THE BGG RESOLUTION AND THE WEYL CHARACTER FORMULA

Fan Zhou
Summer 2020
fanzhou@college.harvard.edu

In this expository paper we will give[1] the classical proof of the Bernstein-Gelfand-Gelfand (BGG) resolution, following the original paper of BGG, and use it to prove the celebrated Weyl character formula (in some sense the BGG resolution is a categorification of the Weyl character formula). I have tried my best to flesh out details which BGG omitted in their original paper, as well as organize the proof and the flow of logic in a manner which I find most motivated and easily understood.

Structurally we will tend to assume facts and prove goals, proving the facts later, i.e., logically this exposition should be read backwards; for example we will begin by proving Weyl assuming BGG. We do this for the sake of clarity and motivation. More specifically, we will prove BGG in this order: we will first show how Three Lemmas imply BGG; then we will show how Weak BGG[2] implies Three Lemmas; then we will prove Weak BGG. Along the way we will assume some theorems not to be proven in this exposition; these facts are enumerated in Section 1.

Some notes about labelling: we will label theorems by their names when appropriate, and we will label statements from BGG by just their number, e.g. writing [10.5] rather than [BGG10.5]. When we need to refer to facts from e.g. Kirillov or Humphreys[3], we will for example say [K8.27] or [H4.2].

Contents

[1] Though we will blackbox some facts in the interest of length, e.g. the conditions for embeddings to exist between Verma modules.

[2] This is Theorem 9.9 in BGG's paper, and they did not call it this; but this seems an appropriate name, as BGG appears to strictly improve upon it.

[3] Unfortunately Humphreys does not label his theorems/propositions/lemmas, instead relying on the fact that there is a unique theorem/proposition/lemma per section; we will rely on context to tell what we mean.

1. A Short Introduction

Some notations/conventions: We will work over $\mathbb{C}$ throughout. Unless otherwise stated, $\mathfrak{g}$ will be a semisimple Lie algebra. We will write Σ for the set of simple roots; if α_i is a simple root, it will be said that $\alpha_i \in \Sigma$, and otherwise α_i denotes any indexed set of roots. In this spirit, we will denote by $I(\Sigma)$ the index set of the simple roots, i.e. $i \in I(\Sigma) \iff \alpha_i \in \Sigma$. An irreducible representation of highest weight λ is commonly called L_λ, and sometimes to emphasize that it is finite-dimensional we may write Π_λ instead. Recall the notion, for $\lambda, \mu \in P$, of

$$\mu \leq \lambda \iff \lambda - \mu \in Q_+.$$

We will write W for the Weyl group and

$$W_k := \{w \in W : \ell(W) = k\}.$$

We will let

$$w \circ \lambda := w(\lambda + \rho) - \rho$$

define the affine action of the Weyl group W on $\mathfrak{h}^*$, where recall $\rho := \frac{1}{2}\sum_{R_+} \alpha = \sum_\Sigma \alpha$, where $\Sigma \subseteq R_+$ denotes the set of simple roots. We will also write

$$\lambda \sim \mu \iff \exists\, w : \lambda = w(\mu),$$

and

$$\lambda \overset{\circ}{\sim} \mu \iff \exists\, w : \lambda = w \circ \mu.$$

In line with the notation of $\mathfrak{g}^\alpha$, we will also write

$$M^\mu := \{v \in M : \mathfrak{h}v = \mu(\mathfrak{h})v\}$$

for the μ-weight space of M; for example for Verma modules this would be written ${M_\lambda}^\mu$ with staggered indices.

Full disclosure: **here are the facts we will be blackboxing** (in addition to some standard homological algebra facts/constructions, such as the Jordan-Holder theorem, which is for example covered in the first two pages of Benson's *Representations and Cohomology I*) in the interest of length:

(1) Verma modules admit finite Jordan-Holder composition series; in fact, the category $\mathcal{O}$ consists of Artinian objects.

(2) Moreover, for $\lambda \in P_+$, the Jordan-Holder decomposition of a Verma module $M_{w\circ\lambda}$ contains irreducibles of form

$$L_{w'\circ\lambda} \in \mathrm{JH}(M_{w\circ\lambda}), \quad w' \geq w.$$

In fact there are more precise conditions (which we won't need), which we will give later.

(3) Maps between Verma modules have

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu) = \begin{cases} 0 & \\ \mathbb{C} & \end{cases};$$

moreover, for $\lambda \in P_+$,

$$\mathrm{Hom}_{\mathfrak{g}}(M_{w_1\circ\lambda}, M_{w_2\circ\lambda}) = \mathbb{C} \iff w_1 \geq w_2$$

where $w_1 \geq w_2$ refers to the Bruhat order (more on this later). In fact there are more precise conditions for these homs, too long to be appropriate for this preamble, which we will state later.

(4) The Harish-Chandra theorem, which states that central characters[4] $\vartheta_\lambda = \vartheta_\mu$ are equal iff $\lambda = w \circ \mu$ for some w, i.e. iff $\lambda + \rho \sim \mu + \rho$.

[4]More of this later.

(5) Various facts about central characters, such as the exactness of the functor $\square^\vartheta$.
(6) Various purely combinatorial facts about Weyl groups, e.g. the existence of squares and the existence of a choice of signs attached to arrows so that the product of signs in each square is -1.

(1) is proved in chapter 1 of Humphreys's $\mathcal{O}$ book, whereas (3) is proved in chapter 4. I have still yet to learn more about the Harish-Chandra theorem; this is something I will do next. (6) is proved in section 11 of BGG, but as it is purely combinatorial we omit it for the sake of length here.

Interestingly I could not find any classical (non-generalization) sources on the BGG resolution aside from the original BGG paper, which we (try to) follow here. The original BGG paper was surprisingly difficult to obtain, and in it BGG uses some conventions/notations which are different from those most use (as noted in Humphreys's book on the category $\mathcal{O}$). Moreover, there were some points in BGG's proofs which I found rather difficult to follow (for example due to omitted details). In this exposition I will try to flesh out these details to the best of my ability and organize the material in a way which is most motivated and easily understood, as well as switch the conventions/notations of BGG to something more familiar. Any errors are, of course, entirely my own.

Among the odd notations of BGG, the most notable is that BGG writes $M_{\lambda+\rho}$ where we would write M_λ for Verma modules. Though changing BGG's $M_{\lambda+\rho}$ notation to M_λ is nothing more than an index shift, I can only hope I've made no errors. Here are some others: it seems to be common for central characters to be denoted χ, whereas BGG uses ϑ; this is a convention I will keep. BGG also writes

$$M_\vartheta = \operatorname{Ker}^\infty \operatorname{Ker} \vartheta$$

for the eventual kernel of $\operatorname{Ker} \vartheta \subseteq Z(U\mathfrak{g})$, which I will instead denote by

$$M^\vartheta$$

since in some sense M admits a "weight decomposition" in this way (more later). Interestingly BGG also writes $w_1 \geq w_2$ implies $\ell(w_1) \leq \ell(w_2)$, so that instead of a unique maximal element in the Weyl group there is a unique minimal element. We will stick to the unique maximal element convention.

Lastly some explanation on my choice of blackbox material: last summer I worked on a resolution for representations of S_n the symmetric group, and this project is one which I have still not finished. To this end I figured learning and writing down the proof of BGG would be helpful, and so I have chosen to commit to paper here the parts of the BGG proof which could possibly carry over to the S_n context with appropriate modifications; for example, the combinatorial facts about the Weyl group hold for S_n, so we omit them here, and similarly homs between permutation representations of S_n no longer enjoy the nice properties of Verma (either zero or $\mathbb{C}$), so we also omit them.

2. The Setting: Category $\mathcal{O}$

For completeness let us describe the setting we work in. We will only be story-telling and won't prove any of the details in this section. This is described in section 8 of BGG, where BGG mostly describes and cites things, and also elaborated upon in chapter 1 of Humphreys, where he proves e.g. $\mathcal{O}$ is Artinian.

Let us first define $\mathcal{O}$:

Definition. Let $\mathcal{O}$ be the full subcategory of the category $\mathsf{Mod}_{U\mathfrak{g}}$ of left $U\mathfrak{g}$-modules, whose objects are all M such that:

(1) M is $U\mathfrak{g}$-finitely-generated.
(2) M is $\mathfrak{h}$-semisimple (i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ has a weight basis).
(3) M is locally $U\mathfrak{n}^+$-finite, i.e.

$$\dim \operatorname{Span}_{U\mathfrak{n}^+}(v) < \infty \quad \forall\, v \in M.$$

Recall that Verma modules lie in this category.

Before going on to give more facts about $\mathcal{O}$, let us describe the notion of central characters:

Definition. For any $M \in \mathsf{Rep}\,\mathfrak{g}$, if $v \in M$ is an eigenvector with respect to all of $Z(U\mathfrak{g})$, then we can find a

$$\vartheta \in \operatorname{Hom}_{\mathsf{Alg}_{\mathbb{C}}}(Z(U\mathfrak{g}), \mathbb{C})$$

such that

$$zv = \vartheta(z)v \quad \forall\, z \in Z(U\mathfrak{g}).$$

These ϑ are called "central characters"; more generally we may refer to any such homomorphism as a central character.

Let us also write

$$\Theta(M) \coloneqq \{\text{such } \vartheta\}$$

for the set of central characters of a module.

That this is a homomorphism of algebras is clear since for example $(z_1 + z_2)v = z_1 v + z_2 v$ so ϑ respects addition, and similarly for multiplication.

For completeness let us cite some facts about $\mathcal{O}$ (see Humphreys):

Proposition. Let $M \in \mathcal{O}$.

(1) All weight spaces of M are finite-dimensional:

$$\dim M^\lambda < \infty.$$

Moreover, the set of weights of M is contained in a finite union of cones $\lambda_i - Q_+$, $\lambda_i \in \mathfrak{h}^*$:

$$\operatorname{wt} M \subseteq \bigcup_{i \,\text{fnt}:\lambda_i \in \mathfrak{h}^*} (\lambda_i - Q_+).$$

(2) $\mathcal{O}$ is both Noetherian and Artinian, i.e. every $M \in \mathcal{O}$ is both Noetherian and Artinian as a $U\mathfrak{g}$-module. In particular this means every $M \in \mathcal{O}$ admits a finite Jordan-Holder decomposition series.

(3) $\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums.

(4) $\mathcal{O}$ is abelian.

(5) For $M \in \mathcal{O}$ and $\dim V < \infty$,

$$M \otimes V \in \mathcal{O}.$$

In particular

$$\dim V < \infty \implies \Box \otimes V : \mathcal{O} \xrightarrow{\text{exact}} \mathcal{O}.$$

(6) M is $U\mathfrak{n}^-$-finitely-generated.

(7) M is $Z(U\mathfrak{g})$-finite:

$$\dim \operatorname{Span}_{Z(U\mathfrak{g})}(v) < \infty \quad \forall\, v \in M.$$

(8) Every irreducible module in $\mathcal{O}$ is of form L_λ, the quotient of M_λ by the maximal submodule ($\lambda \in \mathfrak{h}^*$).

