5}{4}m\dot{y}^2 + mgy \\ &= \frac{1}{14m}(5p_x^2 - 2p_x p_y + 3p_y^2) + mgy. \end{aligned} \quad (15.138)$$

The four Hamilton's equations are then

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} \implies \dot{x} = \frac{1}{7m}(5p_x - p_y), \\ \dot{p}_x &= -\frac{\partial H}{\partial x} \implies \dot{p}_x = 0, \\ \dot{y} &= \frac{\partial H}{\partial p_y} \implies \dot{y} = \frac{1}{7m}(-p_x + 3p_y), \\ \dot{p}_y &= -\frac{\partial H}{\partial y} \implies \dot{p}_y = -mg. \end{aligned} \quad (15.139)$$

The first and third of these simply reproduce Eq. (15.137), while the second and fourth are identical to the Euler-Lagrange equations,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) &= \frac{\partial L}{\partial x} \implies \frac{dp_x}{dt} = \frac{\partial L}{\partial x} \implies \dot{p}_x = 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) &= \frac{\partial L}{\partial y} \implies \frac{dp_y}{dt} = \frac{\partial L}{\partial y} \implies \dot{p}_y = -mg. \end{aligned} \quad (15.140)$$

REMARKS: The conservation of p_x is a consequence of the fact that x is a cyclic coordinate, which in turn is a consequence of our specific choice of the masses. For randomly chosen masses, in general neither coordinate is cyclic.

Using the p_x in Eq. (15.136), $\dot{p}_x = 0$ implies that $\ddot{y} = -3\ddot{x}$ at all times. That is, the downward acceleration of the right mass is three times the upward acceleration of the middle mass. If we plug this relation into the fourth of Hamilton's equations, which says that (using the p_y in Eq. (15.136)) $m\ddot{x}/2 + 5m\ddot{y}/2 = -mg$, we find that the accelerations of the right two masses are $\ddot{x} = g/7$ and $\ddot{y} = -3g/7$. And then the acceleration of the left mass is the negative of the average of these, which equals $g/7$. If you wish, you can verify these accelerations by solving this problem with $F = ma$.

If you want to demonstrate how the Hamiltonian method can be monumentally more cumbersome than the Lagrangian method, you can try to solve this problem in the case of three general masses, m_1, m_2, m_3 . ♣

15.8. Two masses and a spring

The position of the right mass is $x + z$, so the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \dot{z})^2 - \frac{1}{2}kz^2. \quad (15.141)$$

The conjugate momenta are

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = 2m\dot{x} + m\dot{z} \quad \text{and} \quad p_z \equiv \frac{\partial L}{\partial \dot{z}} = m(\dot{x} + \dot{z}). \quad (15.142)$$

Inverting these to solve for \dot{x} and \dot{z} gives

$$\dot{x} = \frac{1}{m}(p_x - p_z) \quad \text{and} \quad \dot{z} = \frac{1}{m}(-p_x + 2p_z). \quad (15.143)$$

Using these expressions to eliminate \dot{x} and \dot{z} in favor of p_x and p_z , you can verify that the Hamiltonian is

$$\begin{aligned} H = p_x \dot{x} + p_z \dot{z} - L &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \dot{z})^2 + \frac{1}{2}kz^2 \\ &= \frac{1}{m} \left(\frac{p_x^2}{2} - p_x p_z + p_z^2 \right) + \frac{1}{2}kz^2. \end{aligned} \quad (15.144)$$

The four Hamilton's equations are then

$$\begin{aligned} \dot{x} = \frac{\partial H}{\partial p_x} &\implies \dot{x} = \frac{1}{m}(p_x - p_z), \\ \dot{p}_x = -\frac{\partial H}{\partial x} &\implies \dot{p}_x = 0, \\ \dot{z} = \frac{\partial H}{\partial p_z} &\implies \dot{z} = \frac{1}{m}(-p_x + 2p_z), \\ \dot{p}_z = -\frac{\partial H}{\partial z} &\implies \dot{p}_z = -kz. \end{aligned} \quad (15.145)$$

The first and third of these simply reproduce Eq. (15.143), while the second and fourth are identical to the Euler-Lagrange equations,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} &\implies \frac{dp_x}{dt} = \frac{\partial L}{\partial x} \implies \dot{p}_x = 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) = \frac{\partial L}{\partial z} &\implies \frac{dp_z}{dt} = \frac{\partial L}{\partial z} \implies \dot{p}_z = -kz. \end{aligned} \quad (15.146)$$

The p_x equation is the statement that the total momentum of the system is conserved, because $p_x = m\dot{x} + m(\dot{x} + \dot{z}) \equiv m\dot{x}_1 + m\dot{x}_2$, where x_1 and x_2 are the positions of the two masses. The $\dot{p}_z = -kz$ equation is equivalent to $(d/dt)(mx_2) = -kz$, which is simply the $F = ma$ statement for the right mass.

