Chapter 8

Exercises

Probability, For the Enthusiastic Beginner (Exercises, Version 1, September 2016)

8.1 Chapter 1

Section 1.2: Permutations

1. **Assigning seats** $*$

   Five girls and three boys are to be assigned to eight seats in a row, with the stipulation that a girl sits in the second seat. How many arrangements are possible?

2. **Assigning seats, again** $*$

   Four girls, two boys, and five cats are to be assigned to 11 seats in a row, with the stipulations that a girl sits in the fourth seat, a boy sits in the sixth seat, and a cat sits in the seventh seat. How many arrangements are possible?

Section 1.3: Ordered sets, repetitions allowed

3. **Cube and octahedron** $*$

   A standard cubical (6-sided) die and an octahedral (8-sided) die are each rolled n times. What is the smallest value of n for which the total number of possible outcomes for the octahedral die is at least twice the number for the cubical die?

Section 1.4: Ordered sets, repetitions not allowed

4. **Subtracting the repeats, 3 chosen from 4** $**$

   Repeat the task of Problem 1.3(a), but now in the case where you pick triplets from four people (instead of five).

5. **Subtracting the repeats, 4 chosen from 4** $**$

   Repeat the task of Problem 1.4, but now in the case where you pick quadruplets from four people (instead of five).

6. **Subtracting the repeats, 4 chosen from N** $**$

   Repeat the task of Problem 1.4 (or Exercise 5), but now in the general case where you pick quadruplets from N people, instead of five (or four).

Section 1.5: Unordered sets, repetitions not allowed

7. **Sum of squares** $*$

   In the spirit of Problem 1.5(b), use mathematical induction to prove that the sum of the squares of the first N integers, $1^2 + 2^2 + 3^2 + \cdots + N^2$, equals $N(N+1)(2N+1)/6$. (This is just a fun math problem; it isn't directly related to combinatorics.)

8. **Orderings of "committee"** $*$

   How many different orderings are there of the nine letters in the word "committee"?

9. **Five and four** $*$

   In the example at the end of Section 1.2, explain logically how the total number of possible orderings with no restriction (namely 9!) can be obtained from the $5! \cdot 4! = 2880$ result in the example.

10. **Two ways of choosing two** $*$

    Show mathematically that

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab. \tag{8.1}$$

    Then explain in words why the relation is true, by imagining a scenario that can be looked at in two different ways.

11. **Comparing the types** $**$

    Four balls are drawn (with replacement) from a box containing N distinct balls labeled A, B, C, ... The ordered result is written down (such as CRCF). Consider two general types of ordered sets: those that have three of the same letter plus a different letter (such as EWEE), and those that have two pairs of letters (such as JDDJ). Which type are there more of? (Remember that the sets are ordered)

12. **Forming teams** $*$

    How many different teams of five people can be formed from six girls and six boys, with the condition that the team has at least one girl and at least one boy?

13. **Forming pairs** $**$

    Ten students are divided into five pairs. How many ways can this be done?

14. **Two different titles** $*$

    From nine people, how many ways can you form a committee of five people consisting of two co-presidents and three regular members?

15. **At least one 7** $*$

    How many different five-card poker hands contain at least one 7?

16. **Mini full houses** $**$

    Three cards are dealt from a standard deck. How many different "mini full-house" hands are possible (two cards of one value and one card of another)?

Section 1.7: Unordered sets, repetitions allowed

17. **$n = 4$ chosen from $N = 4$** $**$

    In the spirit of the examples at the beginning of Section 1.7, determine how many different sets of $n = 4$ letters you can pick (with replacement, and with the order not mattering) from a hat containing $N = 4$ letters: A, B, C, D.

18. **$n = 6$ chosen from $N = 3$** $**$

    Repeat the previous exercise, but now with sets of $n = 6$ letters chosen from $N = 3$ letters.

19. **Reproducing N^n** $**$

    In the spirit of the example at the end of Section 1.7, reproduce the N^n result for the second example at the beginning of Section 1.7, with $n = 3$ and $N = 4$. So your goal is to show that the total number of ordered sets is $4^3 = 64$.

20. **Reproducing N^n again** $**$

    Repeat the previous exercise, but now for the third example at the beginning of Section 1.7, with $n = 5$ and $N = 3$. So your goal is to show that the total number of ordered sets is $3^5 = 243$.