Next some facts regarding the central characters:

(9) For $\lambda \in \mathfrak{h}^*$ and a Verma module M_λ, there is exactly one central character, which we will call ϑ_λ:

$$\Theta(M_\lambda) = \{\vartheta_\lambda\}.$$

(10) For any $M \in \mathcal{O}$,

$$|\Theta(M)| < \infty.$$

(11) For any $\vartheta \in \operatorname{Hom}(Z(U\mathfrak{g}), \mathbb{C})$, its kernel is an ideal $\operatorname{Ker}\vartheta \subseteq Z(U\mathfrak{g})$ which has stabilizing eventual kernel:

$$M \supseteq \{v \in M : (\operatorname{Ker}\vartheta)^n v = 0\} \text{ stabilizes for large } n.$$

This will be denoted

$$M^\vartheta := \operatorname{Ker}^\infty \operatorname{Ker}\vartheta \overset{\mathsf{Rep}}{\subseteq} M,$$

which is a subrepresentation of M.

(12) Moreover,

$$\Theta(M^\vartheta) = \{\vartheta\}$$

and

$$M = \bigoplus_{\vartheta \in \Theta(M)} M^\vartheta$$

and

$$\Box^\vartheta : \mathcal{O} \xrightarrow{\text{exact}} \mathcal{O}$$

is exact.

I think the proof[5] of most of these are not too difficult and can in fact be found in chapter 1.1 of Humphreys, except maybe for showing $\mathcal{O}$ is Artinian (requires citing Harish-Chandra), which is in chapter 1.11 of Humphreys.

There is another theorem about these central characters which we will need to cite:

Theorem (Harish-Chandra)**.** For $\lambda, \mu \in \mathfrak{h}^*$,

$$\begin{aligned} \vartheta_\lambda = \vartheta_\mu &\iff \lambda \overset{\circ}{\sim} \mu \\ &\iff \lambda + \rho \sim \mu + \rho, \end{aligned}$$

where the second line is the definition of $\overset{\circ}{\sim}$.

Interestingly Harish-Chandra is a single person and not the last name of two authors.

3. The BGG Resolution and The Weyl Character Formula

In this section we will state the main results of this exposition: the BGG resolution and the Weyl character formula. We will use the former to prove the latter.

3.1. **The BGG Setup.**

Unfortunately the full statement of the BGG resolution takes a bit of setup. Rather than state a partial result now and use it to prove Weyl, only to give a full statement later, we will begin with this setup and give the full statement right away. Most of the proofs implicit in this setup will be omitted in the interest of length.

Recall that Verma modules were defined as the "universal" highest weight modules,

$$M_\lambda \coloneqq U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}^\lambda,$$

where $\lambda \in \mathfrak{h}^*$. One may ask what homs between such spaces look like, and there is a theorem of Verma and BGG which characterizes this (I will give the statement as it appears in BGG, adjusting for index shifts):

[5] A note to myself: as Humphreys remarks, it is very easy to for example see M^ϑ is a subrep of M, since $\operatorname{Ker}\vartheta \subseteq Z(U\mathfrak{g})$ and so $(\operatorname{Ker}\vartheta)^n v = 0 \implies (\operatorname{Ker}\vartheta)^n \mathfrak{g} v = \mathfrak{g}(\operatorname{Ker}\vartheta)^n v = 0$.

Theorem (Verma). For $\lambda, \mu \in \mathfrak{h}^*$,

$$\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \begin{cases} 0 \\ \mathbb{C} \end{cases},$$

and any nonzero map between two Verma modules is an injection.

Moreover,

$$\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C}$$
$$\Updownarrow$$
$$\begin{aligned} &\exists\, \alpha_1, \cdots, \alpha_k \in R_+ \\ &: \mu + \rho = s_{\alpha_k} \cdots s_{\alpha_1}(\lambda + \rho), \\ &\ \ s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda + \rho) - s_{\alpha_i} \cdots s_{\alpha_1}(\lambda + \rho) \in \mathbb{Z}_{0+}\alpha_i. \end{aligned}$$

In the case that $\lambda \in P_+$, these conditions simplify to

$$\mathrm{Hom}_{\mathfrak{g}}(M_{w_1 \circ \lambda}, M_{w_2 \circ \lambda}) = \mathbb{C}$$
$$\Updownarrow$$
$$w_1 \geq w_2.$$

The proof of this theorem is a long journey through chapter 4 of Humphreys; as one might expect, it it much easier to prove that $\dim \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) \leq 1$ and is injective when nonzero than it is to prove the full criterion.

For our purposes the $w_1 \geq w_2$ will be the relevant condition. As noted earlier, $w_1 \geq w_2$ refers to the Bruhat order on the Weyl group: i.e. meaning there exists a chain

$$w_1 \xrightarrow{s_{i_1}} u_1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_{n-1}}} u_{n-1} \xrightarrow{s_{i_n}} w_2$$

such that

$$\begin{aligned} u_k &= s_{i_{k+1}} u_{k+1}, \\ \ell(u_k) &= \ell(u_{k+1}) + 1, \end{aligned}$$

where we set $u_0 = w_1$ and $u_n = w_2$. We may sometimes suppress the arrow labels s in this notation and instead just write e.g.

$$w_1 \longrightarrow u_1.$$

In view of this theorem, since all nonzero maps are injective, when appropriate, there is a Verma submodule M_μ inside M_λ. Hence, for $w_1 \geq w_2$ let us write

$$\iota_{w_1 \to w_2} \colon M_{w_1 \circ \lambda} \hookrightarrow M_{w_2 \circ \lambda}$$

for the canonical embedding.

There are two other combinatorial facts about the Weyl group which we will quote; proofs may be found in section 11 of BGG. Here we will refer to them by their numbers in BGG.

Consider the (finite) directed graph $\Gamma(W)$ whose vertices are elements of W and whose arrows are precisely those such that $w_1 \xrightarrow{s} w_2$. We will call (w_1, w_2, w_3, w_4) a "square" if

$$\begin{array}{ccccc} & & w_2 & & \\ & \nearrow & & \searrow & \\ w_1 & & & & w_4\ . \\ & \searrow & & \nearrow & \\ & & w_3 & & \end{array}$$

Lemma (10.3,10.4)**.** For $w_1, w_4 \in W$ with $\ell(w_1) = \ell(w_4) + 2$, there are either zero or two vertices that fit in arrows between:

$$\#\{w : w_1 \to w \to w_4\} = \begin{cases} 0 \\ 2 \end{cases} .$$

Moreover, to each arrow $w_1 \to w_2$ of $\Gamma(W)$ we may assign a sign

$$\operatorname{sgn}(w_1, w_2) = \pm 1$$

such that, for all squares (w_1, w_2, w_3, w_4),

$$\prod_{\text{4 arrows in square}} \operatorname{sgn}(w, w') = -1.$$

3.2. **The BGG Resolution.** Now we are in a position to set up the BGG resolution. This resolution will be constructed as follows for $\lambda \in P_+$: grade the graph $\Gamma(W)$ by length $\ell(w)$; place a Verma module $M_{w \circ \lambda}$ at each vertex w of $\Gamma(W)$; define maps in the resolution by putting a map $\operatorname{sgn}(w_1, w_2)\iota_{w_1 \to w_2}$ between $M_{w_1 \circ \lambda}$ and $M_{w_2 \circ \lambda}$ for each arrow $w_1 \to w_2$ (recall that maps between these two is $\mathbb{C}$ since $w_1 \to w_2$, so any map is given by a multiple of the canonical embedding); and lastly direct sum all modules in the same grading (i.e. of same length $\ell(w)$), appropriately combining the maps $\operatorname{sgn}(w_1, w_2)\iota_{w_1 \to w_2}$ to obtain d. Note well that, since each ι is a map of representations, d so defined is also a map of representations.

Theorem (BGG)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$ a finite-dimensional irrep of highest weight $\lambda \in P_+$, there is a resolution by $\mathfrak{g}$-modules of Π_λ:

$$0 \longrightarrow M_{w_0 \circ \lambda} \xrightarrow{\mathrm{d}_{|R_+|}} \cdots \longrightarrow \bigoplus_{w \in W_k} M_{w \circ \lambda} \xrightarrow{\mathrm{d}_k} \cdots \longrightarrow \bigoplus_{i \in I(\Sigma)} M_{s_i \circ \lambda} \xrightarrow{\mathrm{d}_1} M_\lambda \xrightarrow{\mathrm{d}_0} \Pi_\lambda \longrightarrow 0.$$

Note that each term of the complex is given by ($|R_+| \geq k \geq 0$)

$$C_k = \bigoplus_{w \in W_k} M_{w \circ \lambda},$$

out of which d_k ($k \geq 1$) is defined as

$$\mathrm{d}_k\big|_{M_{w \circ \lambda}} = \big(\operatorname{sgn}(w, w')\iota_{w \to w'}\big)_{w' \in W_{k-1}};$$

d_0 is defined as

$$\mathrm{d}_0 \coloneqq \pi \colon M_\lambda \twoheadrightarrow \Pi_\lambda$$

the projection.

Note also that, as $\ell(w_0) = |R_+| = \dim \mathfrak{n}^-$, $M_{w_0 \circ \lambda}$ belongs to the ($|R_+| = \dim \mathfrak{n}^-$)-th term of the sequence (where we take Π_λ to be the -1-th term).

3.3. **The Weyl Character Formula.** We will now prove the Weyl character formula. First some background: recall we had defined

Definition.
$$\mathbb{C}[P] \coloneqq \mathbb{C}\langle e^\lambda : \lambda \in P,\ e^\lambda e^\mu = e^{\lambda+\mu},\ e^0 = 1\rangle,$$
in which lives a character (for $V \in \mathsf{Rep}_{\text{fd}}\,\mathfrak{g}$ finite-dimensional)
$$\chi_V \coloneqq \sum_{\lambda \in \operatorname{wt} V} \dim(V^\lambda) e^\lambda.$$

This character enjoys some basic properties, e.g.

Fact.
$$\begin{aligned} \chi_{\mathbb{C}} &= 1, \\ \chi_{V \oplus W} &= \chi_V + \chi_W, \\ \chi_{V \otimes W} &= \chi_V \chi_W, \\ \chi_{V^*} &= \overline{\chi}_V, \end{aligned}$$
where $\overline{\square}$ is defined by
$$\overline{e^\lambda} \coloneqq e^{-\lambda}.$$

For finite-dimensional representations we know that the above sum must be finite (since there are only finitely many weights of a finite-dimensional V), so there are no issues. In the case of an infinite-dimensional representation, however, we must be more careful; to this end define

Definition.
$$\widehat{\mathbb{C}[P]} \coloneqq \left\{ \sum_{\lambda \in P} c_\lambda e^\lambda : \{\lambda : c_\lambda \neq 0\} \subseteq \bigcup_{\substack{i \text{ fnt} \\ \lambda_i \in P}} (\lambda_i - Q_+) \right\}$$
where we allow infinite sums as long as all nonzero terms lie in a finite union of cones of form $\lambda_i - Q_+$.

Since highest weight representations (those generated by a single $v \in \operatorname{Ker} \mathfrak{n}^+$) have weights $\lambda - Q_+$, a character χ_{V_λ} for a highest weight representation V_λ of highest weight λ will live in $\widehat{\mathbb{C}[P]}$; in fact, the nonzero terms lie in a single cone $\lambda - Q_+$.