15.9. $y = f(x)$ constraint

The horizontal speed is \dot{x} , and the vertical speed is \dot{x} times the slope, or $\dot{x} \cdot f'(x)$. So the Lagrangian is (with f and f' written as shorthand for $f(x)$ and $f'(x)$)

$$L = \frac{1}{2}m(1 + f'^2)\dot{x}^2 - mgf. \quad (15.147)$$

The momentum conjugate to x is $p \equiv \partial L / \partial \dot{x} = m(1 + f'^2)\dot{x}$, so the Hamiltonian is

$$\begin{aligned} H = p\dot{x} - L &= \frac{1}{2}m(1 + f'^2)\dot{x}^2 + mgf \\ &= \frac{p^2}{2m(1 + f'^2)} + mgf. \end{aligned} \quad (15.148)$$

Hamilton's equations are then

$$\begin{aligned} \dot{x} = \frac{\partial H}{\partial p} &\implies \dot{x} = \frac{p}{m(1 + f'^2)}, \\ \dot{p} = -\frac{\partial H}{\partial x} &\implies \dot{p} = \frac{p^2 f' f''}{m(1 + f'^2)^2} - mgf'. \end{aligned} \quad (15.149)$$

The first of these simply reproduces the definition of p . The second one can be rewritten as

$$\begin{aligned} \frac{d}{dt} (m(1 + f'^2)\dot{x}) &= \frac{(m(1 + f'^2)\dot{x})^2 f' f''}{m(1 + f'^2)^2} - mgf' \\ \implies \frac{d}{dt} ((1 + f'^2)\dot{x}) &= f' f'' \dot{x}^2 - gf'. \end{aligned} \quad (15.150)$$

This is the same as the E-L equation, namely

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \implies \frac{d}{dt} ((1 + f'^2)\dot{x}) = f' f'' \dot{x}^2 - g f'. \quad (15.151)$$

REMARK: Note that if we simplify Eq. (15.151), it becomes

$$\begin{aligned} (1 + f'^2)\ddot{x} + (2f' f'' \dot{x})\dot{x} &= f' f'' \dot{x}^2 - g f' \\ \implies (1 + f'^2)\ddot{x} + f' f'' \dot{x}^2 + g f' &= 0. \end{aligned} \quad (15.152)$$

This is just a disguised form of conservation of energy, because in view of the energy given by the first form of the Hamiltonian in Eq. (15.148), the conservation-of-energy statement is

$$\begin{aligned} 0 &= \frac{d}{dt} \left(\frac{1}{2} m (1 + f'^2) \dot{x}^2 + m g f \right) \\ \implies 0 &= (1 + f'^2) \dot{x} \ddot{x} + (f' f'' \dot{x}) \dot{x}^2 + g f' \dot{x} \\ \implies 0 &= (1 + f'^2) \ddot{x} + f' f'' \dot{x}^2 + g f'. \end{aligned} \quad (15.153)$$

It is no surprise that the E-L equation (or Hamilton's second equation) is equivalent to the conservation-of-energy statement, because the E-L equation was used in the derivation of energy conservation back in Chapter 6 (see Eq. (6.53)). Alternatively, we originally derived conservation of E by using $F = ma$ back in Chapter 5 (see the beginning of Section 5.1), and we know that the E-L equation contains the same information as $F = ma$. ♣

15.10. Spring and moving wall

The position of the mass with respect to the location of the wall at $t = 0$ (which we'll arbitrarily take as our origin) is $\ell_0 + z + X_{\text{wall}}$, because the definition of z implies that $\ell_0 + z$ is the position with respect to the wall. So the velocity of the mass is $\dot{z} + \dot{X}_{\text{wall}} = \dot{z} + A\omega \cos \omega t$. The Lagrangian is therefore

$$L = \frac{1}{2} m (\dot{z} + A\omega \cos \omega t)^2 - \frac{1}{2} k z^2. \quad (15.154)$$

The momentum conjugate to z is $p \equiv \partial L / \partial \dot{z} = m(\dot{z} + A\omega \cos \omega t)$, which can be inverted to give $\dot{z} = p/m - A\omega \cos \omega t$. So the Hamiltonian is

$$\begin{aligned} H = p\dot{z} - L &= p \left(\frac{p}{m} - A\omega \cos \omega t \right) - \left(\frac{p^2}{2m} - \frac{1}{2} k z^2 \right) \\ &= \frac{p^2}{2m} - p(A\omega \cos \omega t) + \frac{1}{2} k z^2. \end{aligned} \quad (15.155)$$

Hamilton's equations are then

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p} \implies \dot{z} = \frac{p}{m} - A\omega \cos \omega t, \\ \dot{p} &= -\frac{\partial H}{\partial z} \implies \dot{p} = -kz. \end{aligned} \quad (15.156)$$

The first of these simply reproduces the definition of p , while the second is identical to the Euler-Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) = \frac{\partial L}{\partial z} \implies \frac{dp}{dt} = \frac{\partial L}{\partial z} \implies \dot{p} = -kz. \quad (15.157)$$

This is simply the $F = ma$ statement, because the force on the mass is $-kz$ (and p is indeed the actual linear momentum). If we use $p = m(\dot{z} + A\omega \cos \omega t)$ to write everything in terms of z , the E-L equation becomes

$$m\ddot{z} - mA\omega^2 \sin \omega t = -kz \implies m\ddot{z} + kz = mA\omega^2 \sin \omega t, \quad (15.158)$$

which looks like the equation for a driven undamped oscillator. But note that z represents the stretch of the spring, and not the actual position of the mass.

