Speed, Accuracy, and the Optimal Timing of Choices

Drew Fudenberg, MIT
Philipp Strack, Berkeley
Tomasz Strzalecki, Harvard
Suppose we see an agent’s choices from the menu $A = \{\ell, r\}$. Sometimes she chooses $\ell$, sometimes she chooses $r$.

How to model?

- **choice correspondence**: the agent is indifferent, regardless of the relative probabilities
- **stochastic choice function**: treat choice probabilities as data
New Variable: How long does the agent take to decide?

Time: $\mathcal{T} = [0, \infty)$

Observe: Joint distribution $P \in \Delta(A \times \mathcal{T})$

Question:
- Are fast decisions “better” or “worse” than slow ones?
System I is instinctive and fast, System II is deliberative and slow, so fast decisions are worse (Kahneman)

- we will ignore this and have a one-system story
Are quick decisions better than slow ones?

**Informational Effect:**

- More time $\Rightarrow$ more information $\Rightarrow$ better decisions
  - if forced to stop at time $t$, make better choices for higher $t$
  - seeing more signals leads to more informed choices

**Selection Effect:**

- Time is costly, so you decide to stop depending on how much you expect to learn (option value of waiting)
  - Want to stop early if get an informative signal
  - Want to continue if get a noisy signal

- This creates dynamic selection
  - stop early after informative signals
  - informative signals more likely when the problem is easy
Decreasing accuracy

The two effects push in opposite directions. Which one wins?

Stylized fact: Decreasing accuracy: fast decisions are “better”

- Well established in perceptual tasks (dots moving on the screen), where “better” is objective
- Also in experiments where subjects choose between consumption items
When are decisions “more accurate?”

In cognitive tasks, **accurate** = **correct**

In choice tasks, **accurate** = **preferred**

\[ p(t) := \text{probability of making the correct/preferred choice conditional on deciding at } t \]

**Definition:**

\[ P \text{ displays } \begin{cases} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases} \text{ accuracy iff } p(t) \text{ is } \begin{cases} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases} \]
• $X$: 70 different food items

• Step 1: Rate each $x \in X$ on the scale $-10, \ldots, 10$

• Step 2: Choose from $A = \{\ell, r\}$ (100 different pairs)
  - record choice and decision time

• Step 3: Draw a random pair and get your choice
Decreasing Accuracy

based on data from Krajbich, Armel, and Rangel (2010)
Related Literature: Economics

* stochastic choice conditional on exogenous time
  Caplin and Dean (2011); Natzenzon (2014); Lu (2016);
  Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2017)

* stochastic choice and endogenous timing of decisions
  Che and Mierendorff (2016); Hebert and Woodford (2017);
  Jehiel and Steiner (2017); Woodford (2014);
Related Literature

* Cognitive science: drift diffusion model (DDM)
  e.g. Ratcliff-McKoon (2008); Krajbich et al (2010), Milosavljevic et al (2010); Drugowitsch et al (2012); Clithero- Rangel (2014)

* Probability and statistics: optimal stopping
  e.g. Wald (1947); Arrow, Blackwell, Girshick (1949); Chernoff (1961); Bather (1962); Peskir and Shiryaev (2006)

* Less closely related: classify decisions as “instinctive/heuristic” or “cognitive”
  e.g. Rubinstein (2007); Kahneman (2011); Rand et al (2012); Caplin and Martin (2015)
Model

- choose \( r \)
- choose \( l \)
- continue
- choose \( r \)
- choose \( l \)
- continue
- continue
Learning Model

• unknown utility $\theta = (\theta^\ell, \theta^r) \in \mathbb{R}^2$; prior belief on $\theta$

• observe a two-dimensional signal for $i = \ell, r$

$$Z_t^i = \theta^i t + \alpha B_t^i$$

$B_t^i$ are independent Brownian motions

let $Z_t := Z_t^\ell - Z_t^r$

• examples of prior/posterior families

  • “certain difference”
    - binomial prior: either $\theta = (1, 0)$ or $\theta = (0, 1)$
    - binomial posterior: either $\theta = (1, 0)$ or $\theta = (0, 1)$

