Mirror symmetry for GIT quotients and their subvarieties

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Abstract

These notes are based on an invited mini-course delivered at the 2019 PIMS-Fields Summer School on Algebraic Geometry in High-Energy Physics at the University of Saskatchewan. They give an introduction to mirror constructions for Fano GIT quotients and their subvarieties, especially as relates to the Fano classification program. They are aimed at beginning graduate students. We begin with an introduction to GIT, then construct toric varieties via GIT, outlining some basic properties that can be read off the GIT data. We describe how to produce a Laurent polynomial mirror for a Fano toric complete intersection, and explain the proof in the case of $\mathbb{P}^2$. We then describe conjectural mirror constructions for some non-Abelian GIT quotients. There are no original results in these notes.

1 Introduction

The classification of Fano varieties up to deformation is a major open problem in mathematics. One motivation comes from the minimal model program, which seeks to study and classify varieties up to birational morphisms. Running the minimal model program on varieties breaks them up into fundamental building blocks. These building blocks come in three types, one of which is Fano varieties. The classification of Fano varieties would thus give insight into the broader structure of varieties.

In this lecture series, we consider smooth Fano varieties over $\mathbb{C}$. There is only one Fano variety in dimension 1: $\mathbb{P}^1$. This is quite easy to see, as the degree of a genus g curve is $2-2g$. The curve is Fano if and only if the degree is positive, so only if $g = 0$. The classification of dimension 2 Fano varieties (called *del Pezzo surfaces*) is more difficult, but is known classically. The classification in dimension 3 is due to Iskovskih [15] and Mori–Mukai [23] – it is one of the major results of 20^{th} century mathematics. One of the interesting features of this lists is that they can all be written as certain subvarieties, called *representation theoretic subvarieties*, of GIT quotients of vector spaces [5]. Moreover, the ambient GIT quotients are either *toric varieties* or *quiver flag varieties* (certain non-Abelian GIT quotients generalising type A flag varieties).

The techniques used to classify Fano threefolds do not generalise to higher dimensions. Another suggested approach is via mirror symmetry. Very roughly, mirror symmetry suggests that the symplectic geometry of X should be equivalent to the complex geometry of the mirror of X, often denoted $X^\vee$. When X is a Calabi–Yau variety, then so is its mirror. In particular, X and $X^\vee$ have mirror Hodge diamonds:

$$h^{p,q}(X) = h^{n-p,q}(X^\vee), n = \dim(X).$$

This is the origin of the the name *mirror symmetry*. For Fano varieties, the mirror is a Landau–Ginzburg model. It's expected that this Landau–Ginzburg model can be described as a family of certain Laurent polynomials, related by mutation. Another way to describe the expected picture (as conjectured precisely by Kasprzyk–Tveiten [19]) is as:

$$\{n-\text{dimensional Fano varieties up to deformation}\}$$

$$\leftrightarrow$$

$$\{\text{certain Laurent polynomials in } n-\text{variables up to mutation}\}.$$

Part of the conjectures is precisely what is meant by 'certain'.

Mirror symmetry opens another way of attack in the classification of Fano varieties by proposing to classify the mirrors of Fano varieties instead. This is the Fano classification program of Coates, Corti, Galkin, Golyshev, Kasprzyk and others.

As it happens, mirror symmetry for Fano varieties is best understood for GIT quotients of vector spaces and their subvarieties. These varieties are thus the most accessible testing ground for these conjectures and are also expected to include most (if not all) small dimensional Fano varieties (the first example of a Fano variety which is not such a subvariety is in dimension 66). So it's hoped that understanding mirror symmetry for these types of Fano varieties would be major step towards classifying all Fano varieties.

I'll start by going into slightly more detail about mirror symmetry for Fano varieties, then give an introduction to Geometric Invariant Theory (GIT), with a goal of understanding the GIT quotients in which most (and all dimension less than three) Fano varieties can be found. As mentioned, these come in two types: toric varieties and quiver flag varieties. I'll construct toric varieties as GIT quotients, and then explain the mirror statement for toric varieties, and prove it for $\mathbb{P}^2$. In the last part of these notes, I'll talk about quiver flag varieties and abelianisation, a powerful tool for understanding non-Abelian GIT quotients by reducing to the toric case.

2 Mirror symmetry for Fano varieties and Fano classification

Let X be an n-dimensional smooth complete variety over $\mathbb{C}$. You can think of it as a complex manifold if you are more comfortable with this language. Then X is *Fano* if $-K_X = \wedge^n TX$ is an ample line bundle. A line bundle L is *ample* if it has 'enough' sections in the following sense: there is some tensor power of L, say $L^{\otimes n}$ such that the map

$$i : X \to \mathbb{P}(\Gamma(X, L)^*), x \mapsto ev_x, ev_x(s) = s(x)$$

is an embedding.

In particular, we can see that all Fano varieties are projective.

Example 2.1. $\mathbb{P}^n$ is Fano, as is any hypersurface of degree d, $d < n+1$.

One of the things which makes Fano varieties extremely special is that, up to deformation, there are only finitely many n-dimensional Fano varieties.

Conjecturally, under mirror symmetry, n-dimensional Fano varieties (up to mutation) should correspond to certain Laurent polynomials in n-variables (up to mutation). We can associate to each of f and X a period, defined in each case below. A Fano variety X is mirror to a Laurent polynomial f if the *quantum period* of X is equal to the *regularised classical period* of f. Each period satisfies certain differential equations, so this can also be said as an equality of differential equations or as one period satisfying the differential equations of the other.

The classical period of f Let $f \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$. Then the period of f is

$$\pi_f(t) = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{1}{1-tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

Using the residue theorem, we can re-write this very concretely as

$$\pi_f(t) = \sum_{i=0}^{\infty} \text{Const}(f^i) x^i,$$

where $\mathrm{Const}(f^i)$ is the constant term in the expansion of f^i. The regularised classical period is

$$\pi_f(t) = \sum_{i=0}^{\infty} \frac{\mathrm{Const}(f^i)}{i!} x^i.$$

Mutations are certain birational maps $\phi : (\mathbb{C}^*)^n \dashrightarrow (\mathbb{C}^*)^n$ taking f' to f by pullback. They are constructed so as to preserve period sequences. See [1] for more details.

The quantum period of X The quantum period is much more complicated. We will this discuss in more detail in §7.2, but it is a power series built of genus 0 Gromov–Witten invariants. As a consequence, it is deformation invariant. Therefore, if we can compute it, it can be used to distinguish different Fano varieties (however, it is not known to be a complete invariant).

Example 2.2. The mirror of $\mathbb{P}^2$ is $x + y + \frac{1}{xy}$.

The conjecture has been verified in dimension 2 ([18]). In dimension 3, mirrors to all Fano varieties have been found (Appendix A of [5]). For Fano varieties that are toric complete intersections, one can compute both the quantum period and (usually) find a mirror. For subvarieties of quiver flag varieties, we can compute the quantum period and there are conjectural methods for finding mirrors in certain nice situations. As mentioned, these two cases cover all dimension less than 3 Fano varieties.

3 GIT quotients

In this section, we give a brief introduction to GIT quotients: why one might want to define them, the definition in the affine and projective case, and how one can actually compute them (the Hilbert–Mumford numerical criterion). Good references include Richard Thomas' notes on the arxiv [27], which I mostly follow, together with Newstead [24] and Proudfoot [26].

Let G be a linear algebraic group over $\mathbb{C}$ (i.e. G is a subgroup and subvariety of $\mathrm{GL}(n, \mathbb{C})$). Suppose X is a variety (either projective or affine) such that G acts on X algebraically. We want to define the quotient X/G such that X/G is a variety like X. The following example illustrates why we might not want to just do this naively.

Example 3.1. Consider $G = \mathbb{C}^*$ acting on $\mathbb{C}^2$ by scaling:

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

Note that for any $z = (z_1, z_2)$,

$$\lim_{\lambda \to 0} \lambda \cdot z = 0.$$

The closure of the orbit of z contains 0. This means that if we take the topological quotient, it won't be Hausdorff (or rather separated). If we instead take the quotient $\mathbb{C}^2 - \{0\}$, we obtain $\mathbb{P}^1$.

Some desirable properties for a construction of a quotient are the following:

- X/G is separated.
- X/G satisfies a universal property: A G-invariant map $p : X \to Y$ is a *categorical quotient* if for all G-invariant functions $f : X \to Z$, there exists a unique map $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$.
- Functions on X/G should be the G-invariant functions of X. This is roughly what it means to be an *good quotient*. Let $p : X \to Y$ be a surjective and G invariant morphism. Then p is a good quotient if the following holds:
  1. $p_*(\mathcal{O}_X)^G = \mathcal{O}_Y$.

2. If Z is closed and G-stable, then $p(Z)$ is closed. If Z_1 and Z_2 are closed, G-stable, and disjoint, then $p(Z_1) \cap p(Z_2) = \varnothing$.

- A good quotient $p : X \to Y$ for which orbits of G in X correspond to points of Y is called an *geometric quotient*.

