Problem 1:
Let $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the half-plane and define the Riemannian metric on $H$ by

$$g_0 = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$ 

(This is usually called the hyperbolic metric. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and let $z = x + iy \in \mathbb{C} = \mathbb{R}^2$. Define the map $f_0 : D \rightarrow H$ by

$$f_0(z) = i \cdot \frac{1 - z}{1 + z}.$$ 

1. Prove $f_0$ is a diffeomorphism between $D$ and $H$.

2. Suppose $f : M \rightarrow N$ is a smooth map between manifolds and $g$ is a metric on $N$. Suppose $f(p_0) = q_0$. Let $U \subset M$ be a chart of $p_0$ with $\phi_U : U \rightarrow \mathbb{R}^m$ and let $V \subset N$ be a chart of $q_0$ with $\phi_V : V \rightarrow \mathbb{R}^n$. In the chart $V$, let

$$g = \phi_U^*(g_{ij} \cdot dx^i \otimes dx^j),$$ 

where $g_{ij} : V \rightarrow \mathbb{R}$ is a smooth map. Compute the pull-back metric $f^*g$ on the chart $f^{-1}(V) \cap U$.

3. Compute the pull-back metric $f_0^*g_0$ on $D$.

Problem 2: Let $T \subset \mathbb{R}^3$ be obtained by rotating the circle

$$\{(x, y, z) \mid y = 0, z^2 + (x - 2)^2 = 1\}$$

about the $z$-axis. Let $T$ be parameterized by the coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$

$$f(\theta, \phi) = ((2 + \sin \phi) \cos \theta, (2 + \sin \phi) \sin \theta, \cos \phi).$$

Define the Riemannian metric $g_0$ on $T$ by the induced metric from $\mathbb{R}^3$. Define the metric $g$ on $S^1 \times S^1$ by the pull-back metric $f^*g_0$.

1. Prove that $f$ is $1-1$ and onto as a map from $S^1 \times S^1$ to $T$ such that its differential as map into $\mathbb{R}^3$ is injective. (This implies $f$ is a diffeomorphism.)

2. Compute the inner product $g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}), g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}), g(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$, and the pull-back metric $g$.

Problem 3: Let $U \subset M$ be a chart with $\phi_U : U \rightarrow \mathbb{R}^n$. Suppose $\gamma : I = [0, 1] \rightarrow M$ is a smooth map (called a path). Let

$$\phi_U \circ \gamma : \gamma^{-1}(U) \subset I \rightarrow \mathbb{R}^n$$

be denoted by

$$(\gamma^1(t), \ldots, \gamma^n(t))$$

and let

$$\dot{\gamma}^i = \frac{d\gamma^i}{dt}, \ddot{\gamma}^i = \frac{d^2\gamma^i}{dt^2}.$$ 

The geodesic equation is

$$\ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0,$$

where the indice that appear twice should be taken a sum on $1, \ldots, n$, i.e. $\Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = \sum_{j,k=1}^n \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k$. Note that the Christoffel symbol

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}),$$
where \( \{g^{il}\}_{i,l \in [1,n]} \) is the inverse matrix of \( \{g_{il}\}_{i,l \in [1,n]} \).

Prove that the solution of the geodesic equation is independent of the chart, i.e. given another chart \( V \subset M \) with \( U \cap V \neq 0 \), the path satisfies the geodesic equation on \( U \) iff it satisfies the geodesic equation on \( V \).

**Problem 4:** Let \( S^{n-1} \) be the sphere in \( \mathbb{R}^n \) with radius 1. Let \( g_S \) be the induced metric from the standard metric on \( \mathbb{R}^3 \). (This is called the round metric). Let \( \gamma : I \to S^{n-1} \to \mathbb{R}^n \) be denoted by

\[
(x^1(t), \ldots, x^n(t)).
\]

1. Compute \( g_S \) and the Christoffel symbol \( \Gamma^i_{jk} \).
2. Show that the geodesic equation is

\[
\dddot{x}^i + x^i |\dddot{x}|^2 = 0.
\]

**Problem 5:** Read Appendix 8.1 in Cliff’s book about the vector field theorem. You don’t need to write down anything for this problem.