8.2 Chapter 2

1. **Tossing Heads** $*$

    (a) Two coins are tossed. What is the probability of getting at least one Heads?

    (b) Four coins are tossed. What is the probability of getting at least two Heads?

    (c) Six coins are tossed. What is the probability of getting at least three Heads?
    Which of the above three probabilities is the largest?

2. **Drawing a given ball** $**$

    n balls are in a box. Consider a particular ball; call it B. If balls are drawn in succession *with* replacement, then the probability of drawing B on any given draw is always $1/n$, of course.

    If balls are instead drawn in succession *without* replacement, show that the probability of drawing B on any given draw is still always $1/n$. Do this by imagining drawing the balls in succession.

3. **Three girls and three boys** $*$

   Consider Example 4 (Three girls and three boys) on page 79. Solve the problem in the spirit of the second solution given, but now imagine that first a girl picks her seat, then a boy, then a girl, then a boy, then a girl (and then the last boy's seat is determined).

4. **Red given blue** $**$

   Six red and four blue balls are in a box. Two balls are drawn without replacement. What is the probability that the first ball is red, given that the second ball is blue?

5. **Twins next to each other** $**$

   (a) n people randomly sit in n chairs arranged in a circle. Two of these people are twins. What is the probability that the twins sit next to each other? Answer this by using a counting argument, and then again by using a probability argument (imagine successively assigning the twins' seats).

   (b) Answer the same question, but now with the n chairs arranged in a line.

6. **Alternating letters** $**$

   A string of $2n$ letters is randomly formed from n A's and n B's. What is the probability that an alternating string, like ABABAB. . . , is formed? (The string can start with either letter.) Answer this by using a counting argument, and then again by using a probability argument (imagine successively plopping down the letters).

7. **Birthday $_NP_n$** $*$

   Solve the Birthday Problem in Section 2.4.1 by making use of the $_NP_n$ notation from Section 1.4.

8. **Yahtzee™ rolls** $***$

   On a *single* roll of five dice, what is the probability of obtaining:

   (a) a large straight (five in a row)

   (b) a small straight (four in a row; this one is tricky)

   (c) Yahtzee (five of a kind)

   (d) four of a kind

   (e) three of a kind

   (f) two of a kind (which could be two pairs; this one is tricky)

   (g) a full house (three of one number, two of another)

   (h) none of the above

   In each case, don't count rolls that fall into a harder-to-achieve category. For example, exclude full houses from three of a kinds, and exclude certain small straights from two of a kinds, etc.

8.3 Chapter 3

1. **Roll until a 6** $*$

   If you roll a die until you get a 6, what is the expected total number of rolls you do? (You can follow the strategy in Problem 3.1.)

2. **General expected value** $**$

   Based on the results of Problem 3.1 and the preceding exercise, it's a good bet that n is the expected total number of flips/rolls/etc. that you need to perform to obtain a particular outcome that occurs with probability $1/n$. One proof of this fact follows from the result in Problem 4.7, with $p = 1/n$. Prove this fact in two other ways:

   (a) Let E be the desired expected number. Then E is the appropriate weighted average of the expected values associated with success or failure on the first trial. (Note that if there is failure on the first trial, then you have wasted one trial, and you're back to square one, where the expected number from that point on is E.) Use this fact to generate an equation for E.

   (b) Imagine performing a very large number of trials. On average, how many of these are successes? What then is the expected number of trials needed for each success?

3. **Tetrahedral die** $**$

   (a) The faces of a tetrahedral die are numbered 1, 2, 3, and 4. Let X and Y represent the outcomes of two such dice. By looking at all of the possible outcomes, verify explicitly that $E(XY) = E(X)E(Y)$.

   (b) Repeat the above task, but now for the general case of a die with n faces (all equally likely) labeled 1 through n.

4. **Variance of four coins** $*$

   Four coins are flipped. Using Eq. (3.33), the variance of the number of Heads is $4(1/2)(1/2) = 1$. Reproduce this result by explicitly considering the probabilities of obtaining the various possible numbers of Heads.

5. **Uneven random walk** $**$

   Repeat Problem 3.8, but now for a random walk with a $2/3$ probability of a rightward step, and a $1/3$ probability of a leftward step.

6. **μ_4 for a Gaussian** $**$ *(calculus)*

   At the end of the solution to Problem 3.12, we stated that a Gaussian distribution has $\mu_4 = 3\sigma^4$. Demonstrate this by using Eq. (4.123) in the solution to Problem 4.23.