The Hamiltonian in Eq. (15.155) is *not* the energy, because the relation between the stretch z and the Cartesian coordinate x is $x = z + X_{\text{wall}}$ (plus a constant, depending on the choice of origin). And since $X_{\text{wall}} = A \sin \omega t$, the relation between z and x involves t , thereby causing H to not be the energy, by Theorem 15.1. And H is not conserved, because there is explicit t dependence in L .

REMARK: H isn't conserved, and it isn't the energy. So might the energy in fact be conserved? Let's see. Since p is the actual linear momentum, the energy of the spring-plus-mass system is $E = p^2/2m + kz^2/2$, so we have

$$\frac{dE}{dt} = \frac{d}{dt} \left(\frac{p^2}{2m} + \frac{kz^2}{2} \right) = \frac{p}{m} \dot{p} + kz \dot{z}. \quad (15.159)$$

Using the definition of p and the second of Hamilton's equations to rewrite p and \dot{p} , respectively, this becomes

$$\frac{dE}{dt} = (\dot{z} + A\omega \cos \omega t)(-kz) + kz \dot{z} = (-kz)(A\omega \cos \omega t). \quad (15.160)$$

Since this is not identically equal to zero, the energy is not conserved.

The result in Eq. (15.160) makes sense, because it is the rate at which work is done (that is, the power P) on the end of the spring that is attached to the wall. This is true because

$$P = \frac{dW}{dt} = F \frac{dX_{\text{wall}}}{dt} = (-kz)(A\omega \cos \omega t). \quad (15.161)$$

The force that the wall applies to the spring is $F = -kz$ here, due to Newton's third law and the fact that the spring applies a force of $+kz$ to the wall. ♣

15.11. Bead on a rotating hoop

The velocity components along the hoop and perpendicular to it (in the direction perpendicular to the plane of the page) are $R\dot{\theta}$ and $(R \sin \theta)\omega$. And the potential energy relative to the center is $-mgR \cos \theta$. So the Lagrangian is

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2 \sin^2 \theta) + mgR \cos \theta. \quad (15.162)$$

The momentum conjugate to θ is $p \equiv \partial L / \partial \dot{\theta} = mR^2\dot{\theta}$, so the Hamiltonian is

$$\begin{aligned} H = p\dot{\theta} - L &= \frac{1}{2}m(R^2\dot{\theta}^2 - R^2\omega^2 \sin^2 \theta) - mgR \cos \theta \\ &= \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2 \sin^2 \theta - mgR \cos \theta. \end{aligned} \quad (15.163)$$

Hamilton's equations are then

$$\begin{aligned} \dot{\theta} = \frac{\partial H}{\partial p} &\implies \dot{\theta} = \frac{p}{mR^2}, \\ \dot{p} = -\frac{\partial H}{\partial \theta} &\implies \dot{p} = mR^2\omega^2 \sin \theta \cos \theta - mgR \sin \theta. \end{aligned} \quad (15.164)$$

The first of these simply reproduces the definition of p , while the second is identical to the Euler-Lagrange equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \implies \frac{dp}{dt} = \frac{\partial L}{\partial \theta} \implies \dot{p} = mR^2\omega^2 \sin \theta \cos \theta - mgR \sin \theta. \quad (15.165)$$

Using the definition of p , this can be written as $mR^2\ddot{\theta} = mR^2\omega^2 \sin \theta \cos \theta - mgR \sin \theta$. See Problem 6.11 for the interesting implications of this equation of motion.

The Hamiltonian in Eq. (15.163) is *not* the energy, because the Cartesian coordinates (x, y) in the horizontal plane are related to θ by $(x, y) = R \sin \theta (\cos \omega t, \sin \omega t)$, up to a phase. Since this relation involves t , the Hamiltonian is not the energy. But H is in fact conserved, because there is no t dependence in L .

REMARK: The Hamiltonian in Eq. (15.163) differs from the energy due to the minus sign in the second term. So this means that the energy equals $H + mR^2\omega^2 \sin^2 \theta$. But as noted above, H is conserved. So the energy takes the form of a constant plus $mR^2\omega^2 \sin^2 \theta$. That is, the energy is larger when mass is farther away from the axis of rotation. Where does this additional energy come from? It comes from the work that the hoop does on the bead, due to the normal force perpendicular to the plane of the hoop. We can be quantitative about this as follows.

To find the normal force, note that the angular momentum around the vertical axis is given by $m(R \sin \theta)^2 \omega$. The time derivative of this (which equals the torque) is

$$\tau = 2mR^2\omega \sin \theta \cos \theta \dot{\theta} = (R \sin \theta)(2mR\omega \cos \theta \dot{\theta}). \quad (15.166)$$

Since the “lever arm” is $R \sin \theta$, the normal force must be $2mR\omega \cos \theta \dot{\theta}$. (You can also derive this by using the fact that the normal force must balance the Coriolis force in the rotating frame.)