One can show that a good quotient $p : X \to Y$ is a categorical quotient, so that these definition are increasingly strong conditions. GIT will give us a way to remove certain orbits in order to obtain a quotient which is well-behaved.

Remark 3.2. Note that the property of being a good or geometric quotient is local on the base. That is, a G-invariant morphism $p : X \to Y$ is good/geometric if and only if there is an open cover $\{U_i\}$ of Y such that each restriction $p : p^{-1}(U_i) \to U_i$ is good/geometric.

3.1 The affine case

An affine variety is the zero locus of a finite number of polynomials $p_1, \dots, p_k \in \mathbb{C}[x_1, \dots, x_n]$. The ring of functions is $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k)$, and Spec gives a correspondence between integral domains and affine varieties. Let G be a linear algebraic group acting rationally on an affine variety X (that is, through a representation $X \to \mathrm{GL}(n)$). Then G acts on $\mathcal{O}_X$, and the natural way to construct the quotient is to consider the variety corresponding to $\mathcal{O}_X^G$. But is $\mathcal{O}_X^G$ a finitely generated algebra? In general, the answer is no - that is why we consider only *reductive groups*, where it is always finitely generated.

Let G be a linear algebraic group.

1. G is *reductive* if every smooth connected unipotent normal subgroup of G is trivial. Over $\mathbb{C}$, we can equivalently say that any representation of G splits into direct sums of irreducible representations.

2. G is linearly reductive if for every rational representation V of G, and any non-zero fixed point $v \in V$, there exists a homogeneous G-invariant polynomial f of degree 1 on V such that $f(v) \neq 0$.

3. G is geometrically reductive if for every rational representation V of G, and any non-zero fixed point $v \in V$, there exists a homogeneous G-invariant polynomial f on V such that $f(v) \neq 0$.

Over $\mathbb{C}$, these 4 definitions all coincide, and the last two are more useful in GIT.

Example 3.3. The torus T is reductive, as is $\mathrm{GL}(n)$ and $SO(n)$. The additive group is not reductive.

Theorem 3.4 (Nagata's theorem)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated $\mathbb{C}$-algebra R. Then R^G is finitely generated.*

The following lemma is another very useful property of reductive groups.

Lemma 3.5. *Let G be a geometrically reductive group acting on an affine variety X. Let Z_1 and Z_2 be two closed G-invariant subsets of X such that $Z_1 \cap Z_2 = \varnothing$. Then there exists a G-invariant function $\Psi \in \mathcal{O}_X^G$ such that $\Psi(Z_1) = 1$ and $\Psi(Z_2) = 0$.*

Proof. Let $f_i \in I(Z_i)$ such that $f_1 + f_2 = 1$. The linear subspace spanned by $\{gf_2 | g \in G\}$ is G-invariant and finite dimensional (exercise, or see [24, Lemma 3.1]). Let $h_1, \dots, h_n$ be a basis of this subspace. The action of G on the h_i determine an action of G on $\mathbb{C}^n$, and the map

$$\psi : X \to \mathbb{C}^n, x \mapsto (h_1(x), \dots, h_n(x))$$

is G-equivariant by construction. Then $\psi(Z_1) = 0$ and $v = \psi(Z_2) = (1, \dots, 1)$. Then since G is geometrically reductive, there is a homogeneous $f' \in \mathbb{C}[x_1, \dots, x_n]^G$ such that $f'(v) \neq 0$, and $f(0) = 0$. The composition $f' \circ \psi$ satisfies the requirements of the lemma. □

Now we can construct good quotients for affine varieties.

Theorem 3.6. *Let X be an affine variety, G a reductive group acting on X. Then $p : X \to Y := \operatorname{Spec}(\mathcal{O}(X)^G)$ is a good quotient.*

Proof. First, we show that p is G-invariant. Suppose that there is $x \in X, g \in G$ such that $p(x) \neq p(gx)$. Then, since Y is affine, there is $f \in \mathcal{O}_Y = \mathcal{O}_X^G$ such that $f(p(x)) \neq f(p(gx))$, an obvious contradiction.

Next, we show that p is surjective. Let $y \in Y$, and let $(f_1, \dots, f_k)$ generate the maximal ideal corresponding to y. There exists a maximal ideal of $\mathcal{O}_X$ containing the ideal generated by $f_1, \dots, f_k$ (one needs to check that $f_1, \dots, f_k$ do not generate all of $\mathcal{O}_X$ - this follows from the fact that G is reductive), and the point corresponding to this ideal maps under p to y.

Next, let's show that for any open set $U \subset Y$, $\mathcal{O}_Y(U) = \mathcal{O}_X(p^{-1}(U))$. It suffices to show this for a basis of open sets; in particular, we can consider just open sets D_f for $f \in \mathcal{O}_Y$. That is, we want to show that

$$\mathcal{O}_Y(D(f)) = (\mathcal{O}_X^G)_f = ((\mathcal{O}_X)_f)^G = \mathcal{O}_X(D(f))^G,$$

which is clear.

Now let Z_1 and Z_2 be G-invariant closed subsets which are disjoint. Then by Lemma 3.5, there is a G-invariant function $\Psi \in \mathcal{O}_Y$ such that $\Psi(Z_1) = 1$ and $\Psi(Z_2) = 0$, so in particular $\overline{p(Z_1)} \cap \overline{p(Z_2)} = \varnothing$.

The final thing to show is that if Z is a closed and G-invariant subset of X, then $p(Z)$ is closed. Suppose $y \in \overline{(}p(Z)) - p(Z)$. Then Z and $p^{-1}(y)$ are both closed, and are disjoint, so we can apply the previous statement to arrive at a contradiction. □

Remark 3.7. If G is not reductive, it does not necessarily imply that $\mathcal{O}_X^G$ is not finitely generated. Considerable work has been done on extending GIT to non-reductive groups.

The quotient we have constructed separates orbits as much as possible, by which we mean:

Corollary 3.8. Suppose $p(x_1) = p(x_2)$. Then $\overline{Gx_1} \cap \overline{Gx_2} \neq \varnothing$.

Proof. If $\overline{Gx_1}$ and $\overline{Gx_2}$ were disjoint, then their image under p would also be disjoint. □

If we look back at our first example, of $\mathbb{C}^*$ acting on $\mathbb{C}^2$, this construction still doesn't give us a 'correct' answer, as $\mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}$. To produce projective quotients, we need to generalise this to the Proj construction.

4 Projective GIT

Brief aside on projective varieties There is a correspondence between pairs (X, L) where X is a projective variety and L is an ample line bundle and graded rings (without zero divisors). The graded ring associated to X is

$$R = \oplus_{r=0}^{\infty} H^0(X, L^{\otimes r}).$$

The functor the other way is the Proj functor.

Mimicking what we did in the affine case, we want to take the G-invariant part of this ring and then apply *Proj*. To get an action of G on the section of L, we take L to be *G-linearised.*

Definition 4.1. Let (X, L) be an algebraic variety, and G a linear algebraic group acting on X via $\sigma : G \times X \to X$. Then a G-linearisation of L is a lift of σ to $\overline{\sigma} : G \times L \to L$ which commutes with σ, and such that the zero section is G-invariant.

The group G acts on the sections of a G-linearised line bundle L. Alternatively, if $X = \operatorname{Proj}(R)$, then we can think of a linearisation being a lift of the G-action to R in a way that preserves the grading.

Remark 4.2. When L is very ample, it defines an embedding of X into $\mathbb{P}(H^0(X, L)^*) = \mathbb{P}^k$. A linearisation is then equivalent to saying that G acts on X via a representation $G \to \mathrm{GL}(k+1)$.

We define the GIT quotient of (X, L) by G as

$$X//G = \mathrm{Proj}(\oplus_{r=0}^{\infty} H^0(X, L^{\otimes m})^G).$$

Example 4.3. We construct $\mathbb{P}^n$ as a GIT quotient of $X = \mathbb{C}^{n+1}$ by $\mathbb{C}^*$. Let L be the trivial line bundle on X. Notice that a linearisation is precisely given by a character of $\mathbb{C}^*$. Let $\mathbb{C}^{n+1} \times \mathbb{C}$ be the total space of the dual line bundle; then for $p \in \mathbb{Z} \cong \chi(\mathbb{C}^*)$, define the action to be $(x, y) \mapsto (tx, t^{-p}y)$. Then for $f \in \Gamma(\mathbb{C}^{n+1}, L^{\otimes k}) = \mathbb{C}[x_0, \dots, x_n, y]$, the action of $t \in \mathbb{C}^*$ is given by

$$t \cdot f(x_0, \dots, x_n) = t^{-kp} f(tx_0, \dots, tx_n).$$

Note that if $p < 0$, there are no invariant sections, so $X//G = \varnothing$. If $p = 0$, we get a point. If $p > 0$, then the invariant sections of $L^{\otimes k}$ are the degree kp homogeneous polynomials of $\mathbb{C}[x_0, \dots, x_n]$. So if we take $p = 1$, we obtain $\mathbb{P}^n$ with its usual grading.

