Class 16 Metric compatible covariant derivative (Chap 15)

In weeks 7–8, we study

1. the exterior derivative

\[ d : \Omega^k \rightarrow \Omega^{k+1} \], where \( \Omega^k = C^\infty(M; \Lambda^k T^*M) \)

= \{ sections in \( \Lambda^k T^*M \) \}.

2. the covariant derivative \( \nabla : C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M) \)

3. the exterior covariant derivative

\[ d\nabla : C^\infty(M; E \otimes \Lambda^k T^*M) \rightarrow C^\infty(M; E \otimes \Lambda^{k+1} T^*M) \]

and

4. the curvature \( F_\nabla \) of \( d\nabla \) as a section of \( \text{End}(E) \otimes T^*M \)

In weeks 5–6, we study metric \( g \) on \( E \), which is

a section of \( E^* \otimes E^* \) s.t. at each fiber, it is

symmetric and positive definite.

Using partition of unity, we show the metric and

the covariant derivative exist.

(the space of the metric is convex hence contractible

and the space of covariant derivative is affine

over \( C^\infty(M; \text{End}(E) \otimes T^*M) \))

In this week, we study metric compatible

covariant derivative, i.e., for any sections \( U, V \) of \( E \),

\[ d g(U, V) = g(\nabla U, V) + g(U, \nabla V) \]

(some version of Leibniz rule)
Given \( u, v \in C^\infty(M; \mathbb{R}) \), \( g(u, v) \) is a map \( M \to \mathbb{R} \).

\( dg(u, v) \) is a 1-form (we apply exterior derivative)

\( \nabla u \in C^\infty(M; \mathfrak{e} \otimes T^\ast M), \nabla u \otimes v \in C^\infty(M; \mathfrak{e} \otimes T^\ast M \otimes \mathfrak{e}) \)

\( g(\nabla u, v) \) means we apply \( g \) to two \( \mathfrak{e} \) components

and hence also obtain a section in \( T^\ast M \), i.e., a 1-form.

Following the notation in previous class, we write everything on chart \( U \) and omit subscript \( U \) for \( s_U, \alpha_U \)

Let \( e_1, \ldots, e_n \) be a basis of orthonormal sections

of \( \mathfrak{e} \), where orthonormal means \( g(e_i(p), e_j(p)) = \delta_{ij} \)

It exists by Gram-Schmidt procedure.

Then \( S = \sum_{k=1}^n S_k e_k \)

\[ S_k : U \to \mathbb{R} \]

(In weeks 5-6, we write \( s_1, \ldots, s_n \) for a basis of sections and \( V_1, \ldots, V_k \) as functions \( U \to \mathbb{R} \)

Now the notation follows from those in weeks 7-8)

\[ \nabla S = \sum_{ij} \left( ds_i + \alpha_i^j \right) S_j e_i \]

\( \alpha_i^j \) is component of \( \alpha \) in previous class, \( \alpha_i^j \) is 1-form.

especially \( \nabla e_i = \sum \alpha_i^j e_j \)

\( = \alpha_i^j e_j \) (write \( \alpha_i^j \) rather than \( \alpha_i^j \)

because we want to omit \( \sum \) later

when an index appears twice, we take sum over this index.)
From \( dg(u,v) = g(\nabla u, v) + g(u, \nabla v) \)
we have \( 0 = dg(e_i, e_j) = g(\nabla e_i, e_j) + g(e_i, \nabla e_j) \)
\[ = g(\alpha_i^k e_k, e_j) + g(e_i, \alpha_j^l e_l) \]
\[ = \alpha_i^j + \alpha_j^i \]

i.e. the matrix \( \alpha \) is the 1-form \( \alpha = \{ \alpha^i_j \}_{i,j} \)

satisfies \( \alpha^T + \alpha = 0 \)

Given \( \nabla \) on \( E \), we can define \( \nabla \) on \( E^* \) as follows.
Let \( S \) be a section of \( E \) and \( S^* \) be a section of \( E^* \), we have a pairing \( \langle S, S^* \rangle : M \to \mathbb{R} \)
\( d\langle S, S^* \rangle \) is a 1-form.