Since $\chi_{V \oplus W} = \chi_V + \chi_W$, one may wonder what the characters of finite-dimensional irreducibles are. This is the theorem of Weyl:

Theorem (Weyl Character)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\text{fd}}\,\mathfrak{g}$ the finite-dimensional irreducible of highest weight $\lambda \in P_+$, the character of Π_λ is given by
$$\chi_{\Pi_\lambda} = \sum_{w \in W} \operatorname{sgn}(w) e^{w \circ \lambda} \prod_{\alpha \in R_+} \frac{1}{1 - e^{-\alpha}},$$
where
$$\operatorname{sgn}(w) \coloneqq (-1)^{\ell(w)}$$
and $\frac{1}{1-e^{-\alpha}}$ represents a formal series
$$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots .$$

This will follow directly from computing the characters of Verma modules and applying the BGG resolution. Indeed, the Verma modules have character:

Proposition. For $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has character

$$\chi_{M_\lambda} = e^\lambda \prod_{\alpha \in R_+} \frac{1}{1 - e^{-\alpha}},$$

where $\frac{1}{1-e^{-\alpha}}$ represents a formal series

$$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots .$$

Proof. Recall that the M_λ has weights $\mu \in \lambda - Q_+$, each of which is finite-dimensional. Recall moreover

$$U\mathfrak{n}^- \overset{\mathsf{Vec}}{\cong} M_\lambda$$

via

$$x \longmapsto xv^\lambda.$$

Recall also from a computation[6] on pset 10 that, for $\alpha \in R_+$, f_α brings a vector in the μ-weight space to the $\mu - \alpha$-weight space, for example

$$f_\alpha v^\lambda \in {M_\lambda}^{\lambda - \alpha}. \tag{$*$}$$

Then, since the Verma module is the free $U\mathfrak{n}^-$ module generated on one vector, if linearly independent $x \neq y \in U\mathfrak{n}^-$ both have $xv^\lambda, yv^\lambda \in {M_\lambda}^{\lambda-\delta}$, then xv^λ, yv^λ are linearly independent. Therefore, to compute the dimensional of each weight space, it suffices to enumerate the number of linearly independent elements of $U\mathfrak{n}^-$ which bring v^λ to the right weight space. By the PBW theorem, since $\mathfrak{n}^-$ has an additive basis $\{f_\alpha\}_{\alpha \in R^+}$, we know $U\mathfrak{n}^-$ has a basis given by $\{\prod_\alpha f_\alpha^{n_\alpha}\}_n$; therefore, for $\delta \in Q_+$ the dimension of each weight space is given by

$$\dim {M_\lambda}^{\lambda-\delta} = \#\left\{ \sum_{\alpha \in R_+} n_\alpha \alpha : \sum_{R_+} n_\alpha \alpha = \delta \right\}.$$

By elementary combinatorics this is the coefficient

$$\dim {M_\lambda}^{\lambda - \delta} = [e^{-\delta}] \prod_{\alpha \in R_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots).$$

[6]For completeness, this is saying

$$\mathfrak{h} f_\alpha v^\lambda = [\mathfrak{h}, f_\alpha] v^\lambda + f_\alpha \mathfrak{h} v^\lambda = -\alpha(\mathfrak{h}) f_\alpha v^\lambda + f_\alpha \lambda(\mathfrak{h}) v^\lambda = (\lambda - \alpha)(\mathfrak{h}) f_\alpha v^\lambda.$$

Then we have

$$\begin{aligned}
\chi_{M_\lambda} &= \sum_{\mu\in\lambda-Q_+} \dim({M_\lambda}^\mu)e^\mu \\
&= \sum_{\delta\in Q_+} \dim {M_\lambda}^{\lambda-\delta}e^{\lambda-\delta} \\
&= \sum_{\delta\in Q_+} e^{\lambda-\delta}[e^{-\delta}] \prod_{\alpha\in R_+}(1+e^{-\alpha}+e^{-2\alpha}+\cdots) \\
&= e^\lambda \sum_{\delta\in Q_+} e^{-\delta}[e^{-\delta}] \prod_{\alpha\in R_+}\frac{1}{1-e^{-\alpha}} \\
&= e^\lambda \prod_{\alpha\in R_+}\frac{1}{1-e^{-\alpha}},
\end{aligned}$$

where in the last line we recall Q_+ is precisely the set of all nonnegative spans of positive roots. This is as advertised. ■

We are now in a position to prove Weyl.

Proof of Weyl Character. Recall from linear algebra that the alternating sum of dimensions in an exact sequence of vector spaces is zero. In fact, for $\varphi\colon M \longrightarrow N$ a map of representations, we have

$$\mathfrak{h}\varphi(v^\lambda) = \varphi\mathfrak{h}(v^\lambda) = \varphi\lambda(\mathfrak{h})(v^\lambda) = \lambda(\mathfrak{h})\varphi(v^\lambda),$$

so that

$$\varphi(M^\lambda) \subseteq N^\lambda \tag{$*$}$$

maps of representations preserve weight spaces. Therefore, in an exact sequence of representations we may restrict attention to each weight space and find that, for

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0,$$

the alternating sum of the dimensions of each weight spaces is also zero:

$$\sum_{i=1}^n (-1)^i \dim({V_i}^\lambda) = 0; \tag{$*$}$$

applying this for each weight space gives

$$\sum_{i=1}^n (-1)^i \chi_{V_i} = 0. \tag{$*$}$$

Apply this now to the BGG resolution

$$0 \longrightarrow M_{w_0\circ\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{w\in W_k} M_{w\circ\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{I(\Sigma)} M_{s_i\circ\lambda} \longrightarrow M_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0,$$

whereupon we obtain

$$\begin{aligned}
\chi_{\Pi_\lambda} &= \sum_{k=1}^{|R_+|}(-1)^k \sum_{w:\ell(w)=k} \chi_{M_{w\circ\lambda}} \\
&= \sum_{w\in W} \operatorname{sgn}(w)\chi_{M_{w\circ\lambda}} \\
&= \sum_{w\in W} \operatorname{sgn}(w)e^{w\circ\lambda} \prod_{\alpha\in R_+}\frac{1}{1-e^{-\alpha}},
\end{aligned}$$

as desired. ∎

In some sense, BGG, being a resolution which turns this alternating sum of formal things into an exact sequence of actual representations, is a "categorification" of the Weyl character formula.

4. Proving BGG from Weak BGG: Three Steps

Now comes the daunting task of proving the BGG resolution. First, some brief words on how we will do this: we will first prove BGG assuming three key lemmas (the three main chunks of the proof), 10.5, 10.6, and 10.7 in the paper. We will then prove these three lemmas, assuming (which we will prove in a later section) two statements from section 9 of BGG (Theorem 9.9 and its Corollary (unnumbered)), which may be thought of as a weaker version of the full BGG theorem.

Along the way we will need to assume some facts (e.g. Harish-Chandra) which will not be proven; we will try to make it clear when we do so, and give precise statements of what the claims are.

As mentioned in the beginning, we will prove BGG in a way which is logically backwards, assuming facts and proving them later, for the sake of clarity and motivation.

4.1. Lemmas imply BGG.

4.1. Lemmas imply BGG. We will prove BGG assuming Lemmas 10.5, 10.6, and 10.7 of BGG. Other facts along the way, being easier to see, will be appropriately proved. For ease of reading let us reproduce a statement of BGG here:

Theorem (BGG)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$ with $\lambda \in P_+$, there is a resolution by $\mathfrak{g}$-modules of Π_λ:

$$0 \longrightarrow M_{w_0\circ\lambda} \overset{\mathrm{d}_{|R_+|}}{\longrightarrow} \cdots \longrightarrow \bigoplus_{w\in W_k} M_{w\circ\lambda} \overset{\mathrm{d}_k}{\longrightarrow} \cdots \longrightarrow \bigoplus_{i\in I(\Sigma)} M_{s_i\circ\lambda} \overset{\mathrm{d}_1}{\longrightarrow} M_\lambda \overset{\mathrm{d}_0}{\longrightarrow} \Pi_\lambda \longrightarrow 0,$$

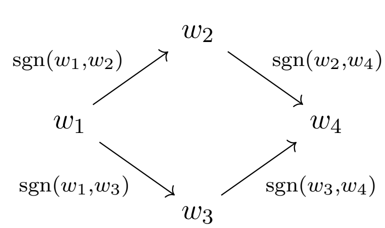
where d_k ($k \geq 1$) is defined as

$$\mathrm{d}_k\big|_{M_{w\circ\lambda}} = \big(\operatorname{sgn}(w,w')\iota_{w\to w'}\big)_{w'\in W_{k-1}}$$

and d_0 is defined as

$$\mathrm{d}_0 \coloneqq \pi\colon M_\lambda \longrightarrow\!\!\!\!\!\rightarrow \Pi_\lambda.$$

Proof of BGG. Step 1: Complex: Let us first see BGG is a complex. We have already described in the last section the "geometry" of the BGG resolution as well as what the maps are. From the way we have defined the maps, it is immediate that BGG gives a complex, i.e. $\mathrm{dd} = 0$: indeed, by the combinatorial lemmas, given two terms in BGG which are two apart, i.e. $\ell(w_1) = \ell(w_4) + 2$, either there are is no path in $\Gamma(W)$ from w_1 to w_4, in which case $\mathrm{d}^2\big|_{w_1\to w_4} = 0$ trivially, or we can write a square with attached signs



such that the product of the four signs is -1; for this to happen, either three are $+1$ and one is -1 or three are -1 and one is $+1$. It's then pretty clear that in any case either the top path is $+1$ and the bottom path is -1 or vice versa, in which case the sum (which is how d is defined) is zero, so that $\mathrm{dd} = 0$ still.

Step 2: Exact Beginning: Next let us see why this sequence is exact[7] at M_λ (position 0) and at Π_λ (position -1).

First some facts:

Fact (K8.27)**.** For $v^\mu \in {M_\lambda}^\mu$,

$$\mathfrak{n}^+ v^\mu = 0 \implies M_\lambda \supseteq \operatorname{Span}_{U\mathfrak{g}}(v^\mu) = \operatorname{Span}_{U\mathfrak{n}^-}(v^\mu) \cong M_\mu$$

is a Verma module.

Proof of K8.27. If $\mathfrak{n}^+$ acts by zero, then clearly instead of acting by all of $U\mathfrak{g}$ it suffices to act on by $U\mathfrak{n}^-$ ($\mathfrak{h}$ will of course act by a character), so that $\operatorname{Span}_{U\mathfrak{g}} v^\mu = \operatorname{Span}_{U\mathfrak{n}^-} v^\mu$.