In a small time dt , the mass moves a distance $(R \sin \theta)(\omega dt)$ horizontally, so the work done by the normal force during this time is $(2mR\omega \cos \theta \dot{\theta})(R \sin \theta \omega dt)$. The total work done by the time the mass reaches an angle θ is therefore

$$\int_0^\theta (2mR\omega \cos \theta (d\theta/dt))(R \sin \theta \omega dt) = \int_0^\theta 2mR^2\omega^2 \sin \theta \cos \theta d\theta = mR^2\omega^2 \sin^2 \theta, \quad (15.167)$$

which agrees with the additional energy we found above. ♣

15.12. $F(x) = x^n$

We have $s \equiv F'(x) = nx^{n-1}$, so $x(s) = (s/n)^{1/(n-1)}$. The Legendre transform of $F(x)$ is then

$$\begin{aligned} G(s) = sx(s) - F(x(s)) &= s \cdot \left(\frac{s}{n}\right)^{1/(n-1)} - \left(\left(\frac{s}{n}\right)^{1/(n-1)}\right)^n \\ &= n \left(\frac{s}{n}\right) \cdot \left(\frac{s}{n}\right)^{1/(n-1)} - \left(\frac{s}{n}\right)^{n/(n-1)} \\ &= (n-1) \left(\frac{s}{n}\right)^{n/(n-1)}. \end{aligned} \quad (15.168)$$

Therefore, $dG(s)/ds = (s/n)^{1/(n-1)}$, which equals x , as expected.

15.13. **Alternate definition of $G(s)$**

For a given value of s , consider the line with slope s that passes through the point $(x_0, F(x_0))$, as shown in Fig. 15.34. From the reasoning in the first paragraph of Section 15.3.2, the negative of the y intercept of this line equals $sx_0 - F(x_0)$. (So $sx_0 - F(x_0)$ is the *distance* below the x axis, as shown.) Now imagine varying the value of x_0 (while keeping s fixed) to obtain the series of lines shown in Fig. 15.35. (Note that s is *not* the slope of the lines at any of the intersection points in this graph, except for the lowest line.) We see that the maximum value of $sx - F(x)$ is achieved in the case of the lowest line which is tangent to the $F(x)$ curve. (If we lower the line any farther, it won't intersect with $F(x)$.) Hence, the present definition of the Legendre transform (where $G(s)$ is defined to be the maximum value of $sx - F(x)$) agrees with the definition in the text (where $G(s)$ was defined to be the negative y intercept of the tangent line with slope s).

15.14. **Equal $G(s)$ values**

The value of $G(s(x))$ as a function of x is $F'(x) \cdot x - F(x)$. We are given that this function has the same value for two different values of x . Call them x_1 and x_2 . The mean value theorem then tells us that there must be a point between x_1 and x_2 where the derivative of $F'(x) \cdot x - F(x)$ is zero (in other words, $F'(x) \cdot x - F(x)$ achieves a max or min somewhere between x_1 and x_2). So there must be an x for which

$$0 = \frac{d(F'(x) \cdot x - F(x))}{dx}$$

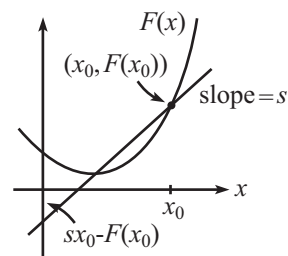


Figure 15.34

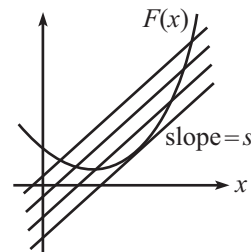


Figure 15.35

$$\begin{aligned}
&= (F''(x) \cdot x + F'(x)) - F'(x) \\
&= F''(x) \cdot x.
\end{aligned} \tag{15.169}$$

Since we are assuming that x_1 and x_2 have the same sign, the factor of x here can't be zero. So we are left with $F''(x) = 0$ as the only option, which means that there exists an inflection point.

It's possible to solve this problem by not calculating the derivative in Eq. (15.169) and instead by just looking at all the possible type of graphs (one of which is shown in Fig. 15.8). But there are a number of cases to consider (depending on which tangent line gets hit first, and whether $F(x)$ is above or below each tangent line), and it's hard to be sure you've covered them all.

REMARK: The $x = 0$ root of Eq. (15.169) is relevant if we remove the restriction that the two x 's have the same sign. If you imagine increasing the value of x in Fig. 15.7, the tangent line "rolls" on $F(x)$ as it migrates from slope s_1 to s_2 . The highest y intercept (which equals the negative of $F'(x) \cdot x - F(x)$) occurs when the tangent line is tangent to $F(x)$ at $x = 0$ (because every other tangent line has a y intercept below this point). So the minimum value of $F'(x) \cdot x - F(x)$ is achieved at $x = 0$, which means that its derivative is zero at $x = 0$, consistent with Eq. (15.169). ♣

15.15. Double-valued $G(s)$

Let x_1 and x_2 be the associated values of x where the lines of slope s are tangent to $F(x)$. Since $x_1 \neq x_2$ (because $F(x)$ is assumed to be well defined), we see that two different values of x yield the same slope. Consider the slope as a function of x . Because the slopes are equal at x_1 and x_2 , the mean value theorem tells us that there must be a point between x_1 and x_2 where the derivative of the slope is zero. In other words, the second derivative of $F(x)$ is zero, as we wanted to show.

