1 Definitions

(25 points)

Exercise 1.1. Give precise definitions of the following

1. the dual of a vector space (5 points)
2. the dual of a linear map (5 points)
3. an exact sequence of vector spaces (5 points)
4. the eigenvector of a linear map (5 points)
5. the dimension of a vector space (5 points)

2 Computations

For these computations, you do not have to justify your answer. (5 points each) (20 points)

Exercise 2.1. Find the eigenvalues of the linear map

$$M = \begin{bmatrix} 1 & 114 & 51 & 25 & 11 \\ 0 & 2 & 141 & 11 & 12 \\ 0 & 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 4 & 13 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Exercise 2.2. Compute the eigenvalues of

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

over the complex numbers

Exercise 2.3. Consider the linear map

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

compute its kernel

Exercise 2.4. Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space and consider complex conjugation:

$$a + ib \mapsto a - ib,$$

as a linear map. Write down the matrix corresponding to this linear transformation of $\mathbb{R}$-vector spaces.

3 True or False

True or false. Explain in one line. (3 points + 2 for explanation) (20 points)

Exercise 3.1. 1. If $\lambda \in k \setminus \{0\}$, then the generalized eigenspace of λ is non-zero if and only if the eigenspace of λ is non-zero.

2. If V is an n-dimensional vector space, then the generalized eigenspace of λ is always n-dimensional.

3. Let $f : V \to W$ be a linear map between finite dimensional vector space. Then the dimension of $\mathrm{Im}(f)$ and $\mathrm{Im}(f^\vee)$ are equal.

4. Every linear map over a finite field always has a nonzero eigenvector.

4 Problems

(35 points)

Exercise 4.1. (15 points) Let k be a field and let V be a finite dimensional vector space with subspaces $U, W \subset V$. Prove that the sequence

$$0 \to U \cap W \to U \oplus W \xrightarrow{(u,w)\mapsto u-w} U + W \to 0,$$

is always exact.

Exercise 4.2. Let V be a vector space over a field k and suppose that $f^n = 0$ and $\dim \ker(f) = 1$.

1. Let $V_k := \ker(f^k)$. Prove that there is a flag

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V.$$

(5 points)

2. Prove that $\dim V_k = k$ for $k = 0, 1, \cdots, n$. (Hint: prove first that $\dim V_{k+1} \leqslant \dim V_k + 1$; also note that the statement is true for $k = 1$ by assumption) (10 points)

3. Conclude that we have a short exact sequence

$$0 \to \ker(f) \to V_k \xrightarrow{f} V_{k-1} \to 0,$$

for $k = 1, \cdots, n$. (5 points)

4. Prove that there is a basis for V such that the matrix of f has 1's above the diagonal and zero everywhere else. (5 points)