Notice that we could have instead embedded $\mathbb{C}^n \subset \mathbb{P}^{n+1}$ via $v \mapsto [v : 1]$, and then used the linearisation given by weights $(1, \dots, 1, -(n+1))$. This would have given the GIT quotient above with $p = n + 1$.

To set this up to be more like the above construction, note that $X = \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_n, y])$, where the x_i have degree 0 and t has degree 1. The action of $\mathbb{C}^*$ on $\mathbb{C}[x_0, \dots, x_n, y]$ is then given by weights 1 on the x_i and weight $-p$ on y.

The GIT quotient $X//G$ is not, strictly speaking, a quotient of X, as there isn't a natural map $X \to X//G$ (as we see in the above example, $\mathbb{P}^n$ is rather a quotient of $\mathbb{C}^{n+1} - \{0\}$). To see what orbits of X appear in the quotient, note that for some $r >> 0$, the quotient $X//G$ is the image of X under the (rational) map $X \dashrightarrow \mathbb{P}((H^0(X, L^{\otimes r})^G)^*)$ which takes $x \in X$ to ev_x. The linear map ev_x takes s to $s(x)$. This map is only defined on points where some G-invariant section of $L^{\otimes r}$ doesn't vanish. It is clearly G-invariant. This motivates the following definition:

Definition 4.4. A point $x \in X$ is *L-semi-stable* if there exists a G-invariant section s of $L^{\otimes r}$ such that $s(x) \neq 0$. A point which is not semi-stable is called *unstable*.

The set of semi-stable points is denoted $X^{ss}(L)$. This is a Zariski open subset of X. Notice that this definition is G-invariant. The set of semi-stable elements is Zariski open.

Theorem 4.5. *There is a G-invariant morphism $p : X^{ss}(L) \to X//G$ such that p is a good quotient, and $X//G$ is quasi-projective. If L is ample, then $X//G$ is projective.*

Sketch. We prove this in the case where L is very ample, so we can assume that we have $X \subset \mathbb{P}^k$, and $R = \mathbb{C}[x_0, \dots, x_k]/I = \mathcal{O}_{\hat{X}}$, where $\hat{X}$ is the cone over X. Note that R^G has no zero divisors, as R doesn't. The inclusion of algebras $R^G \to R$ induces a morphism $\tilde{p} : \mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$; the question is where this map descends to Proj; that is, when a point $\hat{x} \in \mathrm{Spec}(R) - \mathrm{Spec}(R_0)$ lifting $x \in X$ gets mapped away from the irrelevant ideal of R^G. Note that

$$x \in X^{ss}(L) \iff \exists f \in R_m^G \text{ such that } f(\hat{x}) \neq 0, m > 0 \iff \tilde{p}(\hat{x}) \notin \mathrm{Spec}(R_0^G).$$

Since the algebra morphism preserved grading, we obtain a morphism $p : X^{ss}(L) \to \mathrm{Proj}(R^G) = X//G$.

Let $Y = X//G = \mathrm{Proj}(R^G)$; then Y is also covered by subsets Y_f for $f \in R^G$, and $p^{-1}(Y_f) = X_f$. Also, $\mathcal{O}_{Y_f} = ((R^G)_f)_0$. The restriction of $p : X_f \to Y_f$ on coordinate rings is precisely given by the inclusion $(R_f^G)_0 = ((R_f)_0)^G \to (R_f)_0 = \mathcal{O}_{X_f}$. Thus by Theorem 3.6, the GIT quotient $X_f \to Y_f$ is a good quotient. Then since being a good quotient is a property local on the base, this implies that $p : X^{ss} \to X//G$ is a good quotient. □

Remark 4.6 ([26]). Going back to the example of projective space, we see that we removed the unstable locus to obtain a GIT quotient - this was exactly $\mathrm{Spec}(R_0)$, that is, the vanishing locus of the irrelevant ideal. So GIT can be interpreted as giving a geometric reason for the irrelevance of the irrelevant ideal.

The semi-stable points are the points which appear in $X//G$; however, the map may collapse more than just one orbit. We call two semi-stable orbits *GIT equivalent* if they are are collapsed to the same point under p. The main point of GIT is that we can describe this equivalence geometrically. Two semi-stable orbits are GIT equivalent if and only if

$$\overline{G.x_1} \cap \overline{G.x_2} \cap X^{ss} \neq \varnothing.$$

This follows immediately from the fact that this is a good quotient (see proof of Lemma 3.8).

Therefore, G-invariant sections of $L^{\otimes r}$ separate x from nearby orbits exactly when $G.x$ is closed in X^{ss}.

Definition 4.7. A point $x \in X^{ss}(L)$ is stable if $G.x$ is closed in X^{ss} and G_x is finite.

The stable points of X are denoted $X^s(L)$, a G-invariant and Zariski open subset of X. We have essentially proved the following statement:

Theorem 4.8. *The restriction of $p : X^{ss}(L)$, $p : X^s(L) \to X^s/G$, is a geometric quotient.*

4.1 Affine criterion for stability

The power of GIT is that one can find very computable characterizations of stability. We will first show criterion for semi-stable and stable points by looking at the action of G on the total space of L^*. For this section, let's restrict to the case where $X \subset \mathbb{P}^k$, $L = \mathcal{O}(1)$ (so that a lifted point is just a point in $\mathbb{C}^{k+1}$). Let $\hat{x}$ be a lift of $x \in X$. Suppose the closure of the orbit of $\hat{x}$ contains 0. Then the G-invariant sections of L must all vanish on x, and so x is not semi-stable. In fact, this works in both directions, and there is a similar characterisation of stability.

Theorem 4.9. *The point x is semi-stable if and only if $0 \notin \overline{G \cdot \hat{x}}$. Similarly, x is stable if and only if $G \cdot \hat{x}$ is closed in $\mathbb{C}^{k+1}$ and $\hat{x}$ has finite stabilizer.*

Proof. We have already proved one direction of the first statement. Now suppose $0 \notin \overline{G.\hat{x}}$. Then since G is a reductive group, there is a G-invariant polynomial f such that $f(0) = 0$ and $f(\overline{G.\hat{x}}) = 1$. Therefore the constant term of f is zero, and some homogeneous part of it must not vanish on $\hat{x}$. Then f gives a G-invariant section of L which doesn't vanish on x.

Now assume that x is stable. There is an invariant section of L such that $x \in X_f$, that is, f is an invariant homogeneous polynomial not vanishing on $\hat{x}$. By assumption, $G.x$ is closed in X^{ss}, and so clearly $G.x$ is closed in X_f. Let $\alpha = f(\hat{x})$, and consider the closed and G-invariant set

$$Z_\alpha = \{y \in \mathbb{C}^{k+1} Z \mid f(y) = \alpha\}.$$

Then the map $Z_\alpha \to \mathbb{P}^k_f$ is surjective and finite. The inverse image is a finite collection of orbits of the same dimension, so in particular all the orbits are closed in Z_α; in particular, $G.\hat{x}$ is closed. It clearly has finite orbit.

For the converse, suppose $G.\hat{x}$ is closed in $\mathbb{C}^{k+1}$. Then by what we have shown, we know that x is semi-stable. We can construct a surjective and finite map $Z_\alpha \to \mathbb{P}^k_f$ just as above, which maps $G.\hat{x}$ surjectively onto $G.x$. Therefore G_x is finite and $G.x$ is closed in $\mathbb{P}^k_f$; however since we can vary f this shows that $G.x$ is closed in $X^{ss}(L)$. □

Now let's apply what we have done for $G = \mathbb{C}^*$. Suppose G acts linearly on $X \subset \mathbb{P}^k$. Up to change of basis, we can assume that G acts diagonally on $\mathbb{C}^{k+1}$. Suppose it acts with weights $w_0, \ldots, w_k$: that is, for any $t \in \mathbb{C}^*$, and $\hat{x} = (x_0, \ldots, x_k) \in \mathbb{C}^{k+1}$,

$$t \cdot (x_0, \ldots, x_k) = (t^{w_0} x_0, \ldots, t^{w_k} x_k).$$

Define $\mu(x) = \max(-w_i \mid x_i \neq 0)$. Consider $\lim_{t \to 0} t^s t \cdot \hat{x}$: if s is too big, this limit goes to 0, and if s is too small, this limit does not exist. The integer $\mu(x)$ is the unique number such that the limit exists and is not equal 0.

Note that $\mu(x) > 0$ if and only if $\lim_{t\to 0} t \cdot \hat{x}$ does not exist, and $\mu(x) = 0$ if and only if the limit exists and is not zero. Now similarly define $\mu^-(x) = \max\{w_i \mid x_i \neq 0\}$. As above, $\mu^-(x) > 0$ if and only if $\lim_{t\to\infty} t \cdot \hat{x}$ does not exist, and $\mu(x) = 0$ if and only if the limit exists and is not zero.

Proposition 4.10. Let $G = \mathbb{C}^*$ act on $X \subset \mathbb{P}^k$ a projective variety. Then for all $x \in X$,

- x is semi-stable if and only if $\mu(x) \geq 0$ and $\mu^-(x) \geq 0$.
- x is stable if and only if $\mu(x) > 0$ and $\mu^-(x) > 0$.