15.16. Straight lines

We claim that straight lines remain straight if and only if the velocity vector (\dot{q}, \dot{p}) is at most a linear function of the coordinates, that is, if it takes the form,

$$\mathbf{v}(q, p) \equiv (\dot{q}(q, p), \dot{p}(q, p)) = (a_1 + a_2q + a_3p, \quad b_1 + b_2q + b_3p). \tag{15.170}$$

The constant (a_1, b_1) part of this velocity is irrelevant, because it corresponds to a uniform shift of all points in the plane, which doesn't affect the straightness of lines. So we'll ignore it. The velocity is then $\mathbf{v}(q, p) = (a_2q + a_3p, \quad b_2q + b_3p)$. With this form, you can quickly check that (with $\mathbf{r}_i \equiv (q_i, p_i)$)

$$\mathbf{v}(\mathbf{r}_1 + \mathbf{r}_2) = \mathbf{v}(\mathbf{r}_1) + \mathbf{v}(\mathbf{r}_2), \quad \text{and} \quad \mathbf{v}(k\mathbf{r}) = k\mathbf{v}(\mathbf{r}). \tag{15.171}$$

Consider now two points in phase space, \mathbf{r} and $\mathbf{r} + \mathbf{a}$. Any other point lying on the line determined by these two points takes the form of $\mathbf{r} + k\mathbf{a}$, where k is some number.

Our strategy will be to show that after a small time dt , the points \mathbf{r} , $\mathbf{r} + \mathbf{a}$, and $\mathbf{r} + k\mathbf{a}$ still lie on a line (with k being an arbitrary number, thus accounting for the whole line). And since any flow in phase space can be built up from many little steps of time dt , it then follows that a line remains a line for all times if the velocity takes the form in Eq. (15.170). Our three points end up at

$$\begin{aligned}
\mathbf{r} &\longrightarrow \mathbf{r} + \mathbf{v}(\mathbf{r})dt, \\
\mathbf{r} + \mathbf{a} &\longrightarrow \mathbf{r} + \mathbf{a} + \mathbf{v}(\mathbf{r} + \mathbf{a})dt = \mathbf{r} + \mathbf{a} + \mathbf{v}(\mathbf{r})dt + \mathbf{v}(\mathbf{a})dt, \\
\mathbf{r} + k\mathbf{a} &\longrightarrow \mathbf{r} + k\mathbf{a} + \mathbf{v}(\mathbf{r} + k\mathbf{a})dt = \mathbf{r} + k\mathbf{a} + \mathbf{v}(\mathbf{r})dt + k\mathbf{v}(\mathbf{a})dt.
\end{aligned} \tag{15.172}$$

Regrouping, the three new points are

$$\begin{aligned}
&(\mathbf{r} + \mathbf{v}(\mathbf{r})dt), \\
&(\mathbf{r} + \mathbf{v}(\mathbf{r})dt) + (\mathbf{a} + \mathbf{v}(\mathbf{a})dt), \\
&(\mathbf{r} + \mathbf{v}(\mathbf{r})dt) + k(\mathbf{a} + \mathbf{v}(\mathbf{a})dt).
\end{aligned} \tag{15.173}$$

These have the same general form (where the difference between the first and third is a multiple of the difference between the first and second) as the three initial points, so they do indeed lie on a line, as we wanted to show.

Conversely, if the velocity is not of the form in Eq. (15.170), then the relations in Eq. (15.171) fail to hold for at least some positions, causing the final positions to not take the form in Eq. 15.173.

Interestingly, for 1-D Hamiltonians of the standard form, $p^2/2m + V(q)$, the only potentials that lead to phase-space velocities of the form in Eq. (15.170) are $V(q) = C$, $V(q) \propto q$, and $V(q) \propto q^2$, which exactly correspond to the “Constant velocity,” “Falling balls,” and “Harmonic oscillator” examples in Section 15.5.1. So, although these three examples might have led you to the inductive conclusion that straight lines remain straight in any setup, these examples are actually the only ones (with Hamiltonians of the form $p^2/2m + V(q)$) for which this is true.

15.17. No branching

First, note that Hamilton’s equations uniquely determine the velocity vector $(\dot{q}, \dot{p}) = (\partial H/\partial p, -\partial H/\partial q)$ at a given point (q, p) . So the two paths must have the same velocity at the branching point if there is any chance of a branch actually existing.