$\mathfrak{n}^+ v^\mu = 0$ implies the submodule generated[8] by v^μ, $\operatorname{Span}_{U\mathfrak{g}} v^\mu$, is a highest weight representation of highest weight μ, so that it is a quotient of M_μ. Since $U\mathfrak{n}^- \cong M_\mu$ as vector spaces via action, to show $\operatorname{Span}_{U\mathfrak{g}} v^\mu \cong M_\mu$ is suffices to show the map

$$\begin{aligned} U\mathfrak{n}^- &\longmapsto \operatorname{Span}_{U\mathfrak{g}} v^\mu \\ x &\longmapsto x v^\mu \end{aligned}$$

is injective (it is automatically surjective since $\mathfrak{n}^+ v^\mu = 0$). AFSOC $xv^\mu = 0$ for some x; since $v^\mu \in M_\lambda$, there is some $y \in U\mathfrak{n}^-$ such that $yv^\lambda = v^\mu$, so this is saying $xyv^\lambda = 0$, which can only happen if $xy = 0$ since $M_\lambda \cong U\mathfrak{n}^-$; but by PBW

Theorem (PBW)**.** Any Lie algebra $\mathfrak{g}$ with ordered basis

$$\mathfrak{g} = \mathbb{C}(\xi_1, \cdots, \xi_n)$$

has that $U^k\mathfrak{g}$ has basis

$$U^k\mathfrak{g} = \mathbb{C}\{\xi_1^{k_1} \cdots \xi_n^{k_n}\}_{\sum k_i \leq k}.$$

we have $U\mathfrak{n}^-$ has no zero divisors, so this implies $x = 0$ (since $v^\mu \neq 0$, so that $y \neq 0$), which implies the map is injective and therefore surjective. ■

Recall that, for $\alpha_i \in \Sigma$ and $\lambda \in P_+$,

$$f_i^{\lambda(h_i)+1}(\widetilde{v}^\lambda) = 0 \in L_\lambda.$$

In fact this can be made more precise:

Lemma (K8.28a)**.** For $\lambda \in P_+$ and $\alpha_i \in \Sigma$, the submodule inside M_λ generated by $f_i^{\lambda(h_i)+1} v^\lambda$

$$M_\lambda \supseteq \operatorname{Span}_{U\mathfrak{g}}\left(f_i^{\lambda(h_i)+1} v^\lambda\right) = \operatorname{Span}_{U\mathfrak{n}^-}\left(f_i^{\lambda(h_i)+1} v^\lambda\right) \cong M_{s_i \circ \lambda}$$

is isomorphic to a Verma module $M_{s_i \circ \lambda}$.

Proof of K8.28a. We saw earlier that

$$f_\alpha v^\lambda \in {M_\lambda}^{\lambda - \alpha},$$

so in particular

$$f_i^{\lambda(h_i)+1} v^\lambda \in {M_\lambda}^{\lambda - \left(\lambda(h_i)+1\right)\alpha_i} = {M_\lambda}^{s_i \circ \lambda},$$

[7] BGG claims that this follows from Harish-Chandra's theorem on ideals, but I could not quite understand this or locate precisely what statement they were referring to; instead we will give here an alternate and no doubt clumsier (though hopefully not incorrect) argument.

[8] It feels like $U\mathfrak{g} \cdot v^\mu$ would refer to the free module generated by v^μ, so to emphasize that this is a *sub*module generated by v^μ, we will write $\operatorname{Span}_{U\mathfrak{g}}^{M_\lambda} v^\mu$, or just $\operatorname{Span}_{U\mathfrak{g}} v^\mu$ for short.

where

$$s_i \circ \lambda = s_i(\lambda + \rho) - \rho = s_i\lambda + s_i\rho - \rho = s_i\lambda - \alpha_i = \lambda - \big(\alpha_i^*(\lambda) + 1\big)\alpha_i,$$

where we recall that $\rho - s_i\rho = \alpha_i$, and more generally

$$\rho - w(\rho) = \sum_{\alpha \in R_+ : w^{-1}\alpha \in R_-} \alpha.. \tag{$*$}$$

Since

$$f_i^{\lambda(h_i)+1} v^\lambda \in M_\lambda{}^{s_i \circ \lambda},$$

to show the desired by Fact K8.27 it suffices to show $\mathfrak{n}^+$ brings this to zero. Recalling still that $\mathfrak{n}^+$ is generated by e_i for $i \in I(\Sigma)$, it suffices to show this is killed by all e_j for $j \in I(\Sigma)$. For $j \neq i$ this is easy, as $[e_j, f_i] = 0$ and so

$$e_j \cdot f_i^{\lambda(h_i)+1} v^\lambda = f_i^{\lambda(h_i)+1} e_j v^\lambda = 0$$

since, being a highest weight vector, $\mathfrak{n}^+ v^\lambda = 0$. For $i = j$ we must[9] embark on a slightly lengthier computation:

$$\begin{aligned}
e_i f_i^{\lambda(h_i)+1} v^\lambda &= \big([e_i, f_i] + f_i e_i\big) f_i^{\lambda(h_i)} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda \quad + \quad f_i e_i f_i^{\lambda(h_i)} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda \quad + \quad f_i \big([e_i, f_i] + f_i e_i\big) f_i^{\lambda(h_i)-1} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda \quad + \quad f_i^2 e_i f_i^{\lambda(h_i)-1} v^\lambda \\
&\vdots \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda + \cdots + f_i^{\lambda(h_i)} h_i v^\lambda \quad + \quad f_i^{\lambda(h_i)+1} \cancel{e_i v^\lambda} \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda + \cdots + f_i^{\lambda(h_i)} h_i v^\lambda \\
&= (\lambda - \lambda(h_i)\alpha_i)(h_i) f_i^{\lambda(h_i)} v^\lambda \\
&\quad + \big(\lambda - (\lambda(h_i) - 1)\alpha_i\big)(h_i) f_i^{\lambda(h_i)} v^\lambda \\
&\quad\quad + \cdots \\
&= \sum_{k=0}^{\lambda(h_i)} \big(\lambda - (\lambda(h_i) - k)\alpha_i\big)(h_i) \cdot f_i^{\lambda(h_i)} v^\lambda \\
&= f_i^{\lambda(h_i)} v^\lambda \sum_{k=0}^{\lambda(h_i)} \lambda(h_i) - \lambda(h_i)\alpha_i(h_i) + k\alpha_i(h_i) \\
&= f_i^{\lambda(h_i)} v^\lambda \sum_{k=0}^{\lambda(h_i)} \lambda(h_i) - 2\lambda(h_i) + 2k \\
&= f_i^{\lambda(h_i)} v^\lambda \left(-\lambda(h_i)(\lambda(h_i) + 1) + 2 \cdot \frac{1}{2} \lambda(h_i)(\lambda(h_i) + 1) \right) \\
&= 0,
\end{aligned}$$

completing our check that $\mathfrak{n}^+ f_i^{\lambda(h_i)+1} v^\lambda$. Hence, by Fact K8.27, the submodule generated by this vector is a Verma module of the form advertised. ■

[9] Probably I am missing some easier way to do this, but this certainly works.

Recall that L_λ is a quotient of a Verma module by a maximal submodule not containing the highest weight vector; in fact, this can also be "refined" to

Lemma (K8.28b)**.** For $\lambda \in P_+$,

$$\Pi_\lambda = M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda},$$

where by "$M_{s_i\circ\lambda}$" we mean the submodule $\mathrm{Span}_{U\mathfrak{n}^-}(f_i^{\lambda(h_i)+1} v^\lambda) \subset M_\lambda$ constructed in the previous lemma which is isomorphic to $M_{s_i\circ\lambda}$.

Proof of K8.28b. We will show the RHS is a finite-dimensional representation and then appeal to complete reducibility.

To see that

$$\dim_{\mathbb{C}} M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda} < \infty,$$

note

$$M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda} \overset{\mathsf{Vec}}{\cong} U\mathfrak{n}^- \Big/ U\mathfrak{n}^- \langle f_i^{\lambda(h_i)+1} \rangle_{i \in I(\Sigma)}$$

where we have used $M_\lambda \cong U\mathfrak{n}^-$ again, as well as using that $M_{s_i\circ\lambda} \subset M_\lambda$ is generated by $f_i^{\lambda(h_i)+1} v^\lambda$. But recall that an ideal of form $U\mathfrak{n}^-\langle f_i^{\lambda(h_i)+1}\rangle_{i\in I(\Sigma)}$ has finite codimension

$$\mathrm{codim}_{U\mathfrak{n}^-} U\mathfrak{n}^- \langle f_i^{\lambda(h_i)+1} \rangle_{i \in I(\Sigma)} < \infty,$$

which shows finite-dimensionality.

Then, by complete reducibility, we may write

$$M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda} \cong \bigoplus_{\mu \leq \lambda} m_\mu \Pi_\mu,$$

where we know $\mu \leq \lambda$ since the left hand side is a quotient of a Verma module and so its weights must be of form $\lambda - Q_+$. Moreover, by looking at the dimension of the λ-space, since the LHS has $\dim(M_\lambda / \sum_i M_{s_i\circ\lambda})^\lambda = 1$ (the quotient not containing the highest weight vector), we force $m_\lambda = 1$:

$$M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda} \cong L_\lambda \oplus \bigoplus_{\mu < \lambda} m_\mu \Pi_\mu.$$

In particular, this means the highest weight vector $\widetilde{v}^\lambda$ of $M_\lambda / \sum_i M_{s_i\circ\lambda}$ lies inside the factor Π_λ, so that the submodule inside generated by $\widetilde{v}^\lambda$ must be Π_λ (by irreducibility of Π_λ). On the other hand, as v^λ generates M_λ, we have $\widetilde{v}^\lambda$ generates all of $M_\lambda / \sum_i M_{s_i\circ\lambda}$. Hence we have $M_\lambda / \sum_i M_{s_i\circ\lambda} \cong \Pi_\lambda$, as claimed. ∎

Indeed, the failure of the $\sum_{i\in I(\Sigma)}$ to be a direct sum is why the rest of the terms in BGG are necessary. That is, K8.28b allows us to describe Π_λ as a cokernel

$$\bigoplus_{i \in I(\Sigma)} M_{s_i \circ \lambda} \longrightarrow M_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0,$$

but says nothing about what the kernel is. BGG will do this for us. Note well that the map $\bigoplus_{i\in I(\Sigma)} M_{s_i\circ\lambda} \longrightarrow M_\lambda$ is given by the canonical embeddings[10] $M_{s_i\circ\lambda} \subset M_\lambda$, as prescribed by

[10]Here we have chosen $\mathrm{sgn}(s_i, \mathrm{id}) = +1$ for all $i \in I(\Sigma)$, which might seem to be an issue given that in our statement of Lemma 10.4 we do not have any choice on what the signs are; however, by looking at the proof of Lemma 10.4 in section 11 of BGG, one sees that the proof goes by induction on $\ell(w)$, where at the base case (i.e. in the case of $\mathrm{sgn}(s_i, \mathrm{id})$) we are free to choose whatever signs we fancy; so in particular we may take them all to be positive signs, as we do here.

Verma (Section 2 above); this agrees with the differential maps we described in the statement of BGG.