Proof. Note that the closure of $G.\hat{x}$ is the orbit union the two limits, $\lim_{t\to 0} t \cdot \hat{x}$ and $\lim_{t\to\infty} t \cdot \hat{x}$. The claim follows from the topological criteria for stability. □

What makes this perspective powerful is the surprising fact that it suffices to test this only for all 1-parameter subgroups of G. This is called the Hilbert–Mumford numerical criterion.

5 The Hilbert–Mumford numerical criterion

Let (X, L) be a projective variety and L an ample line bundle. Let G be a reductive group acting on X, and suppose L is linearised. A 1-parameter subgroup of G (called a 1-PS) is a morphism $\lambda : \mathbb{C}^* \to G$. This defines an action of $\mathbb{C}^*$ on X. Let $\hat{x}$ be a lift of x to the total space of L^*. We can re-write the topological criteria for stability in this language as

- x is semi-stable if and only if $0 \notin \overline{G.\hat{x}}$ (here 0 means the 0 section).
- x is stable if and only if $G.\hat{x}$ is closed and $\hat{x}$ has finite stabilizer.

Let $x_0 = \lim_{t\to 0} t\cdot x$ (it exists as X is projective, if X is only quasi-projective then we need to assume that it exists). The point $x_0 \in X$ is a fixed point for the G action, so $\mathbb{C}^*$ acts on the fiber of L^* over x_0. Let $\mu_L(x, \lambda)$ be the *negative* of the integer giving the character of this action.

As before, $\lim_{t\to 0} \lambda(t)\hat{x}$ does not exist if and only if $\mu_L(x, \lambda) > 0$; it exists and is not zero if and only if $\mu_L(x, \lambda) = 0$. Therefore, one direction of the Hilbert–Mumford criterion is clear:

Theorem 5.1 (Hilbert–Mumford numerical criterion)**.** *For all $x \in X$, x is semi-stable if and only if for all 1-PS λ, $\mu_L(\lambda, x) \geq 0$. Similarly, x is stable if and only if for all 1-PS λ, $\mu_L(\lambda, x) > 0$.*

Example 5.2 (The complex Grassmannian)**.** Let $r < n$. Consider $\mathrm{GL}(r)$ acting on $V = \mathrm{Mat}(r \times n) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ by multiplication on the left. Let L be the trivial line bundle on V, and linearise with character $1 \in \mathbb{Z}$. If (A, y) is in total space of L^*, $g \cdot (A, y) = (gA, \det(g)^{-1} y)$. We claim that $A \in V$ is stable if and only if it has full rank, and it is unstable otherwise. First, suppose that $\mathrm{rank}(A) < r$, so that A is not surjective. The matrix A is semi-stable if and only if gA is for any g, so we can assume that the last row of A is zero, after multiplying by g. Then we can choose any 1-PS of G given by weights $(w_1, \ldots, w_r)$ such that $w_r < 0$, all the other $w_i > 0$, and $w_1 + \cdots + w_r < 0$. Then clearly $\lim_{t\mapsto 0} t\cdot A$ exists, and λ acts on the fiber with weight $w_1 + \cdots + w_n < 0$. Therefore A is unstable. For the converse, suppose A has full rank. First, note that we only need to consider diagonal 1-PS, as for any g, λ

$$\mu_L(x, g^{-1}\lambda g) = \mu_L(gx, \lambda).$$

Let $\lambda(t)$ be the diagonal 1-PS given by weights $w_1, \ldots, w_r$. After multiplying by an element of G, we can assume that A is in reduced row echelon form. If the limit exists, then we must have that $w_i \geq 0$ for all i. Ignoring the trivial 1-PS, we must have that $\mu_L(A, \lambda) > 0$.

Remark 5.3. To avoid the issues of when the limit exists, we could instead consider $\mathrm{SL}(r)$ acting on the projectivisation of V.

6 Toric Varieties as GIT quotients

In many cases, the stable and semi-stable elements can be computed. One family for which this is true is toric varieties. We will later see another family of examples called quiver flag varieties. A toric variety is a normal variety which contains a torus as an open dense subset, such that the action of the torus on itself extends to the whole variety. Although not all toric varieties can constructed via GIT, from many perspectives, these are the central examples. They are also the type of toric varieties where one can really view them as generalisations of projective space (we won't develop this fully here, but some examples include: homogeneous coordinates, the affine charts, the Euler sequence). We give the GIT quotient construction of a toric variety, and briefly relate it to the fan construction. For a compact description of this construction see [6].

Let $K = (\mathbb{C}^*)^r$ be a torus, and let L be the co-character lattice (the lattice of 1-PS, which is naturally isomorphic to $\mathbb{Z}^r$). Let $L^\vee$ be the dual lattice. For any lattice M, let $M_\mathbb{Q}$ denote $M \otimes_\mathbb{Z} \mathbb{Q}$.

Let $D_1, \dots, D_m \in N^\vee$ be characters of K. This defines an action of K on $V = \mathbb{C}^m$. As all vector bundles on V are trivial, a linearisation is given a character $w \in L^\vee_\mathbb{Q}$. We call w a stability condition. This then defines a GIT quotient $X_w = (\mathbb{C}^m)^{ss}(w)/K$. X_w is a toric variety: the open dense torus is the quotient $T = (\mathbb{C}^*)^m/K$. This torus is often referred to as the big torus, to distinguish it from K.

Example 6.1 ($\mathbb{P}^{m-1}$)**.** Consider $K = \mathbb{C}^*$ and $D_1 = \cdots = D_m = 1 \in \chi(K) \cong \mathbb{Z}$. It is easy to see that for $w > 0$, the set of semi-stable elements is the set of stable elements, which is $\mathbb{C}^m - \{0\}$. The GIT quotient is thus $\mathbb{P}^{m-1}$.

We can describe the semi-stable locus explicitly for a toric variety. To do so, we need the following notation:

- For $I \subset \{1, \dots, m\}$, let $\angle_I$ be the open cone over the $D_i, i \in I$.
- $\mathcal{A}_w := \{I \subset \{1, \dots, m\} | w \in \angle_I\}$.
- $U_w := \bigcup_{I \in \mathcal{A}_w} (\mathbb{C}^*)^I \oplus (\mathbb{C})^{\overline{I}}$.

Assumptions 6.2. *We will make some assumptions that will hold for the rest of these notes:*

1. *Assume that $\{1, \dots, m\} \in \mathcal{A}_w$ (otherwise the semi-stable locus will be empty).*
2. *Assume that the weights $D_1, \dots, D_m$ generate $L^\vee$ (this simplifies the statements).*
3. *Assume that the set $\{1, \dots, m\} - \{i\}$ is an element of $\mathcal{A}_w$ for all i. This assumption is definitely not necessary; however, if it isn't true, then we can write X_w as a quotient by a smaller torus (and potentially a finite group).*

The $\angle_I, I \in \mathcal{A}_w$ are called the *anti-cones.*

Theorem 6.3. $V^{ss}(w) = U_w$.

Proof. We only show one direction, and leave the other to the reader. Let λ be a 1-PS of K, i.e. $\lambda \in L$. The action of $\mathbb{C}^*$ on $\mathbb{C}^m$ is given by

$$a \cdot (z_1, \dots, z_m) = (a^{\langle \lambda, D_1 \rangle} z_1, \dots, a^{\langle \lambda, D_m \rangle} z_m).$$

The limit $\lim_{a \to 0} a \cdot z$ exists if $z_j \neq 0$ implies $\langle \lambda, D_j \rangle \geq 0$. If $z \in U_w$ then $I = \{j | z_j \neq 0\} \in \mathcal{A}_w$, so we can write $w = \sum_{i \in I} b_i D_i$ for $b_i \geq 0$. The 1-PS λ acts on all fibers of L with the character $\langle \lambda, w \rangle$, so

$$\mu_L(z, \lambda) = \langle \lambda, w \rangle = \sum_{i \in I} b_i \langle \lambda, D_i \rangle \geq 0.$$

So every element in U_w is semi-stable. □

The combinatorial data of L and $D_1, \ldots, D_m, w$ encodes a lot of information about the geometry of X_w (many of the results for fans can be translated into the language of weights). For example, X_w is smooth if and only if the set $\{D_i : i \in I\}$ contains a $\mathbb{Z}$-basis for all $I \in \mathcal{A}_w$. If the condition is weakened to assuming that every anti-cone contains a $\mathbb{Q}$-basis, then it isn't hard to show that $V^{ss} = V^s$ and the resulting quotient is orbifold (this last part is because it implies that all semi-stable points have finite stabilizers).

Not all toric varieties can be constructed as GIT quotients in this way. Considering the definition of GIT, we see that any toric variety produced in this way is projective over the affine variety $\mathrm{Spec}(R^K)$, where $R = \mathbb{C}[x_1 \ldots, x_m]$ and K acts on x_i with weight D_i. On the other hand, general toric varieties can have torus factors, for example. The GIT quotient X_w is projective precisely when $\mathbb{C}[x_1, \ldots, x_m]^K = \mathbb{C}$, which can also be characterised as follows:

Proposition 6.4 ([11])**.** X_w is projective if and only if 0 is not contained in the convex hull of the D_i.