Then we define \( \nabla S^* \) by the following equation:
\( d\langle S, S^* \rangle = \langle \nabla S, S^* \rangle + \langle S, \nabla S^* \rangle \) \hspace{1cm} (1)

(this equation holds for any \( S \) and fixed \( S^* \))

We can also extend \( \nabla \) to \( E^* \otimes E^* \) by
\( \nabla (S^1 \otimes S^2) = \nabla S^1 \otimes S^2 + S^1 \otimes \nabla S^2 \) \hspace{1cm} (2)

Then we can define \( \nabla g \in C^\infty(M, E^* \otimes E^* \otimes T^*M) \)

Lem: \( \nabla \) is metric compatible iff \( \nabla g = 0 \)
Pf: To define $\nabla g$, we can first define $\nabla$ on $\text{E} \otimes \text{E}$ by
\[ \nabla(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2 \tag{3} \]
and then define $\nabla$ on $(\text{E} \otimes \text{E})^* = \text{E}^* \otimes \text{E}^*$ by defining $\nabla(s_1^* \otimes s_2^*)$,
\[ d <s_1 \otimes s_2, s_1^* \otimes s_2^*> = <\nabla(s_1 \otimes s_2), s_1^* \otimes s_2^*> + <s_1 \otimes s_2, \nabla s_1^* \otimes s_2^*> \tag{4} \]
for any $s_1 \otimes s_2$ and then extend linearly over $\mathbb{R}$.

Note that when multiply with a map $f: M \to \mathbb{R}$, we define
\[ \nabla f(s_1^* \otimes s_2^*) = (s_1^* \otimes s_2^*)f + f \nabla (s_1^* \otimes s_2^*) \]
\[ = fs_1^* \otimes fs_2^* + f \nabla (s_1^* \otimes s_2^*) \]

We need to check these two definitions are the same i.e. (1-3) implies (4) and (1, 3, 4) $\Rightarrow$ (2).

Use the second definition, we have
\[ dg(u, v) = d <u \otimes v, g> \]
\[ = <\nabla(u \otimes v), g> + <u \otimes v, \nabla g> \]
\[ = <\nabla u \otimes v + u \otimes \nabla v, g> + <u \otimes v, \nabla g> \]
\[ = g(\nabla u, v) + g(u, \nabla v) + <u \otimes v, \nabla g> \]

Then we consider the case $E = TM$, $E^* = T^* M$.

Define $\Lambda$ to be the antisymmetrization.
from $C^\infty(M; \Lambda^1 T^*M) \to C^\infty(M; \Lambda^2 T^*M)$

by $A(w, \delta w_2) = \frac{1}{2} (w, \delta w_2 - \omega_2 \delta w_1)$

and extending, linearly.

For a 1-form $\omega \in C^\infty(M; T^*M)$
we can define $d\omega \in C^\infty(M; \Lambda^2 T^*M)$
and $A(\nabla \omega) = A(d\omega) \in C^\infty(M; \Lambda^2 T^*M)$

(Recall previously we define $d\gamma$ to be a map
$C^\infty(M; E \otimes \Lambda^k T^*M) \to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$

In the special case $E = T^*M$ we compose it with $A$

Cliff still uses the same notation, but we write as $A \nabla$

**Def** The **torsion tensor** $T_\nabla : C^\infty(M; T^*M) \to C^\infty(M; \Lambda^2 T^*M)$
is defined by $T_\nabla \omega = A(\delta \omega) - d\omega$

$\nabla$ (on $T^*M$) is called **torsion free** if $T_\nabla \omega = 0$, i.e.

$A(\nabla \omega) = d\omega$

**Thm.** Given a Riemannian metric $g$ on $TM$, there is
a unique metric compatible, torsion free covariant derivative

$\nabla$ called the **Levi-Civita connection**
Class 17 Levi-Civita connection

Given a Riemannian metric $g$ on $TM$, a covariant derivative $\nabla$ on $TM$ is compatible with $g$ if either
1) $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$ for any vector fields $u, v$

or 2) $\nabla g = 0$ for the covariant derivative on $T^*M \otimes T^*M$ induced by $\nabla (s^1 \otimes s^2) = \nabla s^1 \otimes s^2 + s^1 \otimes \nabla s^2$

$d<s, s^*> = \langle \nabla s, s^*> + \langle s, \nabla s^*>$

$\nabla$ is called torsion-free if the torsion tensor

$T_\nabla = A \nabla - d: C^\infty(M; T^*M) \to C^\infty(M; \Lambda^2 T^*M)$ vanishes

where $A(w_1 \otimes w_2) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1)$

Thm. Given a Riemannian metric $g$ on $TM$, there is a unique metric compatible, torsion free covariant derivative $\nabla$ called the Levi-Civita connection.

Here we don't distinguish $\nabla$ on $TM$ or $T^*M$ because they can induce each other.