7. **Sample variance for two tetrahedral dice rolls** $**$

   Repeat Problem 3.13, but now for tetrahedral dice, with faces numbered 1, 2, 3, and 4.

8.4 Chapter 4

1. **Hypergeometric symmetry** $**$

   It turns out that the hypergeometric distribution in Eq. (4.71), or equivalently in Eq. (4.73), is unchanged if K and n are switched. Demonstrate this mathematically, and then again by giving an argument that explains physically why it is true.

2. **Maximum exponential density** $*$ *(calculus)*

   Consider the exponential density in Eq. (4.27). For a given value of t (call it T), show that if you want the density to be as large as possible when $t = T$, you should pick τ to equal T.

3. **How many shoppers** $*$

   600 shoppers enter a store during an 8-hour day. (Assume, unrealistically, that the process is completely random.) What is the probability that exactly four shoppers enter the store in a given span of five minutes?

4. **Hearing a song** $*$

   Over the course of many years, you estimate that you hear a particular song on the radio about 10 times per year. (Assume that the numbers are fairly steady.) If during the next year you don't hear the song at all, how surprised would you be? Would you think it's just a random fluke?

5. **Probability of zero** $*$ *(calculus)*

   A random process has events occurring at an average rate λ. What is the probability that zero events occur during a given interval of length T? Answer this by using the Poisson distribution, and then again by using the exponential distribution.

6. **Poisson standard deviation** $*$

   We know from Problem 4.13 that the standard deviation of the Poisson distribution is $\sigma = \sqrt{a}$. Explain how this can be quickly deduced from the $\sigma = \sqrt{npq}$ result in Problem 4.5 for the binomial distribution.

7. **Sum of two Poisson random variables** $**$

   A number (call it k_a) is randomly chosen from a Poisson distribution characterized by an a average. Another number (call it k_b) is randomly chosen from a Poisson distribution characterized by a b average. The sum $k_a + k_b$ is recorded. This process is repeated a large number of times. Show that the sums $k_a + k_b$ are distributed according to a Poisson distribution characterized by an $a + b$ average. Do this by giving a physical argument, and then again by working out the math.

8. **Comparing the quantities** $*$

   Consider a Poisson process where the average number of events a is very small (for example, $a = 1/1000$). What is the size order of the three quantities: $P(1)$, $P(\text{at least } 1)$, and the average number a? You will need to use the Taylor approximation $e^x \approx 1 + x + x^2/2$.

8.5 Chapter 5

1. **Unfair coin** $**$

   An unfair coin has a 49% chance of Heads and a 51% chance of Tails. If you flip the coin 10^4 times, what is the probability of getting Heads at least half the time? What if you flip 10^6 coins?

2. **Identical distributions** $*$

   Repeat Problem 5.4, but now instead of standard cubical dice, do the experiment with tetrahedral dice (with faces numbered 1, 2, 3, and 4), where you are dealing with the probability distribution for the number of 4's that appear, relative to the expected number (which is 250).

3. **More than 20 above the mean** $*$

   The expected number of events in a given Poisson process is 100. Approximately what is the probability of having more than 120 events?

4. **30 Heads in 50 flips** $**$

   What is the probability of getting 30 Heads in 50 coin flips? Answer this by using (a) the binomial distribution (exact answer), (b) the Gaussian distribution (approximate answer), and (c) the Poisson distribution (approximate answer).

8.6 Chapter 6

1. **Finding all the quantities** $**$

   Given four (X, Y) points with values $(1, 1)$, $(1, 3)$, $(3, 3)$, $(4, 5)$, calculate (with a calculator) all of the quantities referred to in the five steps listed on page 290. Also calculate the B in Eq. (6.49), and make a rough plot of the four given points along with the regression (least-squares) line.

2. **Scoring higher on a second test** $**$

   For a particular test, the correlation coefficient between the score and innate ability is r. Consider the group of people who score n standard deviations above the mean (plus or minus a hair, to get a sufficient number of data points). Look at a person (call her Jane) whose innate ability is the average of the innate abilities in this group. If Jane retakes the test, what is the probability that her retake score is higher that her original score? Assume that all distributions are Gaussian. (As in Problem 6.7, to give a numerical answer to this problem, you would need to be given r and n. And you would need to use a table or a computer. It suffices here to state the value of the standard deviation multiple that you would plug into the table or computer.)