However, this argument also applies to all higher derivatives. For example, taking the derivative of $\dot{q} = \partial H/\partial p$ yields \ddot{q} in terms of q and p , and also \dot{q} and \dot{p} by the chain rule. But \dot{q} and \dot{p} can be rewritten in terms of q and p via the previous knowledge of the velocity vector (\dot{q}, \dot{p}) . Likewise for \ddot{p} . So the second derivatives can be written in terms of only q and p (that is, no derivatives are needed), and are therefore uniquely determined by a given point (q, p) . In this manner, we can work our way up to all higher derivatives, expressing them in terms of only q and p .

Therefore, at the branching point the derivatives of q and p to all orders for one path must be equal to the corresponding derivatives for the other path. But if two functions have their derivatives equal to all orders, they must actually be the same function. Hence, the two paths are actually the same path, and there is in fact no branching.²⁰ This reasoning also implies that paths can’t merge, because we could imagine time running backwards, in which case we would end up with a fork.

15.18. Liouville for higher N

In the same manner that we obtained Eq. (15.94) in the proof for the $N = 1$ case in the text, we obtain

$$\frac{dV}{dt} = \int_S \mathbf{v} \cdot d\mathbf{A}. \quad (15.174)$$

Here V is the $2N$ -dimensional “volume” enclosed by the $(2N - 1)$ -dimensional surface S . The vector $d\mathbf{A}$ points perpendicular to the surface and has its magnitude equal to the area of a small $(2N - 1)$ -dimensional patch. The velocity vector \mathbf{v} is the $2N$ -dimensional vector $(\dot{q}_1, \dot{p}_1, \dot{q}_2, \dot{p}_2, \dots, \dot{q}_N, \dot{p}_N)$. Combining Eq. (15.174) with the divergence theorem,

$$\int_V \nabla \cdot \mathbf{v} dV = \int_S \mathbf{v} \cdot d\mathbf{A}, \quad (15.175)$$

gives

$$\begin{aligned} \frac{dV}{dt} &= \int_V \nabla \cdot \mathbf{v} dV \\ &= \int_V \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial q_N}, \frac{\partial}{\partial p_N} \right) \cdot (\dot{q}_1, \dot{p}_1, \dots, \dot{q}_N, \dot{p}_N) dV \\ &= \int_V \left(\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{p}_1}{\partial p_1} + \dots + \frac{\partial \dot{q}_N}{\partial q_N} + \frac{\partial \dot{p}_N}{\partial p_N} \right) dV. \end{aligned} \quad (15.176)$$

²⁰There are some classic pathological examples of unequal mathematical functions whose derivatives agree to all orders at a given point (for example, $y = 0$ and $y = e^{-1/x^2}$, at $x = 0$), but we won’t worry about such things here.

But the $2N$ Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \text{for } 1 \leq i \leq N. \quad (15.177)$$

Plugging these into Eq. (15.176) gives

$$\begin{aligned} \frac{dV}{dt} &= \int_V \left(\left(\frac{\partial^2 H}{\partial q_1 \partial p_1} - \frac{\partial^2 H}{\partial p_1 \partial q_1} \right) + \cdots + \left(\frac{\partial^2 H}{\partial q_N \partial p_N} - \frac{\partial^2 H}{\partial p_N \partial q_N} \right) \right) dV \\ &= 0, \end{aligned} \quad (15.178)$$

where we have used the fact that partial derivatives commute.

15.19. **Harmonic oscillator**

The four corners of the initial rectangle have coordinates (x_0, p_0) , $(x_0 + \Delta x, p_0)$, $(x_0, p_0 + \Delta p)$, and $(x_0 + \Delta x, p_0 + \Delta p)$. Using Eq. (15.105), these four corners get mapped into the points, respectively (with $S \equiv \sin \omega t$ and $C \equiv \cos \omega t$),

$$\begin{aligned} (x, p)_1 &= \left(x_0 C + \frac{p_0}{m\omega} S, -m\omega x_0 S + p_0 C \right), \\ (x, p)_2 &= \left((x_0 + \Delta x) C + \frac{p_0}{m\omega} S, -m\omega(x_0 + \Delta x) S + p_0 C \right), \\ (x, p)_3 &= \left(x_0 C + \frac{p_0 + \Delta p}{m\omega} S, -m\omega x_0 S + (p_0 + \Delta p) C \right), \\ (x, p)_4 &= \left((x_0 + \Delta x) C + \frac{p_0 + \Delta p}{m\omega} S, -m\omega(x_0 + \Delta x) S + (p_0 + \Delta p) C \right). \end{aligned} \quad (15.179)$$

Relative to $(x, y)_1$, the corners are therefore located at the points

$$\begin{aligned} (x, p)_1 &= (0, 0), \\ (x, p)_2 &= \left(\Delta x C, -m\omega \Delta x S \right), \\ (x, p)_3 &= \left(\frac{\Delta p}{m\omega} S, \Delta p C \right), \\ (x, p)_4 &= \left(\Delta x C + \frac{\Delta p}{m\omega} S, -m\omega \Delta x S + \Delta p C \right). \end{aligned} \quad (15.180)$$

These points form the vertices of a parallelogram, as shown in Fig. 15.36. The area