Step 3: Exact Everywhere: Now that we know this is a complex which is exact at degrees 0 and -1, we will cite three key lemmas (to be proved later this section) to show BGG is exact everywhere. The proof will be by induction:

$$\text{by induction, assume BGG is exact at degrees } -1, 0, 1, \cdots, k-1.$$

We wish to show exactness at degree k also. Since this is a complex, we already have $\mathrm{d}_{k+1}(C_{k+1}) \subseteq \operatorname{Ker} \mathrm{d}_k \subseteq C_k$; we wish only to show

$$\mathrm{d}_{k+1} \colon C_{k+1} \overset{?}{\twoheadrightarrow} \operatorname{Ker} \mathrm{d}_k$$

is surjective, i.e. $\operatorname{Img} \mathrm{d}_{k+1} = \operatorname{Ker} \mathrm{d}_k$.

In the below, by $U\mathfrak{n}^-$-free, we mean free as a $U\mathfrak{n}^-$-module.

Lemma (10.5)**.** For $M, N \in \mathcal{O}$ such that M is $U\mathfrak{n}^-$-free with generators $v_1, \cdots, v_n$,

$$M = \operatorname{Span}_{U\mathfrak{n}^-}\{v_1, \cdots, v_n\}$$

and

$$\varphi \colon M \xrightarrow{U\mathfrak{n}^-} N$$

a map of $U\mathfrak{n}^-$-modules such that

$$\varphi(v_i) \text{ is a weight vector,}$$

we have

$$\varphi \colon M \twoheadrightarrow N \text{ surj} \iff \widetilde{\varphi} \colon M/\mathfrak{n}^- M \twoheadrightarrow N/\mathfrak{n}^- N \text{ surj.}$$

In particular we will be interested in applying 10.5 for $M = C_{k+1} = \bigoplus_{w \in W_{k+1}} M_{w \circ \lambda}$ and $N = \operatorname{Ker} \mathrm{d}_k$.

Lemma (10.6)**.**

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}/\mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \text{ inj}$$

is injective.

In fact, since we are inducting on k (i.e. to get to the k-th step we have proved this lemma for every $i < k$), this will be true of all $\widetilde{d_i}$ for $i \leq k+1$ (sort of like strong induction).

Lemma (10.7)**.**

$$\dim_{\mathbb{C}} C_{k+1}/\mathfrak{n}^- C_{k+1} = \dim_{\mathbb{C}} \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k < \infty.$$

Assuming these lemmas, it is clear we can now show exactness at k, completing induction. Indeed, 10.6 gives an injection

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}/\mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$$

which by 10.7 is an injection between two finite-dimensional vector spaces of the same dimension; therefore $\widetilde{\mathrm{d}}_{k+1}$ is also surjective

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}/\mathfrak{n}^- C_{k+1} \hookrightarrow\!\!\!\!\!\rightarrow \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k,$$

which by 10.5 implies

$$\mathrm{d}_{k+1} \colon C_{k+1} \twoheadrightarrow \operatorname{Ker} \mathrm{d}_k$$

is also surjective, completing the proof of exactness. This concludes the proof of BGG (Lemmas to be proved later). ■

4.2. **Weak BGG implies Lemmas.** In this subsection we will prove the Lemmas 10.5, 10.6, 10.7 cited above. The title is slightly misleading; 10.5 does not require Weak BGG, 10.6 does, and 10.7 requires Bott's Theorem, a corollary of Weak BGG.

We will restate the Lemmas each time so the viewer does not have to scroll up.

4.2.1. *Lemma 10.5.* First let us show 10.5.

Lemma (10.5)**.** For $M, N \in \mathcal{O}$ such that M is $U\mathfrak{n}^-$-free with generators $v_1, \cdots, v_n$,

$$M = \mathrm{Span}_{U\mathfrak{n}^-}\{v_1, \cdots, v_n\}$$

and

$$\varphi\colon M \xrightarrow{U\mathfrak{n}^-} N$$

a map of $U\mathfrak{n}^-$-modules such that

$$\varphi(v_i) \text{ is a weight vector,}$$

we have

$$\varphi\colon M \twoheadrightarrow N \ \text{ surj} \iff \widetilde{\varphi}\colon M/\mathfrak{n}^- M \twoheadrightarrow N/\mathfrak{n}^- N \ \text{ surj.}$$

Proof of 10.5. $\implies$ It is clear that φ surjective implies $\widetilde{\varphi}$ surjective (since φ commutes with $\mathfrak{n}^-$, it is not possible for a nonzero person in $N/\mathfrak{n}^-N$ to have preimage in $\mathfrak{n}^-M$; indeed, the image of $\mathfrak{n}^-M$ lies inside $\mathfrak{n}^-N$).

$\impliedby$ Now let us see that $\widetilde{\varphi}$ surjective implies φ surjective. We will show that any weight vector in N is actually in the image of φ, which implies the desired[11]; that is, we claim

$$u \in N^\mu \implies u \in \mathrm{Img}\,\varphi \qquad \forall\, \mu.$$

The idea is to proceed by induction/infinite descent (not sure what to call this) on the weights of N. That is, since $N \in \mathcal{O}$, we know the set of weights of N lie in a finite union of cones (see Section 2):

$$\mathrm{wt}\, N \subseteq \bigcup_{i\, \mathrm{fnt}:\lambda_i \in \mathfrak{h}^*} (\lambda_i - Q_+).$$

We will start by showing $N^{\lambda_i} \subseteq \mathrm{Img}\,\varphi$ and work our way downwards. In fact, the base case of the highest weights λ_i is a formal consequence of the inductive step, as we will highlight below.

Hence, for the inductive step let us pick a weight μ so that all weights above μ are contained in $\mathrm{Img}\,\varphi$:

$$\implies \text{pick } \ u \in N^\mu : N^{>\mu} \subseteq \mathrm{Img}\,\varphi.$$

Under projection by $\mathfrak{n}^-$ this goes to

$$\begin{aligned} \pi\colon N &\twoheadrightarrow N/\mathfrak{n}^-N \\ u &\longmapsto \widetilde{u}. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{\varphi} \ \text{ surj} &\implies \widetilde{\varphi}(v_1), \cdots, \widetilde{\varphi}(v_n) \ \text{ generate } N/\mathfrak{n}^-N \\ &\implies \widetilde{u} = \sum_i c_i \widetilde{\varphi}(v_i) \end{aligned}$$

[11]Since weight vectors form an eigenbasis with respect to $\mathfrak{h}$, by definition.

for some coefficients c_i. Note[12] that $\mathfrak{h}$ acts on $N/\mathfrak{n}^-N$, with[13]

$$\pi(u) = \widetilde{u} \in (N/\mathfrak{n}^-N)^\mu,$$

i.e.

$$\sum_i c_i \widetilde{\varphi}(v_i) \in (N/\mathfrak{n}^-N)^\mu.$$

However, from linear algebra we know the sum of eigenvectors with differing eigenvalues is not an eigenvector[14], which forces

$$c_i \neq 0 \implies \mathrm{wt}(\widetilde{\varphi}(v_i)) = \mu.$$

Consider $u - \sum_i c_i\varphi(v_i)$. We've noted in an earlier footnote that quotienting (which a priori commutes with $\mathfrak{n}^-$) commutes with $\mathfrak{h}$, so that

$$\mathrm{wt}(\widetilde{\varphi}(v_i)) = \mu \implies \mathrm{wt}(\varphi(v_i)) = \mu$$

and therefore

$$u - \sum_i c_i\varphi(v_i) \in N^\mu$$

since each u and $\varphi(v_i)$ is in N^μ. Since $\pi(u - \sum_i c_i\varphi(v_i)) = 0$, we also have

$$u - \sum_i c_i\varphi(v_i) \in \mathfrak{n}^-N,$$

so

$$\implies u - \sum_i c_i\varphi(v_i) \in N^\mu \cap \mathfrak{n}^-N.$$

Hence we can write $u - \sum_i c_i\varphi(v_i)$ as

$$u - \sum_i c_i\varphi(v_i) = \sum_{\alpha\in R_+} c_\alpha f_\alpha w^{\mu+\alpha}$$

for some weight vectors $w^{\mu+\alpha} \in N^{\mu+\alpha}$ and constants c_α. Here we know we have factors of f_α since $u - \sum_i c_i\varphi(v_i) \in \mathfrak{n}^-N$, and we know the weights on w must be $\mu+\alpha$ since $u - \sum_i c_i\varphi(v_i) \in N^\mu$ and f_α drops the weight by α.

But

$$w^{\mu+\alpha} \in N^{>\mu} \subseteq \mathrm{Img}\,\varphi,$$

which is a submodule of N, so $f_\alpha w^{\mu+\alpha} \in \mathrm{Img}\,\varphi$, and

$$\implies u - \sum_i c_i\varphi(v_i) = \sum_{\alpha\in R_+} c_\alpha f_\alpha w^{\mu+\alpha} \in \mathrm{Img}\,\varphi$$

$$\implies u \in \mathrm{Img}\,\varphi,$$

completing the proof.

Note well that, in the above argument, if $\mu = \lambda_i$ is a maximal weight, then $w^{\mu+\alpha} = 0$ and we immediately have $u - \sum_i c_i\varphi(v_i) = 0 \implies u = \sum_i c_i\varphi(v_i) \in \mathrm{Img}\,\varphi$, so that indeed the base case is subsumed by the inductive step. ■

[12] This is since

$$\mathfrak{h}\mathfrak{n}^-N = [\mathfrak{h},\mathfrak{n}^-]N + \mathfrak{n}^-\mathfrak{h}N \subseteq \mathfrak{n}^-N + \mathfrak{n}^-N = \mathfrak{n}^-N,$$

so that $\mathfrak{h}$ acts on $\mathfrak{n}^-N$ (i.e. $\mathfrak{n}^-N$ is closed under $\mathfrak{h}$), so that $\mathfrak{h}$ acts on $N/\mathfrak{n}^-N$.

[13] We know $\mathfrak{h}\pi(u) = \pi(\mathfrak{h}u)$ since $\mathfrak{h}\colon \mathfrak{n}^-N \longrightarrow \mathfrak{n}^-N$; then

$$\mathfrak{h}\pi(u) = \pi(\mathfrak{h}u) = \mu(\mathfrak{h})\pi(u) \implies \pi(u) \in (N/\mathfrak{n}^-N)^\mu.$$

[14] In other words, eigenvectors of differing eigevalues are linearly independent.

4.2.2. *Lemma 10.6.* Next let us do 10.6. This lemma will be broken up into two parts, 10.6a and 10.6b (as they are named in BGG). To prove this lemma we will need to cite Weak BGG as well as some facts about the Jordan-Holder decomposition of Verma modules. First let me recall what Jordan-Holder is:

Definition. A "decomposition series" (or composition series) of $M \in \mathsf{Mod}_R$ is a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

such that

$$M_i/M_{i-1} \text{ is simple.}$$

We will denote the set of such simple quotients by

$$\mathrm{JH}(M) \coloneqq \{M_i/M_{i-1}\}_i,$$

which we will call the "Jordan-Holder factors".

It is a theorem that this $\mathrm{JH}(M)$ is well-defined. Concretely one can think about this in two ways: to get a decomposition series, one may either keep taking maximal submodules[15] to obtain the desired filtration, or one may do the following process: take a maximal simple submodule of M, Π_1, and pass to M/Π_1; then take a maximal simple submodule Π_2 of V/Π_1, and pass to $(M/\Pi_1)/\Pi_2$; etc.. The filtration is then $F_0 = 0$, $F_1 = \Pi_1$, $F_2 = \pi_1^{-1}(\Pi_2)$, etc..