Consider the exact sequence

$$0 \to M^\vee \xrightarrow{i} \mathbb{Z}^m \xrightarrow{\pi} L^\vee \to 0$$

where $\pi(e_i) = D_i$. The $e_1, \ldots, e_m$ are the standard basis of $\mathbb{Z}^m$, and $M^\vee$ is defined to the kernel fo π. The lattice M has rank $n := m - r$.

The fan construction For those who are familiar with the fan construction of a toric variety, the fan construction of X_w can be recovered as follows. First, dualise this sequence:

$$0 \to L \to \mathbb{Z}^m \xrightarrow{i^\vee} M \to 0.$$

Let $\rho_i = i^\vee(e_i)$. Let Σ be the fan with rays ρ_i and cone $\sigma_I \in \Sigma$ if $\overline{I} \in \mathcal{A}_w$. The toric variety associated to this fan is X_w.

Given a toric variety X_F associated to a fan F with rays $\rho_1, \ldots, \rho_m$, there's a well-known construction of X_F as a quotient of an open subset of $(\mathbb{C}^*)^m$ by a torus K. The weights $D_1, \ldots, D_m$ of the torus K giving the quotient are defined by reversing the process by which we found the rays (we can assume that the rays generate the lattice by adding extra rays if necessary). However, as mentioned above, there may not be a stability condition for which this open set is the set of semi-stable elements. The existence of such a stability condition is equivalent to the existence of a strictly convex piece-wise linear function on the support of the fan, linear on each cone.

Line bundles and divisors The sequence

$$0 \to M^\vee \to \mathbb{Z}^m \to L^\vee \to 0 \tag{1}$$

also has a very important geometric interpretation. Suppose we have another character ν of K. Consider the construction

$$L_\nu = (\mathbb{C} \times V^{ss})/K \to X_w,$$

where K acts on $\mathbb{C}$ with character ν. When is this a line bundle? If it isn't, the problem lies with semi-stable points (which could be glued together) or non-trivial stabilizers. Suppose for the moment that $V^{ss} = V^s$; that is, X_w has orbifold singularities. Then it follows from Kempf's descent lemma that L_ν is a line bundle if and only if for all $x \in V^{ss}$, $g \in K_x$, g acts trivially on the fiber $\mathbb{C}_x$. So in particular, when X_w is smooth, L_ν is always a line bundle. When X_w is an orbifold, for any ν there is some multiple of ν that produces a line bundle. More generally, a character produces a rank 1 reflexive sheaf on X_w. This construction is done by using that X_w is constructed via Proj, and using modules to construct sheafs (details can be found in [9]).

The middle lattice in the exact sequence (1) can be identified with $Div_T(X_w)$, the group of T-invariant divisors. This identification is given by associating to the standard basis element e_i the divisor in X_w given by setting the i^{th} coordinate zero in $U_w \subset \mathbb{C}^m$, that is, the divisor

$$\{z_i = 0\} \cap (\mathbb{C}^m)^{ss}(w)/K.$$

There is natural map from the lattice of T-invariant divisors to the class group (divisors up to linear equivalence), which is also the group of rank 1 reflexive sheaves. A main result in toric geometry says that this map is in fact π; the above exact sequence can be identified with the sequence

$$0 \to M^\vee \to Div_T(X_w) \to Cl(X_w) \to 0.$$

If X_w is smooth, then all divisors are Cartier and the sequence simplified to

$$0 \to M^\vee \to Div_T(X_w) \to Pic(X_w) \to 0.$$

As suggested by the above discussion, when X_w is an orbifold $Pic(X_w)$ has finite index in $Cl(X_w)$, and T-invariant Cartier divisors $CDiv_T(X_w)$ have finite index in $Div_T(X_w)$. In general, there is an exact sequence

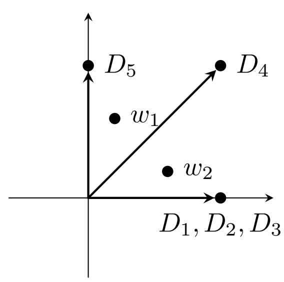
$$0 \to M^\vee \to CDiv_T(X_w) \to Pic(X_w) \to 0.$$

Variation of GIT The D_i define a wall-and-chamber decomposition of $L_{\mathbb{Q}}^\vee$. As the weight w varies, the toric variety X_w doesn't change until it crosses a wall: this is the only way that the semi-stable locus can change, as suggested by Theorem 6.3. This phenomenon is illustrated in the following example.

Example 6.5 (Blow-up of $\mathbb{P}^3$)**.** Let $K = (\mathbb{C}^*)^2$, and let the weight matrix be

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The columns of this matrix are the D_i. Drawing them as vectors in $\mathbb{R}^2$, we see that they divide the positive orthant into two chambers.



A stability condition in the lower chamber (for example, w_2) gives the blow-up of $\mathbb{P}^3$ in a point, and a stability condition chosen in the other chamber gives $\mathbb{P}^3$ (for example, w_1). Notice that w_1 does not satisfy the third assumption in 6.2, and we can write $\mathbb{P}^3$ as a quotient by a 1-dimensional torus.

The wall-and-chamber structure on $L_{\mathbb{Q}}^\vee$ given by the D_i give a wall-and-chamber structure on the cone of effective divisors on X_w, using the identification we have sketched between $L_{\mathbb{Q}}^\vee$ and $Pic(X_w)$. These chambers can be used to identify the cones of ample, moving, and effective divisors. The pseudo-effective cone (the closure of the cone of effective divisors; i.e. divisors with global section) is the positive span of the all of the weights. The ample cone is the cone containing the stability condition w, that is,

$$\bigcap_{I \in \mathcal{A}_w} \angle_I.$$

In the above example, the ample cone if X_{w_2} is the lower chamber. One can show that $-K_X = D_1 + \cdots + D_m$; it is therefore straightforward to check whether a toric variety is Fano. In the above example, as $-K_{X_{w_2}} = [4, 2]$, it is Fano.

Representation theoretic subvarietes The Fano varieties most accessible to mirror constructions (and quantum period calculations) seem to be Fano varieties that are representation theoretic subvarieties of GIT quotients of $V = \mathbb{C}^m$. A representation theoretic subvariety is a subvariety constructed as follows. Let G be a reductive group acting on V. As for toric varieties, a linearisation of G is given by a character $w \in \chi(G)$. Let E be a representation of G and suppose that

$$E_G := E \times V^{ss}(w)/G \to V^{ss}(w)/G$$

is a vector bundle (this will be the case if it is a smooth GIT quotient). If E_G is globally generated, then a generic section $s \in \Gamma(E_G)$ defines a smooth subvariety

$$Z(s) \subset V^{ss}(w)/G = X$$

which is either empty or of dimension $\dim X - \mathrm{rank}(E_G)$. The zero locus $Z(s)$ is called a representation theoretic subvariety. If $G = K$ such a subvariety is a toric complete intersection.

7 Mirror symmetry for Fano toric complete intersections

We now describe how to produce Laurent polynomial mirrors to Fano toric complete intersections. This is given by matching of the quantum period of a Fano variety to the classical period of a Laurent polynomial.

7.1 The Laurent polynomial mirror of a Fano toric variety

Let $D_1, \dots, D_m, K, w$ be the GIT data for a smooth Fano toric variety X_w. We can use the map

$$\mathbb{Z}^m \to L^\vee \to 0$$

to define a map

$$\pi : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^r$$

via

$$\pi(w_1, \dots, w_m) = (\prod_{i=1}^m w_i^{\langle D_i, e_j \rangle})_{j=1}^r).$$

Here e_j is a choice of basis for L.

Let $W = w_1 + \cdots + w_m$. Then (W, π) is a *Landau–Ginzburg model*. The full version of mirror symmetry matches the algebra of fibre-wise critical points of W with quantum cohomology and the Dubrovin connection.

The restriction of $W = w_1 + \cdots + w_m$ to the fiber of π over $(1, \dots, 1) \in (\mathbb{C}^*)^r$ is a Laurent polynomial in $\dim(X_w)$ variables. This Laurent polynomial f is what we define to be the mirror of X_w.

Example 7.1. The GIT data for $\mathbb{P}^2$ is $K = \mathbb{C}^*$, $D_1 = D_2 = D_3 = w = 1$. Therefore $\pi(w_1, w_2, w_3) = w_1 w_2 w_3$. The restriction of $W = w_1 + w_2 + w_3$ to $\pi^{-1}(1) = \{w_1 w_2 w_3 = 1\}$ is

$$f = w_1 + w_2 + \frac{1}{w_1 w_2}.$$

It requires only simple combinatorics to see that

$$\pi_f(t) = \sum_{i=0}^{\infty} \frac{(3i)!}{(i!)^3} t^{3i}.$$

The regularised classical period is thus

$$\tilde{\pi}_f(t) = \sum_{i=0}^{\infty} \frac{1}{(i!)^3} t^{3i}.$$

Now suppose Y is a Fano toric complete intersection. That is, Y is determined by

- the GIT data $D_1, \dots, D_m, K, w$ defining a smooth toric variety, and
- line bundles $L_{\alpha_1}, \dots, L_{\alpha_k}$, where $\alpha_i \in L^\vee$,

satisfying

- each L_{α_i} is a nef line bundle (in fact, this implies that they are globally generated), and
- the divisor $-K_{X_w} - \sum_{i=1}^k \alpha_i$ is in ample cone of X_w (therefore, by the adjunction formula, Y is Fano).