  • “uncertain difference”
    - Gaussian prior: $\theta^i \sim N(X_0^i, \sigma_0^2)$
    - Gaussian posterior: $\theta^i \sim N(X_t^i, \sigma_t^2)$
Interpretation of the Signal Process

- recognition of the objects on the screen
- retrieving pleasant or unpleasant memories
- coming up with reasons pro and con
- introspection
- signal strength depends on the utility difference or on the ease of the perceptual task

In animal experiments, some neuroscientists record neural firing and relate it to these signals

We don’t do this, treat signals as unobserved by the analyst
• $\tau$ is a stopping time (measurable w.r.t $Z_t$)

• conditional on stopping, the agent maximizes expected utility

$$\text{choice}_\tau = \text{argmax}\{\mathbb{E}_\tau \theta^\ell, \mathbb{E}_\tau \theta^r\}$$

**Example:** If stopping is exogenous ($\tau$ is independent of signal $Z_t$), and prior is symmetric, there is **increasing** accuracy: waiting longer gives better information so generates better decisions
Exogenous vs Endogenous Stopping

- Key assumption above: stopping independent of signal
- If stopping is conditional on the signal, this could get reversed
- Intuition: with endogenous stopping you
  
  #1 stop early after informative signals (and make the right choice); wait longer after noisy signals (and possibly make a mistake)
  
  #2 probably faced an easier problem if you decided quickly
The agent chooses a $Z_t$-measurable stopping time $\tau$ to optimize:

$$\max_{\tau} \left[ \max \{ \mathbb{E}_\tau \theta^l, \mathbb{E}_\tau \theta^r \} - c\tau \right]$$

(we focus on the “minimal optimal” stopping time)
The “certain difference” model

* Assumptions:
  - binomial prior: either $\theta = (1, 0)$ or $\theta = (0, 1)$
  - binomial posterior: either $\theta = (1, 0)$ or $\theta = (0, 1)$

* Key intuition: stationarity
  - suppose that you observe $Z_t^l \approx Z_t^r$ after a long $t$
  - you think to yourself: “the signal must have been noisy”
  - so you don’t learn anything $\Rightarrow$ you continue

* Formally, the option value is constant in time
The “certain difference” model

Theorem: (Wald, Arrow, Blackwell, Girshick, Shiryaev)
When the prior is symmetric, the optimal stopping time is

$$\tau^* = \inf \{ t \geq 0 : |Z_t| \geq b \}$$

where $b > 0$. 
\[ \tau^* = \inf\{ t \geq 0 : |Z_t| \geq b \} \]
\[ \tau^* = \inf \{ t \geq 0 : |Z_t| \geq b \} \]
\[ \tau^* = \inf \{ t \geq 0 : |Z_t| \geq b \} \]
Hitting Time Models

- can use this algorithm to generate a distribution $P \in \Delta(A \times T)$ without worrying about optimality
- closed forms for choice probabilities and mean stopping time
- used extensively for perception tasks since the 70’s; pretty well established in psych and neuroscience
- more recently used to study choice tasks by a number of teams of authors including Colin Camerer and Antonio Rangel

- Many versions of the model
  - ad-hoc tweaks (not worrying about optimality)
    - assumptions about the process $Z_t$
    - functional forms for the time-dependent boundary
  - much less often, optimization used:
    - time-varying costs (Drugovitsch et al, 2012)
    - endogenous attention (Woodford, 2014)
Hitting Time Models

* Definition:

- stochastic process $Z_t$ starts at 0
- time-dependent boundary $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$
- hitting time $\tau = \inf\{ t \geq 0 : |Z_t| \geq b(t) \}$