There are three key things we must cite for this subsubsection. The first is the Jordan-Holder Theorem (a standard fact from homological algebra which takes about a page and a half to prove), the second is the Jordan-Holder factors of a Verma module, and the third is Weak BGG. The former two will not be proved in this exposition, while the third will be proved in a later section.

First, the Jordan-Holder Theorem says[16]

Theorem (Jordan-Holder)**.** Given two filtrations of M of possibly different length

$$0 = M_0 \subseteq \cdots \subseteq M_n = M,$$
$$0 = M_0' \subseteq \cdots \subseteq M_m' = M,$$

we may refine them (i.e. stick in extra terms) to obtain two filtrations of equal length

$$0 = N_0 \subseteq \cdots \subseteq N_k = M,$$
$$0 = N_0' \subseteq \cdots \subseteq N_k' = M$$

such that

$$\{N_i/N_{i-1}\}_i \simeq \{N_j'/N_{j-1}'\}_j,$$

where by $\simeq$ we mean the N_i/N_{i-1} are a permutation of the N_j'/N_{j-1}' up to isomorphism.

Moreover, the following are equivalent:

- M admits a decomposition series;
- Every filtration of M can be refined to a decomposition series;
- M is both Noetherian and Artinian.

Here recall from commutative algebra that Noetherian is "ascending chains terminate" and Artinian is "descending chains terminate". Some additional easy facts about JH factors:

[15] If N is a maximal submodule of M, then M/N is simple since the existence of a nontrivial submodule of M/N contradicts the maximality of N.

[16] This statement is taken from the second page of Benson.

Fact. For $M, N \in \mathsf{Mod}_R$ which admit decomposition series,

$$\begin{aligned} \mathrm{JH}(M \oplus N) &= \mathrm{JH}(M) \sqcup \mathrm{JH}(N), \\ \mathrm{JH}(M) &= \mathrm{JH}(M/N) \sqcup \mathrm{JH}(N), \\ \mathrm{JH}(M) &\supseteq \mathrm{JH}(N \subseteq M), \\ \mathrm{JH}(M) &= \bigsqcup_i \mathrm{JH}(M_i/M_{i-1}), \end{aligned}$$

where in the middle two $N \subseteq M$ is a submodule and in the last fact

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

is any filtration (not necessarily a decomposition series).

These facts are pretty easy to exhibit[17].

Second, we must also cite another fact about the Jordan-Holder factors of Verma modules:

Theorem (Jordan-Holder of Verma, 8.12)**.** For $\lambda, \mu \in \mathfrak{h}^*$,

$$L_\mu \in \mathrm{JH}(M_\lambda)$$

$$\Updownarrow$$

$$\begin{aligned} &\exists\ \alpha_1, \cdots, \alpha_k \in R_+ \\ &: \mu + \rho = s_{\alpha_k} \cdots s_{\alpha_1}(\lambda + \rho), \\ &\quad s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda + \rho) - s_{\alpha_i} \cdots s_{\alpha_1}(\lambda + \rho) \in \mathbb{Z}_{0+}\alpha_i. \end{aligned}$$

For $\lambda \in P_+$,

$$L \in \mathrm{JH}(M_{w \circ \lambda}) \implies L = L_{u \circ \lambda},\ u \geq w,$$

with $L_{w \circ \lambda}$ appearing exactly once.

Lastly, we must also cite Weak BGG, to be proved in a later section. To set up Weak BGG we must first define the notion of "type":

---

[17]The first fact follows by taking as filtration

$$0 = \underbrace{M_0 \oplus N_0 \subseteq \cdots \subseteq M_0 \oplus N_n}_{\mathrm{JH}\, N} = \underbrace{M_0 \oplus N \subseteq \cdots \subseteq M_m \oplus N}_{\mathrm{JH}\, M} = M \oplus N,$$

the second follows by taking

$$0 = \underbrace{N_0 \subseteq \cdots \subseteq N_n}_{\mathrm{JH}\, N} = N = \underbrace{M_0 \subseteq \cdots \subseteq M_m}_{\mathrm{JH}(M/N)} = M,$$

the third follows formally from the second, and the fourth follows by using

$$0 = \widetilde{N}_0 \subseteq \cdots \subseteq \widetilde{N}_n = M_i/M_{i-1}$$

to refine

$$0 = M_0 \subseteq \cdots \subseteq M_m = M$$

to

$$\cdots \subseteq M_{i-1} = \underbrace{N_0 \subseteq \cdots \subseteq N_n}_{\mathrm{JH}(M_i/M_{i-1})} = M_i \subseteq \cdots .$$

Definition. For $M \in \mathcal{O}$ and $\Psi = \{\psi_i\}$ a finite set of weights (with multiplicity, i.e. possibly with repetition), we say M is of type Ψ,

$$\operatorname{typ} M = \Psi,$$

if there exists a filtration of M

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

such that the factors

$$M_i / M_{i-1} \cong M_{\psi_i}$$

precisely exhibit all of Ψ.

Now we may state Weak BGG:

Theorem (Weak BGG, 9.9)**.** For $\lambda \in P_+$ and $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$, there is a resolution of $\mathfrak{g}$-modules

$$0 \longrightarrow B_{|R_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

such that

$$\operatorname{typ} B_k = \{w \circ \lambda : w \in W_k\}.$$

Let us remark that the BGG resolution (as one might expect, being stronger than Weak BGG) satisfies this, having terms $C_k = \bigoplus_{w \in W_k} M_{w \circ \lambda}$, so that we may take a filtration consisting e.g. of $M_{w_1 \circ \lambda} \subseteq M_{w_1 \circ \lambda} \oplus M_{w_2 \circ \lambda} \subseteq M_{w_1 \circ \lambda} \oplus M_{w_2 \circ \lambda} \oplus M_{w_3 \circ \lambda} \subseteq \cdots$ realizing $\operatorname{typ} C_k = \{w \circ \lambda\}_{w \in W_k}$.

In the spirit of working backwards, we will state two lemmas, 10.6a and 10.6b in BGG, and show how they imply 10.6, and prove the two smaller lemmas later this section.

Lemma (10.6a)**.**

$$L \in \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) \implies L = L_{w \circ \lambda},\ \ell(w) > k.$$

Lemma (10.6b)**.** Let $w_0 \in W$, $M \in \mathcal{O}$ be such that

$$\ell(w) \geq \ell(w_0) \quad \forall\, L_{w \circ \lambda} \in \mathrm{JH}(M).$$

Then, for

$$\varphi \colon M_{w_0 \circ \lambda} \xrightarrow{U\mathfrak{g}} M$$

a map of representations, we have

$$\varphi(v^{w_0 \circ \lambda}) \neq 0 \implies \widetilde{\varphi(v^{w_0 \circ \lambda})} \neq 0 \in M / \mathfrak{n}^- M.$$

Note that the statement in 10.6b can also be written

$$\varphi(v^{w_0 \circ \lambda}) \neq 0 \implies \varphi(v^{w_0 \circ \lambda}) \notin \mathfrak{n}^- M.$$

Let us see how this implies Lemma 10.6:

Lemma (10.6)**.**

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1} \big/ \mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker} \mathrm{d}_k \big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \quad \text{inj}$$

is injective.

Proof of 10.6, assuming 10.6a/b. Since each term C_{k+1} is $C_{k+1} = \bigoplus_{w \in W_{k+1}} M_{W \circ \lambda}$ with $M_{w \circ \lambda}$ generated over $U\mathfrak{n}^-$ by $v^{w \circ \lambda}$ the highest weight vector of weight $w \circ \lambda$, we can write

$$C_{k+1} = U\mathfrak{n}^- \{v^{w \circ \lambda}\}_{w \in W_{k+1}}$$

where by not writing Span we are indicating that this is the free[18] module generated over $U\mathfrak{n}^-$. Then, modding out by the ideal $\mathfrak{n}^- \overset{\bullet}{\subset} U\mathfrak{n}^-$ gives

$$\implies C_{k+1}/\mathfrak{n}^- C_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$$

is isomorphic to a vector space with basis $\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$.

The claim is that, to show $\operatorname{Ker}\widetilde{\mathrm{d}}_{k+1} = 0$, it suffices to show $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0$; that is,

$$\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0 \ \ \forall\, w \in W_{k+1} \implies \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) \neq 0 \ \ \forall\, \widetilde{v} \neq 0.$$

To see this, note[19] that $\widetilde{\mathrm{d}}_{k+1}$ commutes with $\mathfrak{h}$, and that the basis of the domain of $\widetilde{\mathrm{d}}_{k+1}$, $\{\widetilde{v}^{w\circ\lambda}\}$, consists of eigenvectors of different[20] weights; then, since $\widetilde{\mathrm{d}}_{k+1}$ commutes with $\mathfrak{h}$, the nonzero vectors $\{\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda})\}$ are still eigenvectors in $\operatorname{Ker}\mathrm{d}_k/\mathfrak{n}^-\operatorname{Ker}\mathrm{d}_k$ of differing eigenvalues.

AFSOC some nonzero $\widetilde{v} = \sum_w c_w \widetilde{v}^{w\circ\lambda} \neq 0$ had $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) = 0$; then

$$\begin{aligned} \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) &= 0 \\ \widetilde{\mathrm{d}}_{k+1}\left(\sum_w c_w \widetilde{v}^{w\circ\lambda}\right) &= \\ \sum_w c_w \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) &= \end{aligned}$$

i.e. a nontrivial linear combination of $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda})$ vanishes, contradicting that eigenvectors of differing eigenvalues are linearly independent. Hence we see that $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0 \implies \widetilde{\mathrm{d}}_{k+1}(\neq 0) \neq 0$, so it suffices to show $\widetilde{\mathrm{d}}_{k+1}$ does not vanish on weight vectors.

Now let us appeal to 10.6a and 10.6b. Take $M = \operatorname{Ker}\mathrm{d}_k$ with any $w_0 \in W_{k+1}$ in the setup of 10.6b; this satisfies the problem conditions by 10.6a. To be more explicit, by 10.6a, $L_{w\circ\lambda} \in \mathrm{JH}(\operatorname{Ker}\mathrm{d}_k) \implies \ell(w) \geq k+1 = \ell(w_0)$.

By our work at the beginning of this proof, it suffices to show that $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w_0\circ\lambda}) \neq 0$ for any $w_0 \in W_{k+1}$. Since $C_{k+1} = \bigoplus_{w_0\in W_{k+1}} M_{w_0\circ\lambda}$, it suffices to show the differential restricted to each summand

$$\mathrm{d}_{k+1}\big|_{w_0} \colon M_{w_0\circ\lambda} \longrightarrow \operatorname{Ker}\mathrm{d}_k$$

has

$$\mathrm{d}_{k+1}(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k.$$

But this follows from 10.6b, since by construction $\mathrm{d}_{k+1}(v^{w_0\circ\lambda}) \neq 0$, as recall $\mathrm{d}_{k+1}|_{w_0}$ is defined by the direct sum of the canonical embeddings of Verma modules.

This concludes 10.6. ■

And now we must prove 10.6a and 10.6b. For these two we shall require the Jordan-Holder stuff, and for 10.6a we shall moreover need Weak

---

[18]The different Verma modules don't talk to each other.