The Landau–Ginzburg model of Y is obtained from the Landau–Ginzburg model of X_w by adding extra restrictions. These extra restrictions are given by choosing a partition of the weights, that is a partition $S_0 \sqcup S_1 \sqcup \cdots \sqcup S_k = \{1, \dots, m\}$ such that for $i = 1, \dots, k$

$$\sum_{j \in S_i} D_j = \alpha_i.$$

The new constraints placed on the LG model π, W are, for all $i = 1, \dots, k$,

$$F_i := \sum_{j \in S_i} w_j = 1.$$

The S_0 are called the unused divisors. In certain circumstances (one of which is that the unused divisors contain a basis of $L^\vee$), we can solve these constraints in order to write down a Laurent polynomial mirror in $dim(Y) = n - k$ variables. This is called the *Przyjalkowski method.* We'll illustrate it in an example.

Example 7.2. Consider the complete intersection in $\mathbb{P}^5$ cut out by a section of $\mathcal{O}(2) \oplus \mathcal{O}(3)$. That is, the weight matrix is $[1,1,1,1,1,1]$ and the line bundles are given by the weights $[2]$ and $[3]$. If we take $S_0 = \{1\}$, $S_1 = \{2,3\}$ and $S_2 = \{4,5,6\}$ then constraint coming from the ambient space is

$$w_1 \cdots w_6 = 1$$

and the constraints coming from the line bundles are

$$w_2 + w_3 = 1, w_4 + w_5 + w_6 = 1.$$

Let's solve the first constraint by replacing w_1 with $1/(w_2 \cdots w_6)$. For the second set, introduce new variables y_1, y_2 and y_3 and set

$$w_2 = \frac{1}{1+y_1}, w_3 = \frac{y_1}{1+y_1}, w_4 = \frac{1}{1+y_2+y_3}, w_5 = \frac{y_2}{1+y_2+y_3}, w_6 = \frac{y_3}{1+y_2+y_3}.$$

We now write $W - 2$ using these substitutions, and obtain

$$\frac{(1+y_1)^2(1+y_2+y_3)^3}{y_1 y_2 y_3}.$$

This is a Laurent polynomial in 3 variables.

In general, for each S_i corresponding to a line bundle, one uses the pattern above to solve the constraint and reduce the number of variables by 1. One then uses the original constraints coming from $\pi(w) = 1$ to solve for the basis elements. The conditions that ensure this procedure outputs a Laurent polynomial (and not just a rational function) is that the partition $S_0 \sqcup \cdots S_k$ is a *convex partition*: namely, that each L_i can be written as a positive combination of the basis elements. See [7] for full details, and [3] for the relationship of Laurent polynomial to the LG model.

Let Y be a Fano toric complete intersection as above, and suppose f is a Laurent polynomial produced via the method outlined above. The mirror theorem for Fano toric complete intersections (which is a corollary of stronger mirror statements involving the full Landau–Ginzburg model) is then the following statement.

Theorem 7.3. *[13]The regularised classical period of the Laurent polynomial mirror of f is equal to the quantum period of Y.*

We haven't yet defined the quantum period. This is what we will do now, before proving the theorem for $\mathbb{P}^2$.

7.2 The quantum period of a Fano variety

Good references on Gromov–Witten invariants and quantum cohomology for Fano varieties include [4], [25], [14] and [13]. We can define it directly as a power series with coefficients given by genus 0 descendent invariants:

$$G_X(t) = \sum_{d=0}^{\infty} \sum_{\langle \beta, -K_X \rangle = d} \int_{[\overline{M}_{0,1}(X,\beta)]} ev_1^*[pt]\psi^{d-2}.$$

However, it is easier to understand the differential equations that the quantum period satisfies. These differential equations are called the *quantum differential equations*. The mirror theorem can be rephrased as the statement that the regularised classical period satisfies the quantum differential equations.

Below, we explain how to find the quantum differential equations of a smooth n-dimensional Fano variety X using quantum cohomology.

Quantum cohomology Quantum cohomology is a way of "deforming" the cup product for every $t \in H^2(X, \mathbb{C})$. Given $a, b \in H^*(X, \mathbb{C})$, to define $a \star_t b$, it suffices to define $(a \star_t b, c)$ is for all c, where $(,)$ is the intersection pairing. We set

$$(a \star_t b, c) = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{\int_\beta t} \langle a, b, c \rangle_{0,3,\beta}.$$

$\langle a, b, c \rangle_{0,3,\beta}$ is a genus 0 Gromov–Witten invariant. Very approximately, it is a count of the number of genus 0 curves in X with three marked points such that one of each of three marked points lies in the class Poincaré dual to a, b, and c. To define this properly requires a definition of the moduli space of stable maps:

$$\overline{M}_{0,3}(X, \beta)$$

which parametrizes maps $f : C \to X$ up to isomorphism where

- C is a genus 0, possibly nodal curve with three marked points x_1, x_2, x_3
- $f_*([C]) = \beta$.
- There are stability conditions ensuring that f has a finite stabilizer.

This space comes with natural evaluation maps $ev_1, ev_2, ev_3 : \overline{M}_{0,3}(X, \beta)$ mapping $ev_i(f) = f(x_i)$. This moduli space may be very poorly behaved: it may have components of different dimension. However, it is equipped with virtual fundamental class which is a cohomology class of the expected dimension $\dim X + \langle -K_X, \beta \rangle$ (although in the case of $\mathbb{P}^n$, the virtual fundamental class is not needed as the moduli space is sufficiently nice). For an introduction to the virtual fundamental class, see [8]. Given this, however, we make sense of the definition

$$\langle a, b, c \rangle_{0,3,\beta} = \int_{[\overline{M}_{0,3}(X,\beta)]^{virt}} ev_1^* a \cup ev_2^* b \cup ev_3^* c.$$

In fact, $\star_t$ defines an associative, commutative product.

Example 7.4. The quantum cohomology ring of $\mathbb{P}^2$ is

$$\mathbb{C}[H]/(H^3 - e^t).$$

This isn't hard to compute because almost all of the Gromov–Witten invariants vanish for dimensional reasons, or can be reduced to the usual pairing. We leave this as an exercise to the reader.

7.3 The Dubrovin connection

Consider the trivial vector bundle

$$H^*(X,\mathbb{C}) \times H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C}).$$

We can define a connection on it using the quantum product. Recall that a connection on a vector bundle is a way of differentiating sections. Given V a section of the tangent bundle, and s a section of this vector bundle, we define

$$\nabla_V(s)|_t = V \cdot s(t) - V \star_t s(t).$$

In fact, ∇ is a flat connection: that means $\nabla^2 = 0$. Flat sections – that is, sections such that $\nabla(s) = 0$ – satisfy differential equations (or more precisely, they define a matrix differential equation, and we can consider the associated higher order scalar differential equations). We define the quantum differential equations to be the differential equations satisfied by flat sections of $\nabla|_{\mathbb{C}\cdot -K_X}$: that is, we restrict this entire picture to the line generated by $-K_X$.

Example 7.5. Let $X = \mathbb{P}^2$. Choose generators of the cohomology given by $T_0 = 1, T_1 = 3H$, and $T_2 = 9H^2$. Let t_i be coordinates for this basis. This basis is chosen because we want to find the differential equations over the line generated by $3H = -K_X$. Let $s = (s_0, s_1, s_2)$ be a section of trivial vector bundle (so that s_i is a function of t_0, t_1, t_2). Then in this basis, the Dubrovin connection is given by

$$\nabla_{\frac{\partial}{\partial t_i}} s = \frac{\partial}{\partial t_i} s - sA_i$$

where A_i is the matrix of multiplication by T_i. By example 7.4, the matrix of multiplication by T_1 is given by

$$\begin{bmatrix} 0 & 0 & 27e^{3t_1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

If s is a flat section, we can see that

$$\frac{\partial}{\partial t_1}^3 s_0 = 27e^{3t_1} s_0.$$

Let $t = e^{t_1}$, so that $D = t\frac{d}{dt} = \frac{\partial}{\partial t_i}$. The differential equation is then $D^3 - 27t^3$.

The mirror theorem as we have stated now follows from the easy check that the regularised classical period of $x + y + 1/(xy)$ satisfies the differential equation $D^3 - 27t^3$.