- choice $= \begin{cases} 
\ell & \text{if } Z_\tau = +b(\tau) \\
r & \text{if } Z_\tau = -b(\tau)
\end{cases}$
Drift Diffusion Model (DDM)

Special case where the process $Z_t$ is a diffusion with constant drift and volatility

$$Z_t = \delta t + \alpha B_t$$

(could eliminate the parameter $\alpha$ here but it’s useful later)

**Definition**: $P$ has a **DDM representation** if it can be represented by a stimulus process $Z_t = \delta t + \alpha B_t$ and a time-dependent boundary $b$. We write this as $P = P(\delta, \alpha, b)$. 
Definition: $P$ has an average DDM representation $P(\mu, \alpha, b)$ with $\mu \in \Delta(\mathbb{R})$ if $P = \int P(\delta, \alpha, b) d\mu(\delta)$.

- in an average DDM model the analyst does not know $\delta$, but has a correct prior
- intuitively, it is unknown to the analyst how hard the problem is for the agent
Proposition: Any Borel $P \in \Delta(A \times T)$ has a hitting time representation where the stochastic process $Z_t$ is a time-inhomogeneous Markov process and the barrier is constant.

Remarks:

- this means that the general model is without loss of generality
- in particular, it is without loss of content to assume that $b$ is independent of time
- however, in the general model the process $Z_t$ may have jumps
- From now on we focus on the DDM special cases
**Definition:** Accuracy in DDM is the probability of making the choice which agrees with the signal

\[
p(t) = \mathbb{P}[\text{sgn } Z_\tau = \text{sgn } \delta \mid \tau = t].
\]

- In DDM \( p \) is the probability of making the modal choice.
- If the correct choice is part of the data, this is the probability of making the correct choice.
Theorem: Suppose that $P = P(\delta, \alpha, b)$.

$P$ displays \begin{align*}
\begin{cases}
\text{increasing} \\
\text{decreasing} \\
\text{constant}
\end{cases}
\end{align*}
accuracy iff $b$ is \begin{align*}
\begin{cases}
\text{increasing} \\
\text{decreasing} \\
\text{constant}
\end{cases}
\end{align*}

Intuition for decreasing accuracy: this is our selection effect #1

- higher bar to clear for small $t$, so if the agent stopped early, $Z$ must have been very high, so higher likelihood of making the correct choice
Accuracy in DDM models

**Theorem:** Suppose that $P = P(\mu, \alpha, b)$, with $\mu = \mathcal{N}(0, \sigma_0)$

$P$ displays
\[
\begin{cases}
\text{increasing} \\
\text{decreasing} \\
\text{constant}
\end{cases}
\]
accuracy iff $b(t) \cdot \sigma_t$ is
\[
\begin{cases}
\text{increasing} \\
\text{decreasing} \\
\text{constant}
\end{cases}
\]

where $\sigma_t^2 := \frac{1}{\sigma_0^{-2} + \alpha^{-2}t}$

**Intuition for decreasing accuracy:** this is our selection effect #2

- $\sigma_t$ is a decreasing function; this makes it an easier bar to pass
Proposition: Suppose that \( \mu = \mathcal{N}(0, \sigma_0) \), and \( b(t) \cdot \sigma_t \) non-increasing. Then \( |\delta| \) decreases in \( \tau \) in the sense of FOSD, i.e. for all \( d > 0 \) and \( 0 < t < t' \)

\[
P[|\delta| \geq d \mid \tau = t] > P[|\delta| \geq d \mid \tau = t']
\]

- larger values of \( |\delta| \) more likely when the agent decides quicker
- problem more likely to be "easy" when a quick decision is observed
- this is a selection coming from the analyst not knowing how hard the problem is
* So far, only the constant boundary $b$ was microfounded

* Do any other boundaries come from optimization?

* Which boundaries should we use?