[19]Indeed, more generally for a map

$$\varphi\colon M \xrightarrow{U\mathfrak{n}^-} N$$

which commutes with $\mathfrak{h}$, we have

$$\widetilde{\varphi}\colon M/\mathfrak{n}^-M \longrightarrow N/\mathfrak{n}^-N$$

also commutes with $\mathfrak{h}$ since $\mathfrak{h}\widetilde{\varphi}(\widetilde{v}) = \mathfrak{h}((\varphi(v) + \mathfrak{n}^-N) = \mathfrak{h}\varphi(v) + \mathfrak{h}\mathfrak{n}^-N = \varphi(\mathfrak{h}v) + \mathfrak{h}\mathfrak{n}^-N \subseteq \varphi(\mathfrak{h}v) + \mathfrak{n}^-N = \widetilde{\varphi}(\mathfrak{h}\widetilde{v})$, where we recall from an earlier footnote (I think it's footnote 11) that $\mathfrak{h}\mathfrak{n}^-N \subseteq \mathfrak{n}^-N$.

In this case d_{k+1} is certainly a map of representations and so commutes with $\mathfrak{h}$, and therefore so does $\widetilde{\mathrm{d}}_{k+1}$.

[20]$w \circ \lambda = w' \circ \lambda \implies w = w'$.

Proof of 10.6a. Since $C_i = \bigoplus_{W_i} M_{w\circ\lambda}$, the JH factors of C_i is

$$\operatorname{JH} C_i = \operatorname{JH} \bigoplus_{w\in W_i} M_{w\circ\lambda} = \bigsqcup_{w\in W_i} \operatorname{JH} M_{w\circ\lambda}.$$

By Weak BGG, we get a resolution

$$0 \longleftarrow \Pi_\lambda \longleftarrow B_0 \longleftarrow \cdots \longleftarrow B_{|R_+|} \longleftarrow 0,$$

where since $\operatorname{typ} B_k = \{w\circ\lambda\}_{w\in W_k}$, there exists a filtration of B_k whose quotients are $M_{w\circ\lambda}$, so by a previously noted fact about JH factors we have

$$\operatorname{JH} B_i = \bigsqcup_j \operatorname{JH}(B_i^j/B_i^{j-1}) = \bigsqcup_{w\in W_i} \operatorname{JH} M_{w\circ\lambda},$$

whereupon

$$\implies \operatorname{JH} C_i = \operatorname{JH} B_i.$$

The idea is sort of like DNA/RNA[21], where we only know $C_\bullet$ is exact in degrees $\le k-1$, whereas $B_\bullet$ is exact everywhere. Let us begin:

$$\begin{aligned}
\Pi = B_0/\operatorname{Ker} \mathrm{d}_0^B = C_0/\operatorname{Ker}\mathrm{d}_0 &\implies \operatorname{JH}(B_0/\operatorname{Ker}\mathrm{d}_0^B) = \operatorname{JH}(C_0/\operatorname{Ker}\mathrm{d}_0)\\
(\operatorname{JH} B_0 = \operatorname{JH} C_0) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_0^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_0)\\
(\text{exact at } 0) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_1^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_1)\\
&\implies \operatorname{JH}(B_1/\operatorname{Ker}\mathrm{d}_1^B) = \operatorname{JH}(C_1/\operatorname{Ker}\mathrm{d}_1)\\
(\operatorname{JH} B_1 = \operatorname{JH} C_1) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_1^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_1)\\
(\text{exact at } 1) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_2^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_2)\\
&\implies \operatorname{JH}(B_2/\operatorname{Ker}\mathrm{d}_2^B) = \operatorname{JH}(C_2/\operatorname{Ker}\mathrm{d}_2)\\
(\operatorname{JH} B_2 = \operatorname{JH} C_2) &\implies \cdots\\
&\ \vdots\\
(\text{exact at } k-1) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_k^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_k)\\
&\implies \operatorname{JH}(B_k/\operatorname{Ker}\mathrm{d}_k^B) = \operatorname{JH}(C_k/\operatorname{Ker}\mathrm{d}_k)\\
(\operatorname{JH} B_k = \operatorname{JH} C_k) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k).
\end{aligned}$$

At this point we cannot proceed any further since we do not know $C_\bullet$ is exact at k. However, $B_\bullet$ *is* exact at k, so we get

$$\begin{aligned}
\operatorname{JH}(\operatorname{Ker}\mathrm{d}_k) &= \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k^B)\\
&= \operatorname{JH}(\operatorname{Img}\mathrm{d}_{k+1}^B)\\
&= \operatorname{JH}(B_{k+1}/\operatorname{Ker}\mathrm{d}_{k+1}^B)\\
&\subseteq \operatorname{JH} B_{k+1}\\
&= \bigsqcup_{w\in W_{k+1}} \operatorname{JH} M_{w\circ\lambda},
\end{aligned}$$

i.e.

$$\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k) \subseteq \bigsqcup_{w\in W_{k+1}} \operatorname{JH} M_{w\circ\lambda}.$$

[21] I don't remember any high school biology, but I think I remember the word "polymerase" or something...

By theorem 8.12, we know that $\mathrm{JH}\, M_{w\circ\lambda}$ contains things of form $L_{u\circ\lambda}$, where $u \geq w$; hence

$$L \in \mathrm{JH}(\mathrm{Ker}\, \mathrm{d}_k) \implies L = L_{u\circ\lambda},\ u \geq w \text{ for some } w \in W_{k+1},$$

which in particular means

$$L \in \mathrm{JH}(\mathrm{Ker}\, \mathrm{d}_k) \implies L = L_{u\circ\lambda},\ \ell(u) \geq k+1,$$

which is the statement of 10.6a. ■

10.6b, while also involving Jordan-Holder, will not involve Weak BGG.

Proof of 10.6b. We will prove this by induction on the number of JH factors, i.e. induction on $|\mathrm{JH}(M)|$.

Since $M \in \mathcal{O}$, the set of weights of M is contained in some finite union of cones $\lambda_i - Q_+$, so we may pick a vector of maximal weight:

$$\text{pick } \ v \in M^\mu : \mathfrak{n}^+ v = 0.$$

Denote by N the submodule inside M generated by v:

$$N := \mathrm{Span}_{U\mathfrak{g}}(v) \subseteq M;$$

as this is a submodule generated by a single highest weight vector, this is a highest weight representation which must therefore be a quotient of a Verma module

$$\implies N \cong M_\mu/\text{smth}.$$

In particular

$$\mathrm{JH}\, N \subseteq \mathrm{JH}\, M_\mu.$$

Since $v \in M^\mu$ is a[22] highest weight vector, it (or rather, its image under an appropriate quotient) also generates an irreducible L_μ "inside" N; that is, we can quotient $N \cong M_\mu/\text{smth}$ by something else to get to the quotient by the maximal submodule not containing the highest weight vector, which is L_μ. Hence

$$\implies L_\mu \in \mathrm{JH}\, N.$$

We will break into two cases, the second of which is induction/reduction. Consider

$$\varphi(v^{w_0\circ\lambda});$$

either this is contained in N or it isn't.

Case 1: Suppose it is

$$\varphi(v^{w_0\circ\lambda}) \in N.$$

Since φ is a map of representations, it preserves weights so that we have $\varphi(v^{w_0\circ\lambda}) \in M^{w_0\circ\lambda}$; moreover, it commutes with $\mathfrak{n}^+$, so that $\mathfrak{n}^+\varphi(v^{w_0\circ\lambda}) = \varphi(\mathfrak{n}^+ v^{w_0\circ\lambda}) = 0$. Therefore $\varphi(v^{w_0\circ\lambda})$ generates a submodule in N which is a highest weight module and therefore isomorphic to a quotient of $M_{w_0\circ\lambda}$. This quotient of $M_{w_0\circ\lambda}$ can be further quotiented to obtain $L_{w_0\circ\lambda}$, so that

$$L_{w_0\circ\lambda} \in \mathrm{JH}\, N \subseteq \mathrm{JH}\, M_\mu.$$

By Theorem 8.12 of BGG (JH factors of Verma), this implies

$$\implies \mu = w\circ\lambda \quad \text{for some } w \leq w_0.$$

But we saw earlier that $L_{w\circ\lambda} = L_\mu \in \mathrm{JH}\, N \subseteq \mathrm{JH}\, M$. By the conditions of 10.6b, this implies

$$\implies \ell(w) \geq \ell(w_0).$$

Combined with $w \leq w_0$, this forces

$$\implies w = w_0, \text{ and } \mu = w_0 \circ \lambda.$$

[22] Not *the*, but *a*.

Then

$$\varphi(v^{w_0\circ\lambda}) \in M^{w_0\circ\lambda} = M^{\mu}$$

is a vector inside a weight space of maximal weight, which implies

$$\implies \varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- M.$$

This concludes Case 1.

Case 2: We will reduce to a smaller JH size. Suppose

$$\varphi(v^{w_0\circ\lambda}) \notin N.$$

Then

$$\widetilde{\varphi(v^{w_0\circ\lambda})} \neq 0 \in M/N$$

where the tilda refers to the equivalence class under quotient $\pi\colon M \to M/N$. Since $\mathrm{JH}(M/N) \subset \mathrm{JH}(M)$ (clearly N is a nontrivial submodule), we have

$$\implies |\mathrm{JH}(M/N)| < |\mathrm{JH}(M)|,$$

so by applying the inductive hypothesis to

$$\pi \circ \varphi\colon M_{w_0\circ\lambda} \xrightarrow{U\mathfrak{g}} M/N$$

we have $\pi \circ \varphi(v^{w_0\circ\lambda}) \neq 0 \in M/N$ implies

$$\implies \pi\varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^-(M/N)$$

$$\implies \varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- M.$$

This completes the induction and therefore the lemma. ■

Having paid off our debts to 10.6a and 10.6b, we may continue to the third lemma in our path.

4.2.3. *Lemma 10.7.* The proof of this lemma will not directly involve Weak BGG, but will require Bott's Theorem (unnamed Corollary of Theorem 9.9 in BGG) on cohomology, which is a corollary of Weak BGG. First let us state this corollary:

Theorem (Bott)**.** For $\Pi \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$ a finite-dimensional irrep,

$$\dim H^k(\mathfrak{n}^- : \Pi) = |W_k|.$$

Here recall that, for any $\mathfrak{g} \in \mathsf{LieAlg}$, Lie algebra cohomology was secretly the same as Ext, i.e.

$$H^k(\mathfrak{g} : M) = \mathrm{Ext}^k_{U\mathfrak{g}}(\mathbb{C}, M) \tag{$*$}$$

are canonically isomorphic. We will show how this follows from Weak BGG later.