Pf. 1 (in the next pages, using orthonormal basis)
In the last class, we compute in a locally orthonormal basis $e_1, \ldots, e_n$. Suppose
\[ \nabla e_i = \sum_j e_j \otimes \hat{\alpha}_i^j = \sum_j \alpha_i^j e_j = \alpha_i^j e_j \] for short.

$\alpha^j_i$ 1-forms, $\nabla$ is compatible with $g$ iff $\alpha^j_i = -\alpha^i_j$ (1)

Consider the dual basis $e^i = e_i^*$
\[ \langle e_i, e^j \rangle = \delta_{ij} : M \to \mathbb{R} \] Suppose
\[ \nabla e^i = e^j \otimes \beta^i_j = e^j \otimes \beta^i_{jk} e^k = \beta^i_{jk} e^j \otimes e^k \]

$\beta^i_j$ 1-form, $\beta^i_{jk} : M \to \mathbb{R}$

0 = $d \langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla e^j \rangle$
= $\langle \alpha_i^k e_k, e^j \rangle + \langle e_i, e^j \otimes \beta^i_j \rangle$
= $\alpha_i^j + \beta^i_j$

$\Rightarrow \nabla$ is compatible with $g$ iff $\beta^i_j = -\beta^i_j$ (1')

$\nabla e^i = \Lambda (\beta^i_{jk} e^j \otimes e^k)$
= $\beta^i_{jk} \left( \frac{1}{2} (e^j \otimes e^k - e^k \otimes e^j) \right)$
= $\frac{1}{2} (\beta^i_{jk} - \beta^i_{kj}) e^j \otimes e^k$
Suppose \( \text{de}^i = \sum_{j<k} \gamma^{i}_{jk} e^j \wedge e^k \)

\[
= \sum_{j<k} \gamma^{i}_{jk} \frac{1}{2} (e^j \wedge e^k - e^k \wedge e^j) = \sum_{j<k} \gamma^{i}_{jk} e^j \wedge e^k
\]

where we set \( \gamma^{i}_{jk} = -\gamma^{i}_{kj} \) for \( j \neq k \), \( \gamma^{i}_{jj} = 0 \)

Then \( \nabla \text{de}^i = A \nabla e^i - \text{de}^i = \frac{1}{2} (\beta^{i}_{jk} - \beta^{i}_{kj} - \gamma^{i}_{jk}) e^j \wedge e^k \)

\( \nabla \) is torsion free iff \( \beta^{i}_{jk} - \beta^{i}_{kj} - \gamma^{i}_{jk} = 0 \) (2)

Change indices, we have \( \beta^{j}_{ki} - \beta^{j}_{ik} - \gamma^{j}_{ki} = 0 \) (2')

\( \beta^{k}_{ij} - \beta^{k}_{ji} - \gamma^{k}_{ij} = 0 \) (2'')

Also from (1'), we have \( \beta^{i}_{ik} = -\beta^{i}_{jk} \)

(2') + (2'') - (2'')

\[
\Rightarrow \beta^{i}_{jk} = \frac{1}{2} (\gamma^{i}_{jk} + \gamma^{j}_{ki} - \gamma^{k}_{ij})
\]

If there is another solution \( (\beta^{1})_{jk} \), suppose

\[
\eta^{i}_{jk} = \beta^{i}_{jk} - (\beta^{1})_{jk}, \text{ then } (2) \Rightarrow \eta^{i}_{jk} = \eta^{i}_{kj}
\]

we have

(1') \Rightarrow \eta^{j}_{ik} = -\eta^{j}_{jk}

\[
\eta^{j}_{ik} = -\eta^{j}_{jk} = -\eta^{j}_{kj} = \eta^{j}_{ij} = \eta^{j}_{ij} = \eta^{j}_{ij} = -\eta^{j}_{ik} = -\eta^{j}_{ik}
\]

\[
\Rightarrow \eta^{j}_{ik} = 0, \text{ so the solution is unique.}
\]
In the next two pages

Pf 2 (Using coordinate basis $\frac{\partial}{\partial x^i}$ and $dx^i$.)

Note that we don’t have $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}$ in this case.

But $d(dx^i) = 0$, so we don’t have $\Gamma^i_{jk}$ terms.