* We now derive the optimal boundary
The “uncertain difference” model

* Assumptions:
  - Gaussian prior: $\theta^i \sim N(\mu^i_0, \sigma^2_0)$
  - Gaussian posterior: $\theta^i \sim N(\mu^i_t, \sigma^2_t)$

* Key intuition: **nonstationarity**
  - suppose that you observe $Z^l_t \approx Z^r_t$ after a long $t$
  - you think to yourself: “I must be indifferent”
  - so you have learned a lot $\Rightarrow$ you stop

* Formally $\sigma^2_t = \frac{1}{\sigma^2_0 + \alpha^2 t}$ so option value is decreasing in time

* Intuition for the difference between the two models:
  - interpretation of signal depends on the prior
The “uncertain difference” model

Theorem:

1. There is a strictly decreasing, strictly positive \( k^* : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\tau^* = \inf \{ t \geq 0 : |X^l_t - X^r_t| \geq k^*(t) \}.
\]

Moreover \( \lim_{t \to \infty} k^*_t = 0 \).

2. If \( X^l_0 = X^r_0 \), there is a strictly positive \( b^* : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\tau^* = \inf \{ t \geq 0 : |Z^l_t - Z^r_t| \geq b^*(t) \},
\]

where \( b^*(t) = \alpha^2 \sigma_t^{-2} k^*(t) \). Furthermore, we have the following bounds on the slope of \( b^* \)

\[
-b^*(t)\sigma_t^2 \leq b^*(t) \leq \frac{1}{2} b^*(t)\sigma_t^2
\]
Part 1. follows from the principle of optimality for continuous time processes and the shift invariance property of the value function, which is due to the normality of the posterior.

Part 2. describes the optimal strategy $\tau^*$ in terms of stopping regions for the signal process $Z_t := Z^l_t - Z^r_t$. This facilitates comparisons with the simple DDM, where the process of beliefs lives in a different space and is not directly comparable.
Intuitions

- $k^*$ strictly decreasing because belief updating slows down

- $k^*$ decreases all the way to 0 because otherwise the agent would have a positive subjective probability of never stopping and incurring an infinite cost
  
  - note: in the simple DDM, the agent is sure that the absolute value of the drift of the signal is bounded away from 0, so she believes she will stop in finite time with probability 1 even though the boundaries are constant
**Proposition:** The average “uncertain difference” DDM has decreasing accuracy, i.e. the probability that the agent makes the correct choice

\[ \mathbb{P} \left[ \text{sgn} \left( X_{\tau^*}^l - X_{\tau^*}^r \right) = \text{sgn} \delta \mid \tau^* = t \right] \]

decreases in \( t \).
Endogenous Attention

- The agent can choose attention levels
  \[ \beta^l_t, \beta^r_t \geq 0 \]

- Attention influences the signals \( Z^1_t, Z^2_t \)
  \[ dZ^i_t = \beta^i_t \theta^i_t dt + dB^i_t. \]

- Fixed attention budget \( \beta^l_t + \beta^r_t \leq 2 \)

- \( \beta^l_t = \beta^r_t = 1 \) leads to the same signal process as before

- \( \alpha = 1 \) for simplicity here
**Theorem:** The optimal attention strategy pays equal attention to both signals

\[ \beta_t^l = \beta_t^r = 1 \]

and thus leads to the same choice process as the exogenous attention model.
Theorem: The optimal attention strategy pays equal attention to both signals
\[ \beta^l_t = \beta^r_t = 1 \]
and thus leads to the same choice process as the exogenous attention model.