The proof in general The proof of Theorem 7.3 for a Fano toric complete intersection is a corollary of a stronger theorem due to Givental [13] and Lian-Liu-Yau [21], which gives a closed formula for the generating function for more general descendent invariants of X (so that the quantum period is just once piece of it). This generating function is called the *small J-function.* It satisfies the quantum differential equations given by the Dubrovin connection when one does not restrict to the line generated by $-K_X$. The closed formula for the J-function is a hypergeometric function called the I-function. The I-function and the Landau–Ginzburg model are closely related. The closed formula allows one to write down the closed formula for the quantum period of a Fano toric complete intersection. For a Fano toric complete intersection Y given by the GIT data $T, D_1, \ldots, D_m, L_{\alpha_1}, \ldots, L_{\alpha_k}$, and satisfying the conditions of 7.1 it is

$$G_Y(t) = e^{-ct} \sum_{\beta \in H_2(X,\mathbb{Z}), \langle \beta, D_i \rangle \geq 0} \frac{t^{\langle \beta, -K_X \rangle} \prod_{i=1}^k \langle \beta, \alpha_i \rangle!}{\prod_{i=1}^m \langle \beta, D_i \rangle!}.$$

Here c is the unique rational number ensuring that the coefficient of t is 0. The mirror theorem as we have stated then follows from this closed formula. For a proof of the case of a Fano toric complete intersection, see [8].

7.4 The mirror conjectures

To a smooth Fano toric variety X, we can now associate a Laurent polynomial f such that the classical period of f computes the quantum period of X. By construction, the exponent vectors appearing in the monomials of f are the generators of the rays of the fan defining X. Let P be the *Newton polytope* of f, that is, the polytope which is the convex hull of the exponent vectors of the monomials of f. To rephrase, the spanning fan of the Newton polytope of f is the fan of the toric variety X.

If Y is a Fano toric complete intersection, then the Przyjalkowski method produces a Laurent polynomial f – in fact, one for every choice of convex partition, of which there might be many. Two natural questions arise.

1. How does the Newton polytope of a Laurent polynomial mirror relate to Y?
2. How are these different Laurent polynomial mirrors related?

In answer of the first question, Doran–Harder [12] show that there is a toric degeneration of Y to the toric variety defined by the spanning fan of the Newton polytope of a mirror f. A toric degeneration of Y to a toric variety Y' is a flat family $\pi : \mathfrak{X} \to \mathbb{C}$ such that the general fiber is Y and the special fiber is Y' (for our purposes, we also require that the family is $\mathbb{Q}$-Gorenstein). This is expected to hold more generally: if Y is any smooth Fano variety (not necessarily a toric complete intersection) with a Laurent polynomial mirror f, then conjecturally there is a toric degeneration of Y to the toric variety whose fan is the spanning fan of the the Newton polytope of f. Thus, given a Fano variety Y with a toric degeneration to a Fano toric variety, we expect to be able to find a mirror of Y by assigning coefficients to the lattice points of the polytope (that is, the polytope whose spanning fan defines Y). But how should we assign coefficients? This motivates the third question:

3. What class of Laurent polynomials are mirror to smooth Fano varieties?

Kasprzyk–Tveiten [19] conjecture answers to the last two questions. Laurent polynomial mirrors of a fixed Fano variety Y are related via *mutations*, and the class of Laurent polynomials is the class of *rigid maximally mutable* Laurent polynomials. See [1] and [16] for the precise definitions of mutations and rigid maximally mutable Laurent polynomials. Roughly speaking, however, mutations are combinatorial moves on polytopes that induce a transformation of any Laurent polynomial f to a rational function. If this rational function is in fact a Laurent polynomial, then we say that f is compatible with the mutation. Mutations preserve period sequences. A Laurent polynomial is maximally mutable if it is compatible with a maximal set of mutations of its Newton polytope, and it is rigid maximally mutable if f is the only such Laurent polynomial. We'll discuss some evidence for this conjecture that arises in the search for mirrors for Fano quiver flag zero loci.

Quiver flag zero loci can be considered the next level of complexity after toric complete intersections: while they are still GIT quotients of vector spaces, the quotienting group is no longer abelian. The sets of Fano toric complete intersections and Fano quiver flag zero loci together contain all dimension less than 3 Fano varieties, and are expected to contain at least most small dimensional Fano varieties.

8 Quiver flag varieties

Quiver flag varieties are a generalisation of type A flag varieties introduced by Craw [10] based on work of [20]. They are a family of very nicely behaved GIT quotients that are also fine moduli spaces.

A quiver flag variety $M(Q, \mathbf{r})$ is determined by a quiver Q and a dimension vector $\mathbf{r}$. The quiver Q is assumed to be finite and acyclic, with a unique source. Let $Q_0 = \{0, 1, \ldots, \rho\}$ denote the set of vertices of Q and let Q_1 denote the set of arrows. Without loss of generality, after reordering the vertices if necessary, we may assume that $0 \in Q_0$ is the unique source and that the number n_{ij} of arrows from vertex i to vertex j is zero unless $i < j$. Write $s, t : Q_1 \to Q_0$ for the source and target maps, so that an arrow $a \in Q_1$ goes from $s(a)$ to $t(a)$. The dimension vector $\mathbf{r} = (r_0, \ldots, r_\rho)$ lies in $\mathbb{N}^{\rho+1}$, and we insist that $r_0 = 1$.

8.1 Quiver flag varieties as GIT quotients.

Consider the vector space

$$\operatorname{Rep}(Q,\mathbf{r}) = \bigoplus_{a\in Q_1} \operatorname{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}})$$

and the action of $\mathrm{GL}(\mathbf{r}) := \prod_{i=0}^{\rho} \mathrm{GL}(r_i)$ on $\operatorname{Rep}(Q,\mathbf{r})$ by change of basis. The diagonal copy of $\mathrm{GL}(1)$ in $\mathrm{GL}(\mathbf{r})$ acts trivially, but the quotient $G := \mathrm{GL}(\mathbf{r})/\mathrm{GL}(1)$ acts effectively; since $r_0 = 1$, we may identify G with $\prod_{i=1}^{\rho} \mathrm{GL}(r_i)$. The quiver flag variety $M(Q,\mathbf{r})$ is the GIT quotient $\operatorname{Rep}(Q,\mathbf{r}) /\!\!/_\theta G$, where the stability condition θ is the character of G given by

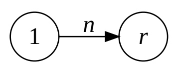
$$\theta(g) = \prod_{i=1}^{\rho} \det(g_i), \qquad\qquad g = (g_1,\dots,g_\rho) \in \prod_{i=1}^{\rho} \mathrm{GL}(r_i).$$

For the stability condition θ, all semi-stable points are stable. To identify the θ-stable points in $\operatorname{Rep}(Q,\mathbf{r})$, set $s_i = \sum_{a\in Q_1, t(a)=i} r_{s(a)}$ and write

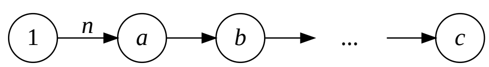
$$\operatorname{Rep}(Q,\mathbf{r}) = \bigoplus_{i=1}^{\rho} \operatorname{Hom}(\mathbb{C}^{s_i}, \mathbb{C}^{r_i}).$$

It is only slightly more difficult than the example 5.2 to show that a point $w = (w_i)_{i=1}^{\rho}$ is θ-stable if and only if w_i is surjective for all i (see [10] for the proof).

Example 8.1. Consider the quiver Q given by



so that $\rho = 1$, $n_{01} = n$, and the dimension vector $\mathbf{r} = (1,r)$. Then $\operatorname{Rep}(Q,\mathbf{r}) = \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$, and the θ-stable points are surjections $\mathbb{C}^n \to \mathbb{C}^r$. The group G acts by change of basis, and therefore $M(Q,\mathbf{r}) = \operatorname{Gr}(n,r)$, the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$. More generally, the quiver



gives the flag of quotients $\operatorname{Fl}(n,a,b,\dots,c)$.

Theorem 8.2. *$M(Q,\mathbf{r})$ is smooth and projective. It is a fine moduli space for θ-stable representations of the quiver.*

As fine moduli spaces, quiver flag varieties carry universal bundles W_i, where W_0 is the trivial line bundle. In the Grassmannian case, W_1 is the tautological quotient bundle.

As for toric varieties, elements of $\chi(G)$ define line bundles on $M(Q,\mathbf{r})$. Craw also proves that $M(Q,\mathbf{r})$ is a Mori dream space, which in particular gives an isomorphism $Pic(Q) \cong \chi(G)$ and implies that we can characterise the nef and effective cones using GIT (much as we did for toric varieties – toric varieties are also Mori dream spaces).

Higher dimensional representations of G give the representation theoretic subvarieties we are interested in. Representation theoretic subvarieties of quiver flag varieties are called *quiver flag zero loci.* We have expressed the quiver flag variety $M(Q,\mathbf{r})$ as the geometric quotient by G of the stable locus $\operatorname{Rep}(Q,\mathbf{r})^{ss} \subset \operatorname{Rep}(Q,\mathbf{r})$. A representation E of G, therefore, defines a vector bundle $E_G \to M(Q,\mathbf{r})$ with fiber E; here $E_G = E \times_G \operatorname{Rep}(Q,\mathbf{r})^{ss}$. The Fano varieties we are interested in are subvarieties of quiver flag varieties cut out by regular sections of such bundles. If E_G is globally generated, a generic section cuts out a smooth subvariety.