Suppose $\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k$

(We use $\Gamma^j_{ik}$ because we will show this is indeed the Christoffel symbol)

$\Gamma^j_{ik} = \frac{1}{2} g^{il} \left( \partial_k g_{lj} + \partial_l g_{jk} - \partial_j g_{lk} \right)$

From the pairing $\nabla \left( \frac{\partial}{\partial x^i}, dx^j \right)$, we have

$\nabla dx^i = -\Gamma^j_{ik} dx^j \otimes dx^k$

$\nabla \nabla = 0 \Rightarrow \Gamma^2_{jk} = \Gamma^2_{kj}$

Then we need to show $\nabla$ is compatible with $g$

Suppose $g = g_{ij} dx^i \otimes dx^j$, $g_{ij} = g_{ji} : U \to \mathbb{R}$

$\{g_{ij}\} \Rightarrow (g_{ij})^T \quad 0 = \nabla g = \nabla (g_{ij} dx^i \otimes dx^j)$

$= (\nabla g_{ij}) dx^i \otimes dx^j + g_{ij} (\nabla dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\nabla dx^j)$

For simplicity, we can consider $\nabla g = \langle \frac{\partial}{\partial x^i}, \nabla g \rangle$

Note that $\nabla_i g_{ij} = \frac{\partial g_{ij}}{\partial x^i} = \partial_i g_{ij}$

$\nabla_i dx^i = -\Gamma^k_{ik} dx^k$
\[ 0 = \partial_k g_{ij} \, dx^i \otimes dx^j - g_{ij} \left( \Gamma^k_{ip} \, dx^p \otimes dx^i + \Gamma^i_{jq} \, dx^q \otimes dx^i \right) \]

We can change indices and solve \( \Gamma^i_{jk} \) as in the first proof, and also prove the uniqueness similarly. Here for simplicity I just show the Christoffel symbol satisfies the equation.

Recall \( \Gamma^j_{ik} = \frac{1}{2} g^{jl} \left( \partial_l g_{ik} + \partial_k g_{il} - \partial_i g_{lk} \right) \)

\[ g_{ij} \Gamma^i_{kp} = \frac{1}{2} g_{ij} g^{ir} \left( \partial_r g_{kp} + \partial_p g_{ir} - \partial_i g_{rp} \right) \]

\[ = \frac{1}{2} \left( \partial_r g_{jp} + \partial_p g_{jr} - \partial_j g_{rp} \right) d x^p \otimes d x^j \]

because \( g_{ij} g^{ir} = \delta^r_j \)

Then \( g_{ij} \left( \Gamma^k_{ip} \, dx^p \otimes dx^i + \Gamma^i_{jq} \, dx^q \otimes dx^i \right) \)

\[ = \frac{1}{2} \left( \partial_r g_{jp} + \partial_p g_{jr} - \partial_j g_{rp} \right) d x^p \otimes d x^j \]

\[ + \frac{1}{2} \left( \partial_r g_{qi} + \partial_i g_{qr} - \partial_r g_{qi} \right) d x^q \otimes d x^i \]

\[ p \mapsto i \]

\[ q \mapsto j \]

\[ = \frac{1}{2} \left( \partial_i g_{ij} + \partial_j g_{ij} - \partial_j g_{ij} + \partial_i g_{ij} + \partial_j g_{ij} - \partial_i g_{ij} \right) d x^i \otimes d x^j \]

\[ = \partial_i g_{ij} \, dx^i \otimes dx^j \]
Class 18 Covariantly constant section and curvature

Last time, we showed there is a unique covariant derivative on TM and $T^*M$ that is compatible with a given metric $g$ and torsion free, which is called the Levi-Civita connection $\nabla$. We proved the existence and uniqueness using two different bases:

1. Orthonormal basis $e_1, \ldots, e_n$, $e^i = e_i^*$ metric compatible condition is easier, but $d e^i$ may not be zero.

2. Coordinate basis $\frac{\partial}{\partial x^i}$, $d x^i$:

$$d(d x^i) = 0$$ so the torsion free condition is easier, but $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ may not be $\delta_{ij}$.

In the second proof, we notice that

$$\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes d x^k, \nabla d x^i = -\Gamma^j_{ik} d x^j \otimes d x^k$$

where $\Gamma^j_{ik} = \frac{1}{2} g^{jl} \left( d_i g_{lk} + d_k g_{il} - d_l g_{ik} \right)$ is the Christoffel symbol. Recall it appears in the geodesic equation (Week 5-6)

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \quad \text{or} \quad \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

What is the relation between these two?
First, we give a definition for general U.b.

Def: Let \( r: [0,1] \rightarrow M \) be a smooth path.