Intuition: This strategy minimizes the posterior variance of the difference in posterior means \( X^l_t - X^r_t \) at every point in time \( t \) and thus maximizes the speed of learning.
The Chernoff (1961) model

**Regret Minimization:** for any stopping time $\tau$ the objective function is

$$Ch(\tau) := \mathbb{E} \left[ -1_{\{x^l_\tau \geq x^r_\tau\}} (\theta^r - \theta^l)^+ - 1_{\{x^r_\tau > x^l_\tau\}} (\theta^l - \theta^r)^+ - c\tau \right]$$

the agent gets zero for making the correct choice and is penalized the foregone utility for making the wrong choice
The Chernoff (1961) model

**Theorem:** For any stopping time $\tau$

$$\text{Ch}(\tau) = \mathbb{E} \left[ \max\{X_{\tau}^l, X_{\tau}^r\} - c\tau \right] + \kappa,$$

where $\kappa$ is a constant independent of $\tau$; therefore, these two objective functions induce the same choice process.
The Chernoff (1961) model

**Theorem:** For any stopping time $\tau$

$$\text{Ch}(\tau) = \mathbb{E} \left[ \max\{X_{\tau}^l, X_{\tau}^r\} - c\tau \right] + \kappa,$$

where $\kappa$ is a constant independent of $\tau$; therefore, these two objective functions induce the same choice process.

**Intuition:** Subtracting the expected value of the optimal choice $\mathbb{E}[\max\{\theta^l, \theta^r\}]$, using that $\tau$ is a stopping time and applying the law of iterated expectations conditional on either choice being correct yields the result.
**Theorem:** Consider either the Certain or the Uncertain-Difference DDM. For any finite boundary $b$ and any finite set $G \subseteq \mathbb{R}_+$ there exists a cost function $d : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $b$ is optimal in the set of stopping times $T$ that stop in $G$ with probability one

$$\inf \{ t \in G : |Z_t| \geq b(t) \} \in \arg \max_{\tau \in T} \mathbb{E} \left[ \max \{ X_{\tau}^1, X_{\tau}^2 \} - d(\tau) \right].$$
• data of Krajbich, Armel and Rangel (2010)

• 39 subjects making choices between food items

• asked to refrain from eating for 3 hours before the experiment

• each subject asked to make 100 pairwise choices

• also separately elicited ratings of these items (on the scale from -10 to +10)
Fitting to a closed-form boundary

• We consider two functional forms:
  
  • \( b(t) = \frac{1}{g+ht} \) which is the approximately optimal boundary
  
  • \( b(t) = g \exp(-ht) \) used in Milosavljevic et al. (2010)

• In any case, the parameters are \((\delta, \alpha, g, h)\)
  
  • \( \delta \) is the drift; we take it to be the difference in the numerical ratings of the two items in the choice set
  
  • \( \alpha \) is the volatility of \( Z_t \)
  
  • \((g, h)\) are the parameters of the boundary
For each $({\delta}, {\alpha}, g, h)$ need to compute the joint probability density of stopping and choice (the likelihood function)

- to compute the distribution of hitting times, we used Monte Carlo simulations with 1 million random paths (this takes about a week on a cluster)
- the conditional choice probabilities as a function of stopping time are given in closed form

Then use the gradient descent algorithm to find maximum

**Findings:** for 30 out of 39 subjects the approximately optimal boundary is a better fit than the exponential boundary
Fitting to the optimal boundary

• Additional computation: the optimal boundary

• We computed this by imposing a large finite terminal time, discretizing time and space on a fine grid, and solving backwards. This computation only needs to be done for a single parameter constellation, due to a result in the online appendix, and takes less than two hours on a laptop

• Then compute the likelihood function as above

• **Findings:**
  • there is substantial heterogeneity between the subjects
  • two out of 39 subjects have a non-monotone boundary
Figure: Marginal distributions of $\alpha$, $c$, and $\sigma_0$ and the correlation between the average stopping time for each subject and their estimated cost $c$. 
Figure: Estimated optimal boundaries for different subjects.
Recap

• Observables: joint distribution over choices and decision times

• General DDM: Brownian signals and arbitrary boundary
  – characterize when earlier decisions better

• DDM derived from optimal stopping: Gaussian prior
  – allows agent to learn the choice is a toss-up
  – resulting boundary better fits the data than the constant boundary of simple DDM
  – explains why quicker choices are often better
Thank you!