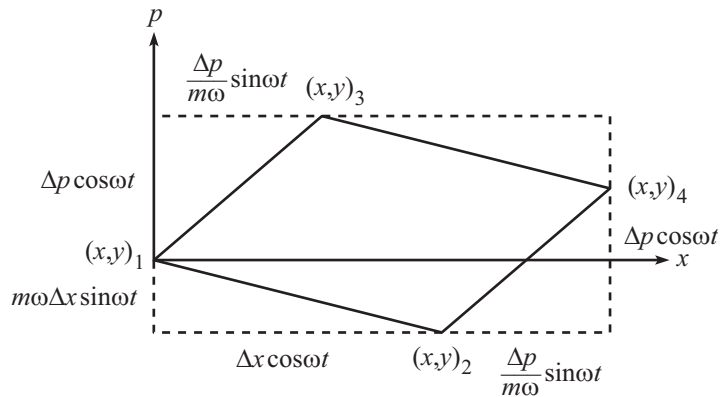


Figure 15.36

of this parallelogram equals the area of the dotted rectangle minus the area of the four triangles, so we have

$$\begin{aligned}
 A &= \left(\Delta x C + \frac{\Delta p}{m\omega} S \right) \left(m\omega \Delta x S + \Delta p C \right) \\
 &\quad - 2 \left(\frac{1}{2} \Delta x C \cdot m\omega \Delta x S + \frac{1}{2} \frac{\Delta p}{m\omega} S \cdot \Delta p C \right) \\
 &= \Delta x C \cdot \Delta p C + \frac{\Delta p}{m\omega} S \cdot m\omega \Delta x S \quad (\text{only the cross terms survive}) \\
 &= \Delta x \Delta p (C^2 + S^2) \\
 &= \Delta x \Delta p,
 \end{aligned} \tag{15.181}$$

as desired. This result for the harmonic oscillator holds for a rectangle of any size, so in particular it holds for an infinitesimal rectangle. And since any region of arbitrary shape can be built up from infinitesimal rectangles, we see that the area of any region remains constant as it flows (and invariably distorts) through phase space.

15.20. **Harmonic oscillator, easier method** **

If we write the time development in matrix form, we have

$$\begin{pmatrix} z(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} z_i \\ p_i \end{pmatrix}. \tag{15.182}$$

This is the familiar matrix that represents a clockwise rotation in the plane through an angle ωt . So all points in the z - p plane move in circles with the same frequency ω , which means that any given initial region R_i simply rotates around the origin in the z - p plane, as shown in Fig. 15.37. Therefore, since every region keeps its same shape as it rotates in the plane, areas are conserved in the z - p plane.

But what about the x - p plane? To see how regions evolve in this plane, we simply have to shrink the z - p plane by a factor of $1/m\omega$ on the horizontal axis (because $x \equiv z/m\omega$), while keeping the vertical axis the same. Fig. 15.38 shows the result for the case where $m\omega = 2$ (in appropriate units). The transformation in the x - p plane doesn't represent a rotation anymore (the shape of the region clearly gets distorted). But the area *does* remain constant, because the uniform horizontal scaling by the factor $1/m\omega$ changes the areas of *both* R_i and $R(t)$ by this same factor. (Imagine slicing each region up into many thin horizontal rectangles; every rectangle has its area decreased by the factor $1/m\omega$, so the same result must be true for the entire region.) And since we showed above that the areas are equal in the z - p plane, they must therefore also be equal in the x - p plane.

15.21. **Liouville's theorem**

If every point moves through the plane with the same velocity, then the initial rectangle simply gets translated into another rectangle, and the area trivially remains the same. However, if the velocities differ throughout the plane, then the area need not remain the same. For example, if points on the right side of the rectangle have a larger \dot{x} than points on the left, then the rectangle gets stretched, and the area increases (due to this effect, at least).

Let's be quantitative about this. From the definition of the partial derivative of \dot{x} , the value of \dot{x} at point 2 in Fig. 15.39 is larger than the value at point 1 by $(\partial \dot{x} / \partial x) \Delta x$ (to leading order in Δx). So after an infinitesimal time dt , the difference in the x values of points 1 and 2 has grown by $(\partial \dot{x} / \partial x) \Delta x dt$ compared with what it was initially (which was Δx). The difference in the x values at time dt is therefore $\Delta x + (\partial \dot{x} / \partial x) \Delta x dt$. Likewise, the difference in the p values at time dt is $0 + (\partial \dot{p} / \partial x) \Delta x dt$. By analogous reasoning (or by simply switching x and p), The difference in the p values of points 1 and 3 at time dt is $\Delta p + (\partial \dot{p} / \partial p) \Delta p dt$, and the difference in the x values is $0 + (\partial \dot{x} / \partial p) \Delta p dt$.

