This final exam is due at 11:59 pm on Monday December 5, 2022.

We always suppose \( M \) is a smooth manifold.

**Problem 1:** Suppose \( f : M \to \mathbb{R} \) is a smooth function. Describe the definition of \( df \) and prove it is well-defined. (Hint: first define it locally on a chart and use partition of unity to define globally, then prove it is independent of the chart.)

**Problem 2:** Let \( T \subset \mathbb{R}^3 \) be obtained by rotating the circle 
\[
\{(x, y, z) \mid y = 0, z^2 + (x - 2)^2 = 1\}
\]
about the \( z \)-axis. Let \( T \) be parameterized by the coordinates \((\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]\)
\[
f(\theta, \phi) = ((2 + \sin \phi) \cos \theta, (2 + \sin \phi) \sin \theta, \cos \phi).
\]
Define the Riemannian metric \( g_0 \) on \( T \) by the induced metric from \( \mathbb{R}^3 \). Define the metric \( g \) on \( S_1 \times S_1 \) by the pull-back metric \( f^*g_0 \).

1. Compute the inner product \( g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}), g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}), g(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) \).

2. Let \( x^1 = \theta, x^2 = \phi \). Compute the Christoffel symbol
\[
\Gamma^i_{jk} = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}),
\]
for all \( i, j, k \in \{1, 2\} \), where \( \{g^{il}\}_{i, l \in \{1, n\}} \) is the inverse matrix of \( \{g_{il}\}_{i, l \in \{1, 2\}} \).

3. Find the geodesic from \( p = (1, 0, 0) \) to \( q = (3, 0, 0) \) in \( T \).

**Problem 3:** Suppose \( B^n \subset \mathbb{R}^n \) is the unit ball and \( S^{n-1} \) is its boundary. Prove the isomorphism between de Rham cohomology
\[
H^k_{dR}(B^n \setminus \{0\}) \cong H^k_{dR}(S^{n-1}).
\]

**Problem 4:** Let \( E \) be a vector bundle over \( M \) and let \( C^\infty(M; E) \) be the space of smooth sections \( s : M \to E \).

1. Describe the definition of covariant derivative \( \nabla \) on \( E \) and prove the space of covariant derivatives on \( E \) is an affine space over \( C^\infty(\text{End}(E) \otimes T^*M) \).

2. When \( E = M \times \mathbb{R}^n \) and \( \nabla = \nabla^0 + \alpha \) with \( \alpha \in C^\infty(\text{End}(E) \otimes T^*M) \), describe the definition of the induced connection \( \nabla^* \) on the dual vector bundle \( E^* \). If we write \( \nabla^* = \nabla^{0*} + \beta \) with \( \beta \in C^\infty(\text{End}(E^*) \otimes T^*M) \), describe and prove the relation between \( \alpha \) and \( \beta \).

**Problem 5:** Construct a connection \( A \) on the product principal bundle \( P = \mathbb{R} \times G \) over \( \mathbb{R} \), where \( G \) is a Lie group. To do this, you need to describe a \( g \)-valued 1-form at each point \( (t, g) \in \mathbb{R} \times G \). Compute the curvature \( F_A \) of the connection \( A \).