A section of \( E|_r = T^1(\gamma) \) is parallel if

\[ \nabla \dot{\gamma} S := \langle y^x \frac{\partial}{\partial t}, \nabla S \rangle \] vanishes, i.e.

the covariant derivative at tangent direction of \( Y \) vanishes. For a basis \( e_i \) of \( E \)

we write \( \nabla S = (ds_i + \alpha^i_{jk} S_j) e_i \)

\[ y^x \frac{\partial}{\partial t} = \frac{d\gamma^c}{dt} \frac{\partial}{\partial x^c} \quad \alpha^i_{jk} = \alpha^i_{jk} \quad dx^k \quad \alpha^i_{jk}: U \rightarrow \mathbb{R} \]

Then \( \nabla \dot{\gamma} S = \left( \frac{ds_i}{dt} + \alpha^i_{jk} \frac{ds_j}{dt} S_k \right) e_i \)

This is a section of \( E|_U \)

\( \nabla \dot{\gamma} S = 0 \) iff \( \langle \nabla \dot{\gamma} S, e_i \rangle = 0 \) for any \( i \)

In particular if \( E = TM \)

\[ e_i = \frac{\partial}{\partial x^i} \quad S = y^x \frac{\partial}{\partial t} \]

\( \nabla = \nabla_c \) Then \( S \circ \gamma = \frac{d\gamma^i}{dt} \)

\[ \nabla \frac{\partial}{\partial x^i} = \left( \frac{d}{dx^i} \circ \gamma \right) + \alpha^i_{jk} \left( \frac{d}{dx^j} \circ \gamma \right) \frac{\partial}{\partial x^k} = \alpha^i_{lk} \frac{\partial}{\partial x^k} \]

\( (\frac{d}{dx^i} \circ \gamma = S_i (\gamma) ) \Rightarrow \alpha^i_{lk} = \Gamma^i_{lk} \)

\( \nabla \dot{\gamma} = 0 \Leftrightarrow \frac{d^2 \gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0 \)

i.e. For a geodesic \( \gamma \), \( y^x \frac{\partial}{\partial x^i} \) is a parallel section with respect to \( \nabla_c \)
Recall in weeks 5-6, we use vector field thm to show that given any point \( p = \gamma(0) \) \( \nu_0 = TM|_{\gamma(0)} \), we can always find a unique solution of \( \gamma \) satisfying the geodesic equation.

Similarly, for a general vector bundle, given a path \( \gamma \) and \( \nu_0 \in E|_{\gamma(0)} \), we can find a unique parallel section \( s \) on \( E|_{\gamma} \). (Solving (**) )

Let \( \nu_1 = s(\gamma(1)) \), it is called the parallel transportation of \( \nu_0 \) along \( \gamma \).

Moreover, if \( \nu_0, \nu_1, \ldots, \nu_n \) a basis of \( E|_{\gamma(0)} \), we obtain a basis \( \nu_1, \ldots, \nu_n \) of \( E|_{\gamma(1)} \) after parallel transport:

i.e. \( \gamma \) and \( \tilde{\gamma} \) induces an isomorphism between \( E|_{\gamma(0)} \) and \( E|_{\tilde{\gamma}(1)} \).

Question. Does the isomorphism depend on \( \gamma \) for given \( \gamma(0) = p \), \( \gamma(1) = q \)?
Answer: This is true when the curvature \( F_0 = 0 \).

We will come back to this topic after introducing principal bundle.

Then let's discuss curvature for metric compatible covariant derivative.
Let \( E_1, \ldots, E_n \) be an orthonormal basis of \( E \).

\[
\nabla e_i = \alpha_k^i e_j \quad \alpha_k^i = \frac{d \alpha_k^i}{dx^j} \quad \alpha^l_j = \frac{d \alpha^l}{dx^j}
\]

\( \alpha^l_j = -\alpha^l_j \) if \( \nabla \) is compatible with \( g \).