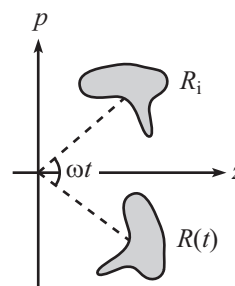


Figure 15.37

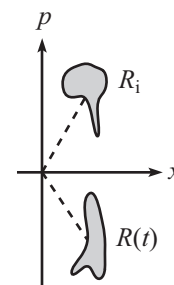


Figure 15.38

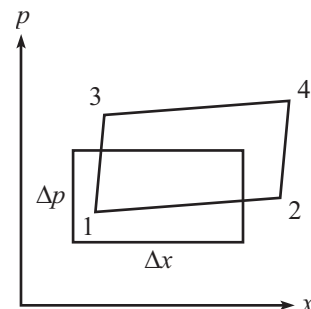


Figure 15.39

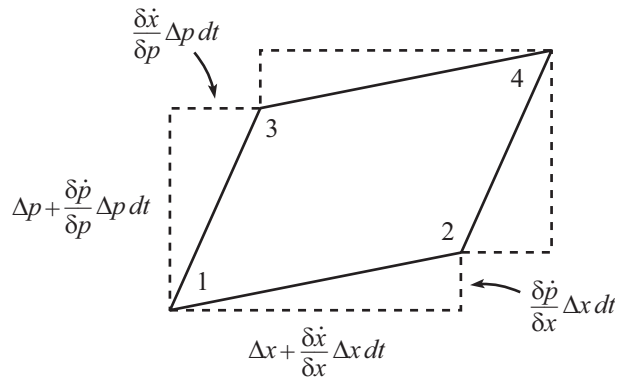


Figure 15.40

Finally (continuing to use the definition of the partial derivative), the difference in the x values of points 1 and 4 at time dt is $\Delta x + (\partial\dot{x}/\partial x)\Delta x dt + (\partial\dot{x}/\partial p)\Delta p dt$, and the difference in the p values is $\Delta p + (\partial\dot{p}/\partial p)\Delta p dt + (\partial\dot{p}/\partial x)\Delta x dt$. We therefore end up with the parallelogram shown in Fig. 15.40.

One method of finding the area of this parallelogram is to calculate the area of the “circumscribing” rectangle and then subtract off the area of the four triangles shown in Fig. 15.40, along with the two tiny rectangles at the upper left and lower right corners. To first order in dt , this does indeed yield the desired area for the parallelogram, given in Eq. (15.107), as you can check. (To keep things from getting too messy, you should immediately drop all terms of order dt^2 . In particular, the two tiny rectangles are irrelevant.) But let’s find the area in a nicer geometric way.

Consider the situation shown in Fig. 15.41. To more easily compare the initial and final areas, we have translated the parallelogram 1,2,3,4 (which doesn’t affect its area, of course) so that its lower left vertex coincides with the lower left corner of the original rectangle A, B, C, D . We can now imagine chopping off the lightly-shaded triangle from the parallelogram shown on the right side of the figure, and pasting into the left side of the figure, as shown. Likewise with the darker triangle. We therefore see that the area of the parallelogram equals the area of the rectangle A', B', C', D' . You might object that we have double counted the region where the shaded triangles overlap in the upper right part of the figure. However, using the

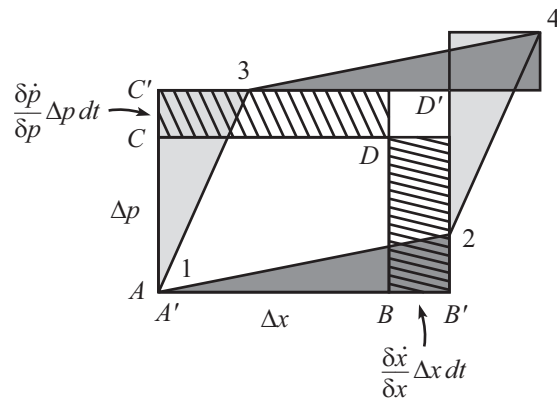


Figure 15.41

lengths shown in Fig. 15.40, the area of this region is of order $\Delta x \Delta p dt^2$, so it can be ignored due to the dt^2 .

Our task therefore reduces to finding the area of rectangle A', B', C', D' . This area equals the area of the original rectangle A, B, C, D (which is $\Delta x \Delta p$) plus the areas of the two striped rectangles. (We are ignoring the tiny rectangle in the upper right corner because its area is of order dt^2 .) Using the lengths shown, the sum of the areas of these two rectangles is

$$\Delta p \cdot \frac{\partial \dot{x}}{\partial x} \Delta x dt + \Delta x \cdot \frac{\partial \dot{p}}{\partial p} \Delta p dt = \Delta x \Delta p \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{p}}{\partial p} \right) dt. \quad (15.183)$$

Adding on the area $\Delta x \Delta y$ of rectangle A, B, C, D , we see that (to leading order) the area of rectangle A', B', C', D' (which is essentially the same as the area of the parallelogram) equals

$$\Delta x \Delta p \left(1 + \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{p}}{\partial p} \right) dt \right), \quad (15.184)$$

as desired. Note that in the case of Hamiltonian flow in phase space, we know that the two partial-derivative terms sum to zero, so one of them must be negative. This means that one of the striped rectangles in Fig. 15.41 actually has negative area. That is, B' and D' lie to the left of B and D , or C' and D' lie below C and D .