The representation theory of $G = \prod_{i=1}^{\rho} \mathrm{GL}(r_i)$ is well-understood, and we can use this to write down the bundles E_G explicitly. Irreducible polynomial representations of $\mathrm{GL}(r)$ are indexed by partitions (or Young diagrams) of length at most r. The irreducible representation corresponding to a partition

α is the Schur power $S^\alpha\mathbb{C}^r$ of the standard representation of $\mathrm{GL}(r)$. For example, if α is the partition (k) then $S^\alpha\mathbb{C}^r = \mathrm{Sym}^k\,\mathbb{C}^r$, the kth symmetric power, and if α is the partition $(1,1,\dots,1)$ of length k then $S^\alpha\mathbb{C}^r = \bigwedge^k\mathbb{C}^r$, the kth exterior power. Irreducible polynomial representations of G are therefore indexed by tuples $(\alpha_1,\dots,\alpha_\rho)$ of partitions, where α_i has length at most r_i. The tautological bundles on a quiver flag variety are representation theoretic: if $E = \mathbb{C}^{r_i}$ is the standard representation of the i^{th} factor of G, then $W_i = E_G$. More generally, the representation indexed by $(\alpha_1,\dots,\alpha_\rho)$ is $\bigotimes_{i=1}^\rho S^{\alpha_i}\mathbb{C}^{r_i}$, and the corresponding vector bundle on $M(Q,\mathbf{r})$ is $\bigotimes_{i=1}^\rho S^{\alpha_i}W_i$.

8.2 Abelianisation

One important tool for understanding a quiver flag varieties is to relate it to a toric variety via abelianisation. Given any GIT quotient $X//G$, one can instead consider the GIT quotient $X//T$, where T is a maximal torus of G. These two GIT quotients are of different dimensions, and on the face of it may seem quite unrelated - however, it turns out, you can learn a lot about $X//G$ from $X//T$. If $X = V$ is a vector space, then $V//T$ is a toric variety, and as we have seen, many properties of toric varieties can be read off from the associated GIT data. For example, the cohomology of a toric variety can be easily described (see for example [9]), and one of the first results relating $X//T$ and $X//G$ is a theorem of Martin [22] describing the cohomology ring of $X//G$ from that of $X//T$.

Remark 8.3. We have already seen that associating a toric variety Y' to a variety Y via toric degeneration can be a powerful tool. Abelianisation is another way of associating a toric variety in the special case that Y is a GIT quotient of a vector space. Toric degenerations are in general not particularly easy to work with – abelianisation however is easily described.

We now describe abelianisation carefully when $X//G$ is a quiver flag variety. Let $T \subset G$ be the diagonal maximal torus. Then the action of G on $\mathrm{Rep}(Q,\mathbf{r})$ induces an action of T on $\mathrm{Rep}(Q,\mathbf{r})$, and the inclusion $i : \chi(G) \hookrightarrow \chi(T)$ allows us to interpret the special character θ as a stability condition for the action of T on $\mathrm{Rep}(Q,\mathbf{r})$. The Abelian quotient is then $\mathrm{Rep}(Q,\mathbf{r})/\!\!/_{i(\theta)}T$.

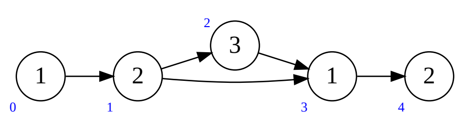
Remark 8.4.

In fact, $\mathrm{Rep}(Q,\mathbf{r})/\!\!/_\theta T$ is a toric quiver flag variety. The quiver is Q^{ab} with vertices

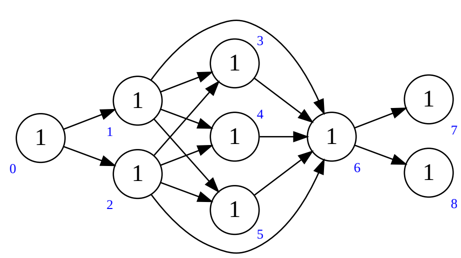
$$Q_0^{ab} = \{v_{ij} : 0 \le i \le \rho, 1 \le j \le r_i\}$$

and the number of arrows between v_{ij} and v_{kl} is the number of arrows in the original quiver between vertices i and k. We call Q^{ab} the *Abelianized* quiver.

Example 8.5. Let Q be the quiver



Then Q^{ab} is



The inclusion of character groups $\chi(G) \to \chi(T)^W$ induces an inclusion of Picard groups. Here W is the Weyl group, in this case $\prod_{i=1}^\rho Sym_{r_i}$.

Proposition 8.6 ([17]). The map $i : \mathrm{Pic}(M_Q) \to \mathrm{Pic}(M_{Q^{ab}})$ induces an inclusion of the effective cones. The wall-and-chamber decomposition of Q is just the one restricted from Q^{ab}. The ample cone of M_Q is the Weyl invariant part of the ample cone of $M_{Q^{ab}}$.

Ciocan-Fontanine–Kim–Sabbah conjectured a relationship between the J-functions and quantum cohomology rings of $X//G$ and $X//T$, for a general X, which they proved for flag varieties [2]. The conjecture also holds for quiver flag varieties [17] and for general GIT quotients of vector spaces [28]. Restricting to the quantum period, we obtain the following theorem.

Theorem 8.7 (The Abelian/non-Abelian correspondence). *The quantum period of a Fano quiver flag zero locus can be computed from the associated Fano toric complete intersection.*

For the precise statement, one needs to use the J-function. As for toric complete intersections, this allows us to explicitly compute terms of the quantum period. However, it does not give a closed formula for the quantum period. There also is no known way of moving from the abelianisation to a Laurent polynomial mirror to the quiver flag zero locus.

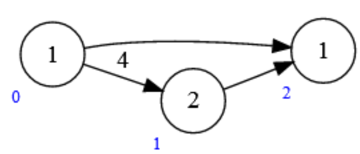
Using this theorem, with Coates and Kasprzyk, we completed an exhaustive computer search for all Fano fourfolds which are quiver flag zero loci in codimension at most four. We found 141 new Fano varieties.

8.3 Finding mirrors to Fano quiver flag zero loci

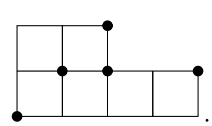
As mentioned, there is no known way to use abelianisation to find a Laurent polynomial mirror to a quiver flag zero locus. Instead, we must use toric degenerations. Toric degenerations are not easy to produce in general. However, for quiver flag varieties, this is made easier via their description as a GIT quotient. In [16], I describe a toric degeneration of a family of quiver flag varieties called Y-shaped quiver flag varieties. The degeneration is to a special type of toric variety: a toric variety which is also a canonical quiver variety (a quiver variety where the stability condition is the anti-canonical class). The representation theoretic bundles E_G degenerate in some sense to direct sums of rank 1 reflexive sheaves (Weil divisors). If a convex partition exists, one can then formally apply the methods for toric complete intersections described to find a Laurent polynomial.

Of the 141 four dimensional Fano quiver flag varieties mentioned above, the method works for 99 of them [16]. The Laurent polynomials produced, however, are often *not* rigid maximally mutable but also not mirrors to the Fano variety. If, however, we adjust the coefficients of the Laurent polynomial (i.e. keep the same Newton polytope) so that the Laurent polynomial is rigid maximally mutable, then in all examples the resulting Laurent polynomial has the correct period sequence, up to 20 terms. This provides significant evidence towards rigid maximally mutable Laurent polynomials being the correct class to consider, as it precisely allows us to write down the correct coefficients once given a polytope. We conclude with an example of this in action.

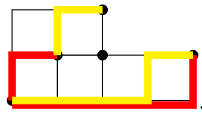
Example 8.8. Consider the quiver flag zero locus X given by the quiver



with bundle $W_1 \otimes W_2$. The diagram below describes a quiver by taking vertices to be the black dots, and arrows to be paths going up and right between vertices not passing through any other vertices. The associated canonical toric quiver variety is a toric degeneration of the ambient quiver flag variety.



Each arrow in the diagram describes Weil divisor on the toric degeneration; a collection of arrows corresponds to the sum of the divisors associated to each arrow in the collection. The bundle $W_1 \otimes W_2$ corresponds to the two divisors associated to the red collection and the yellow collection below.



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Using the Przyjalkowski method of [7], we can produce a Laurent polynomial from this data, which has Newton polytope P. The following rigid maximally mutable Laurent polynomial f also has Newton polytope P:

$$x + yw + y + z + w + 1/x + 1/(xw) + 1/(xz) + z/(xyw) + 1/(xy) + 1/(xyw) + 1/(xyz).$$

Up to the first twenty terms, the period sequence of f matches with the period sequence of the quiver flag zero locus X, which means that this Laurent polynomial is almost certainly a mirror of X.

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