Recall \( F_0 = d\alpha + \alpha \wedge \alpha \in C^\infty(M; \text{End}(E) \otimes \Lambda^2 T^*M) \)

\[
F_0 e_i = e_j \delta^i_j + e_k \otimes \alpha_k^l \wedge \alpha^l_i
\]

\[
= e_j \delta^i_j (d\alpha^i_k + \alpha^l_k \wedge \alpha^l_i)
\]

\[
(F_0)_i^j = d\alpha^i_j + \alpha^l_j \wedge \alpha^l_i \quad \alpha^l_j = \frac{d \alpha^l}{dx^j}
\]

\[
(F_0)_j^i = d\alpha^j_i + \alpha^l_i \wedge \alpha^l_j = -d\alpha^i_j + (\alpha^l_i \wedge \alpha^l_j)
\]

\[
= -d\alpha^i_j - \alpha^l_i \wedge \alpha^l_j = -(F_0)_j^i
\]

For the dual basis \( e^i \), we have \( \nabla e^i = -\alpha^l_j e^j \)

\[
F_0 e^i = e_j \delta^i_j (-d\alpha^2_j + \alpha^l_i \wedge \alpha^l_j)
\]

\[
(F_0)_j^i = -d\alpha^2_j + \alpha^l_i \wedge \alpha^l_j = d\alpha^2_j + \alpha^l_i \wedge \alpha^l_j
\]

This is compatible with previous notation.
For \( \Omega = \Omega_{CL} \), write \((F_\Omega)_j^i = \frac{1}{2} R^i_{jkl} e^k \wedge e^l \)

\( R^i_{jkl} = - R^i_{kjl} \quad R^i_{jlk} = - R^i_{jkl} \)

Since \( \Omega_{CL} \) is torsion-free \( d\Omega = dw \) for 1-form \( w \) where \( d\Omega \) denote \( \Lambda(\Omega w) \) and \( \Lambda \) is the antisymmetrization

We can generalize \( \Lambda \) for \( k \)-tensor s.t.

\[ \Lambda(dx^1 \otimes \ldots \otimes dx^n) = dx^1 \wedge \ldots \wedge dx^n \]

Define \( \square \) on \( \Lambda^k T^*M \) by \( \square(w_1 \wedge w_2) = \Lambda dw_1 \wedge w_2 + (-1)^{deg w_1} w_1 \wedge \Lambda dw_2 \)

\[ \Lambda \square(w_1 \wedge w_2) = (\Lambda \square w_1) \wedge w_2 + (-1)^{deg w_1} w_1 \wedge (\Lambda \square w_2) \]

Note that \( d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{deg w_1} w_1 \wedge dw_2 \)

So \( \Lambda \square w = dw \) for any form

We have \( F\Omega e^i = d_\square e^i = e^j \wedge (F_\Omega)^j_i \)

From \( \Lambda(F\Omega e^i) = \Lambda d_\square e^i = (\Lambda \square)^i_j e^j = d^2 e^i = 0 \)

We have \( \Lambda(e^j \otimes F_\Omega^i_j) = e^j \wedge (F_\Omega^i_j) = 0 \)

This means \( R^i_{jkl} e^j \wedge e^k \wedge e^l = 0 \)

So \( R^i_{jkl} + R^i_{kjl} + R^i_{lij} = 0 \) \((1)\)

Change indices we have

\( R^j_{ilk} + R^i_{jik} + R^j_{ikl} = 0 \) \((2)\)

\( R^k_{jil} + R^k_{jil} + R^k_{lij} = 0 \) \((3)\)

\( R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0 \) \((4)\)

Since \( R^i_{jkl} = -R^k_{lij} = R^k_{lij} \)

\((1)-(2)-(3)+(4) \implies R^i_{jkl} = R^k_{lij} \)
From the definition of $R_{jkl}^i$, we have 
\[ F_0 = \frac{1}{2} R_{jkl}^i e_i \otimes e_j \otimes (e_k \wedge e_l) \]

Since $e_i$ are orthonormal basis, we write 
\[ R_{jkl} = R_{jkl}^i (\text{In } \frac{\partial}{\partial x^i} \text{ basis}, R_{jkl} = g^{im} R_{jkl}^m) \]

Define the Riemannian tensor.
\[ \text{Riem} = \frac{1}{2} \text{Rijkl} e^i \otimes e^j \otimes (e^k \wedge e^l) \]
\[ = \frac{1}{4} \text{Rijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l) \]
\[ \nabla \text{Riem} = \frac{1}{4} \nabla_m \text{Rijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l) \otimes e^m \]
Since \( \nabla e^i = \alpha^i_j e^j \), we have
\[ \nabla_m \text{Rijkl} = \partial_m \text{Rijkl} + \alpha^i_m Rijkl + \alpha^j_m Rinkl + \alpha^k_m Rijnl + \alpha^l_m Rijkn \]
Define the Ricci tensor by \( \text{Ric} = \text{Ric}_{ik} e^i \otimes e^k \)
\[ \text{Ric}_{ik} = \sum_j \text{Rijkj} \]
Define the scalar curvature \( R : M \to \mathbb{R} \) by trace of \( \text{Ric} : \)
\[ R = \sum_i \text{Rici} = \sum_{ij} \text{Rijij} \]
\( (\text{Ric}_{ik} - \frac{1}{2} R \delta_{ik}) e^i \otimes e^k \) is called the Einstein tensor. (However, the Einstein equation in physics does not use Riemannian metric)
Last time, we computed the curvature for $\nabla$.

Suppose $e^1, \ldots, e^n$ are orthonormal basis of $TM$ (locally).

Write $F^i = \partial^2_\nabla e^i = e^j \circ (F^i)_j$

$(F^i)_j = \frac{1}{2} R^i_{jkl} e^k \wedge e^l \quad R^i_{jkl} : U \to \mathbb{R}$

We proved the following identities:

**Prop.** $R^i_{jkl} = -R^i_{klj}$ (from metric compatible condition)

2) $R^i_{jlk} = -R^i_{klj}$ (from definition)

3) $R^i_{jkl} + R^i_{kjl} + R^i_{ljk} = 0$ (from torsion-free condition)

4) $R^i_{jkl} = R^k_{lij}$ (from 1-3)

Today we first prove 5) $\nabla_m R^i_{jkl} + \nabla_k R^i_{jlm} + \nabla_l R^i_{kmj} = 0$

This is equivalent to $d\delta F^i = 0$ (we omit the pt of equivalence).

Recall $F^i = d\alpha + \alpha \wedge \alpha$

This means $F^i \circ s = d^2 s = (p, (d\alpha u + \alpha u \wedge \alpha u) \cdot su)$

$d^2 (d\alpha \circ s) = (p, d((d\alpha u + \alpha u \wedge \alpha u) \cdot su) + \alpha u \wedge (d\alpha u + \alpha u \wedge \alpha u) \cdot su)$

$= d(d\alpha u + \alpha u \wedge su) + (d\alpha u + \alpha u \wedge du) \cdot dsu + \alpha u \wedge (d\alpha u + \alpha u \wedge du) \cdot su$
\[
L = \left( d(ddu + d(u \wedge au)) + au \wedge (dau + auuau) \\
- au(uau + auuuau) + (dau + au \wedge auuau) \wedge dSu \right)
\]

\[
= \text{(d}^2 uu + d(uuuau) + au \wedge dau + auuuauuuau) \wedge dSu \\
- auuauau + auuuuuauuuuuau \wedge dSu \\
+ (dau + au \wedge auuau) \wedge dSu
\]

Then we have \( d_u(F_0s) = F_0u \wedge d_u Su \)

Note \( d_u(F_0s) = d_uF_0 + (-1)^{\deg F_0} F_0 \wedge d_u Su \)

So \( d_uF_0 = 0 \) (Note \( d_u(F_0s) \neq d_u(F_0s) \))
Chern classes.

For a complex vector bundle \( E \) with \( E|_0 \cong \mathbb{C}^n \) we can also define covariant derivative \( \nabla \) and the curvature \( F \).

Define the \( k \)-th Chern class to be

\[
C_k(E) = \frac{1}{(2\pi i)^k} \text{tr}(F \wedge \ldots \wedge F)
\]

\( k \) times

We define the trace. First, choose a basis \( e_1, \ldots, e_n \) for \( E \) and dual basis \( e^1, \ldots, e^n \) for \( \text{End}(E) \).

For any section \( s \) of \( E \otimes E^* \), we can write \( s = s_i^j e_i \otimes e^j \), define \( \text{tr}(s) = s_i^i \).

It is independent of basis.

If \( w \in \mathcal{C}^\infty(\text{End}(E) \otimes \Lambda^k T^* M) \),
we can check \( d(\text{tr} w) = \text{tr}(d\Lambda w) \).

If \( w_i \in \mathcal{C}^\infty(\text{End}(E) \otimes \Lambda^{k_i} T^* M) \)

define \( w_1 \wedge w_2 \in \mathcal{C}^\infty(\text{End}(E) \otimes \Lambda^{k_1+k_2} T^* M) \)
by \((w_1 \wedge w_2) e_1 = w_1 \wedge (w_2 e_1)\).
Then \( \text{tr} (\omega \wedge \omega) \in C^\infty (\Lambda^{k+1} T^* M) \)

Note \( F_\omega \in C^\infty (\text{End}(E) \otimes \Lambda^2 T^* M) \)

\[
\text{tr} (\underbrace{F_\omega \wedge \cdots \wedge F_\omega}_k) \in C^\infty (\Lambda^{2k} T^* M)
\]

\[
d (\text{tr} (\underbrace{F_\omega \wedge \cdots \wedge F_\omega}_k)) = k \text{tr} (d \nabla F_\omega \wedge \cdots \wedge F_\omega) 
\]

\[
= 0 \quad \text{(Bianchi identity)}
\]

Thm. the de Rham cohomology class

\[
[\text{tr} (\underbrace{F_\omega \wedge \cdots \wedge F_\omega}_k)] \text{ does not depend on }
\]

the choice of \( \nabla \), i.e. \( c_k \) only depends on \( E \)

pf. Recall any two covariant derivative \( \nabla, \nabla' \)

are differed by \( \alpha \in C^\infty (\text{End}(E) \otimes T^* M) \)

Consider \( \nabla^t = \nabla + t \alpha \)

\[
F_\omega^t = F_\omega + t d \nabla \alpha + t^2 \alpha \wedge \alpha
\]

\[
d \nabla^t \alpha = d \nabla \alpha + (t \alpha \wedge \alpha + \alpha \wedge t \alpha)
\]

\[
\frac{d}{dt} F_\omega^t = d \nabla^t \alpha
\]
\[ \frac{d}{dt} \text{tr}(F_0^t \wedge F_0^t) \]

\[ = k \cdot \text{tr}(d_F t \wedge F_0^t - \Lambda F_0^t) \quad \text{(Since) } d_F t = 0 \]

\[ = k \cdot d \cdot \text{tr}(\alpha \wedge F_0^t \wedge \cdots \wedge F_0^t) \]

Hence \( C_k(F_0^t) = C_k(F_0) \)

\[ = d \int_0^1 \frac{k}{2\pi (t-1)^k} \text{tr}(\alpha \wedge F_0^t \wedge \cdots \wedge F_0^t) \]

Cor. If \( C_k(E) = 0 \) for some \( k \), then \( F_0 \equiv 0 \) for any \( \nabla \) on \( E \)

Note that the product bundle \( M \times \mathbb{C}^n \) with \( \nabla = \partial \) has \( F_0 = 0 \), so \( C_k(M \times \mathbb{C}^n) = 0 \)

Since \( C_k(E) \) is independent of the choice of \( \nabla \), we can compute \( \text{tr}(F_0 \wedge \cdots \wedge F_0) \) by any simple \( \nabla \)

Rem. Indeed Chern classes can be defined in \( H_{\text{sing}}^*(M; \mathbb{Z}) \), i.e. it is an integer class,

if we integrate the class over submanifolds, we will get integers. That's why we need \( \frac{1}{(2\pi i)^k} \) in the def
Ex 1. $C_1(E) \neq 0$ for tautological $C$-bundle over $\mathbb{CP}^n$

2. $\Sigma$ is an orientable surface in $\mathbb{R}^3$ of genus $g$

$T\Sigma$ can be regarded as a complex vector bundle of dim 1 if we consider multiplying by $i$ as rotation $90^\circ$

$C_1(T\Sigma) = (2-2g)$, generator in $H^2(Sym^1(\Sigma; \mathbb{Z}))$

$C_1(T\Sigma) \neq 0$ unless $g = 1$

The tautological bundles over complex Grassmannians have nonvanishing $C_1, \ldots, C_{dimE}$

For real vector bundle $E$, we can define Pontryagin class:

$\text{Class } p_k(E) = C_{2k}(E \otimes \mathbb{C}) \in H^{4k}(\text{M}; \mathbb{R})$

Roughly, $E \oplus \overline{E}$

The odd Chern class of complexification is determined by Stiefel-Whitney class of the original bundle

For more discussion of characteristic classes, see [Hitchin: Vector bundles and K-theory, Chap 3]

Roughly speaking, Chern classes are obstructions of the triviality of the bundle: Chern class, Pontryagin class, Stiefel-Whitney, Euler class

$W_k \in H^k(M; \mathbb{Z})$, $e \in H^2(M; \mathbb{Z})$
Property of Chern class

\[ f : M \to N \quad \pi : E \to N \quad C_k(f^*E) = f^*C_k(E) \]

\[ C_k(E^*) = (-1)^k C_k(E) \]

\[ C(E) = 1 + c_1(E) + c_2(E) + \cdots \]

\[ C(E_1 \otimes E_2) = c(E_1) \wedge c(E_2) \]