We are now in a position to prove

Lemma (10.7)**.**

$$\dim_{\mathbb{C}} C_{k+1}\big/\mathfrak{n}^- C_{k+1} = \dim_{\mathbb{C}} \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k < \infty.$$

Proof of 10.7. We noted in the beginning of the proof of 10.6 that

$$C_{k+1}\big/\mathfrak{n}^- C_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$$

so that $C_{k+1}/\mathfrak{n}^- C_{k+1}$ is in particular finite-dimensional. Similarly, $\operatorname{Ker} \mathrm{d}_k \subseteq C_k = \bigoplus_{w \in W_k} M_{w \circ \lambda}$ is a submodule of a finitely-generated module over $U\mathfrak{n}^-$, which is a Noetherian ring; therefore[23][24], $\operatorname{Ker} \mathrm{d}_k$ is also finitely-generated over $U\mathfrak{n}^-$, which implies

$$\operatorname{Ker} \mathrm{d}_k / \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \quad \text{is a finite-dimensional vector space;}$$

let us take weight vector generators $v_1, \cdots, v_n \in \operatorname{Ker} \mathrm{d}_k$ so that

$$\operatorname{Ker} \mathrm{d}_k \Big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}_1, \cdots, \widetilde{v}_n\}.$$

In summary this gives

$$\implies \dim_{\mathbb{C}} C_{k+1} \Big/ \mathfrak{n}^- C_{k+1}, \dim_{\mathbb{C}} \operatorname{Ker} \mathrm{d}_k \Big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k < \infty.$$

Now we will show these dimensions are moreover equal by passing through $\dim \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda)$ and using Bott's Theorem. Define

$$D_{k+1} := U\mathfrak{n}^- \{g_1, \cdots, g_n\}$$

the free module generated over $U\mathfrak{n}^-$ with a map

$$\begin{aligned} \delta_{k+1} \colon D_{k+1} &\longrightarrow \operatorname{Ker} \mathrm{d}_k \\ g_i &\longmapsto v_i. \end{aligned}$$

Then, since

$$D_{k+1} \Big/ \mathfrak{n}^- D_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{g}_1, \cdots, \widetilde{g}_n\}$$

has the same dimension as $\operatorname{Ker} \mathrm{d}_k / \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$, we have

$$\widetilde{\delta}_{k+1} \colon D_{k+1} \Big/ \mathfrak{n}^- D_{k+1} \overset{\sim}{\hookrightarrow\!\!\!\!\rightarrow} \operatorname{Ker} \mathrm{d}_k \Big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$$

is in particular surjective. By 10.5[25] this implies

$$\implies \delta_{k+1} \colon D_{k+1} \twoheadrightarrow \operatorname{Ker} \mathrm{d}_k \quad \text{surj},$$

so that the sequence

$$D_{k+1} \xrightarrow{\delta_{k+1}} C_k \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

is exact[26].

The idea is to extend this exact sequence even further. Now let us take a free resolution (in the category of $\mathfrak{n}^-$-modules, i.e. the terms are free over $U\mathfrak{n}^-$) of $\operatorname{Ker} \delta_{k+1}$:

$$\cdots \longrightarrow D_{k+3} \longrightarrow D_{k+2} \xrightarrow{\delta_{k+2}} \operatorname{Ker} \delta_{k+1} \longrightarrow 0$$

so that we may extend

$$\cdots \longrightarrow D_{k+2} \xrightarrow{\delta_{k+2}} D_{k+1} \xrightarrow{\delta_{k+1}} C_k \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \Pi_\lambda \longrightarrow 0 \qquad \text{exact.}$$

Note well that this resolution has terms which are $U\mathfrak{n}^-$-free modules: the $D_\bullet$ are $U\mathfrak{n}^-$-free by construction, and the $C_\bullet$ are $U\mathfrak{n}^-$-free since they are direct sums of Vermas, which are $U\mathfrak{n}^-$-free.

---

[23]Being a submodule of a Noetherian module; recall Noetherian module is equivalent to all submodules being finitely-generated.

[24]Alternatively, $C_k \in \mathcal{O}$ since it is a direct sum of modules in $\mathcal{O}$, and $\operatorname{Ker} \mathrm{d}_k$, being a submodule of an object in $\mathcal{O}$, is also in $\mathcal{O}$, which implies it is $U\mathfrak{g}$-finitely-generated; moreover it is locally $U\mathfrak{n}^+$-finite, so that when we mod out by the ideal action by $\mathfrak{n}^-$, we obtain a finite-dimensional space.

[25]This clearly satisfies the conditions of 10.5, as D_{k+1} is a free $U\mathfrak{n}^-$-module with the image of each generator $\delta_{k+1}(g_i) = v_i$ a weight vector.

[26]Recall it is exact from -1 to $k-1$ by induction.

Recall that in general $M \otimes_R R/I \cong M/IM$; in particular, since $\mathbb{C} \cong U\mathfrak{n}^-/(\mathfrak{n}^- U\mathfrak{n}^-)$ as $\mathfrak{n}^-$-modules, we have

$$\mathbb{C} \otimes_{U\mathfrak{n}^-} M \cong M\big/\mathfrak{n}^- M.$$

Let us compute

$$\mathrm{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda)$$

by resolving the second term like we did above. Then by definition Tor is the homology of the complex

$$\cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+2} \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+1} \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} C_k \longrightarrow \cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} C_0 \longrightarrow 0,$$

which is

$$\cdots \longrightarrow D_{k+2}\big/\mathfrak{n}^- D_{k+2} \xrightarrow{\widetilde{\delta}_{k+2}} D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} C_k\big/\mathfrak{n}^- C_k \longrightarrow \cdots \longrightarrow C_0\big/\mathfrak{n}^- C_0 \longrightarrow 0.$$

Hence

$$\begin{aligned} \Longrightarrow \ \mathrm{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) &= H_{k+1}(\to \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+1} \to) \\ &= H_{k+1}(\to D_{k+1}/\mathfrak{n}^- D_{k+1} \to) \\ &= \mathrm{Ker}\, \widetilde{\delta}_{k+1}\big/\mathrm{Img}\, \widetilde{\delta}_{k+2}. \end{aligned}$$

We claim that

$$\widetilde{\delta}_{k+1} = \widetilde{\delta}_{k+2} = 0.$$

To see the second one, apply $\mathbb{C} \otimes \square$ to the exact sequence

$$D_{k+2} \xrightarrow{\delta_{k+2}} D_{k+1} \xrightarrow{\delta_{k+1}} \mathrm{Ker}\, \mathrm{d}_k \longrightarrow 0 \qquad \text{exact}$$

$$\Big\Downarrow \mathbb{C} \otimes_{\mathfrak{n}^-} \square$$

$$D_{k+2}\big/\mathfrak{n}^- D_{k+2} \xrightarrow{\widetilde{\delta}_{k+2}} D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} \mathrm{Ker}\, \mathrm{d}_k\big/\mathfrak{n}^- \mathrm{Ker}\, \mathrm{d}_k \longrightarrow 0 \qquad \text{exact,}$$

where we know the result is still exact since in general tensor products are right-exact. But we saw earlier that $\widetilde{\delta}_{k+1}$ is an isomorphism, which forces

$$\Longrightarrow \ \widetilde{\delta}_{k+2} = 0.$$

To see the first one, similarly apply $\mathbb{C} \otimes_{\mathfrak{n}^-} \square$ to

$$D_{k+1} \xrightarrow{\delta_{k+1}} C_k \xrightarrow{\mathrm{d}_k} \mathrm{Ker}\, \mathrm{d}_{k-1} \longrightarrow 0 \qquad \text{exact}$$

$$\Big\Downarrow \mathbb{C} \otimes_{\mathfrak{n}^-} \square$$

$$D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} C_k\big/\mathfrak{n}^- C_k \xrightarrow{\widetilde{\mathrm{d}}_k} \mathrm{Ker}\, \mathrm{d}_{k-1}\big/\mathfrak{n}^- \mathrm{Ker}\, \mathrm{d}_{k-1} \longrightarrow 0 \qquad \text{exact.}$$

Strong induction (that is, to get to the point where we know exactness at $-1, \cdots, k-1$, we must have along the way shown Lemma 10.6[27], that $\widetilde{\mathrm{d}}_{i+1}$ is injective, for all $i < k$) on Lemma 10.6 tells us $\widetilde{\mathrm{d}}_k$ is injective, which also forces

$$\Longrightarrow \ \widetilde{\delta}_{k+1} = 0.$$

That $\widetilde{\delta}_{k+1} = \widetilde{\delta}_{k+2} = 0$ implies

$$\mathrm{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) = \mathrm{Ker}\, 0/\,\mathrm{Img}\, 0 = D_{k+1}\big/\mathfrak{n}^- D_{k+1},$$

[27] And 10.5 and 10.7 as well, but 10.6 is the relevant one here.

which, since (recall from the beginning of this proof) $D_{k+1}/\mathfrak{n}^- D_{k+1} \cong \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$, implies

$$\implies \operatorname{Ker} \mathrm{d}_k \big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \cong \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda).$$

Now we need to cite[28] a fact from homological algebra:

Fact. For any Lie algebra $\mathfrak{g}$ and M, N left $U\mathfrak{g}$-modules, let $M^\dagger$ be the right $U\mathfrak{g}$-module defined by $v \cdot \xi := -\xi \cdot v$ (for $\xi \in \mathfrak{g}$, extended to $U\mathfrak{g}$ appropriately) and $M^* = \operatorname{Hom}(M, \mathbb{C})$ be the right $U\mathfrak{g}$-module which is the dual representation. Then

$$\operatorname{Ext}^n_{U\mathfrak{g}}(M, N) \cong \operatorname{Tor}_n^{U\mathfrak{g}}(N^*, M)^* \cong \operatorname{Tor}_n^{U\mathfrak{g}}(M^\dagger, N^{\dagger,*})^*.$$

In particular, let us apply this to $M = \mathbb{C}$ and $N = \Pi$. Since $\mathbb{C}$ is the trivial representation, we have $\mathbb{C} = \mathbb{C}^* = \mathbb{C}^\dagger$. Meanwhile, since Π_λ^* is the representation on which ξ acts by $-\rho_\Pi(\xi)$, we have $\Pi^{*,\dagger} \cong \Pi^*$ as vector spaces is also irreducible (that Π_λ^* is irreducible requires complete reducibility, so that $\operatorname{End}(\Pi_\lambda^*) = \mathbb{C} \implies \Pi_\lambda^* \in \mathsf{irRep}\,\mathfrak{g}$). Then we have

$$\begin{aligned} \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) &\cong \operatorname{Ext}^{k+1}_{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda^{*,\dagger})^* \\ &\cong H^{k+1}(\mathfrak{n}^- : \Pi_\lambda^{*,\dagger})^*, \end{aligned}$$

whereupon by Bott's Theorem

$$\implies \dim \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) = \dim H^{k+1}(\mathfrak{n}^- : \Pi_\lambda^{*,\dagger}) = |W_{k+1}| = \dim C_{k+1}\big/\mathfrak{n}^- C_{k+1},$$

which combined with the previous $\operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \cong \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda)$ implies

$$\implies \dim \operatorname{Ker} \mathrm{d}_k \big/ \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k = \dim \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) = |W_{k+1}| = \dim C_{k+1}\big/\mathfrak{n}^- C_{k+1},$$

precisely as claimed by 10.7. This concludes. ■

At last, having proved 10.5, 10.6, and 10.7, we have shown how Weak BGG implies the full BGG. It lastly remains to show Weak BGG.

5. Proving Weak BGG

In this section we shall prove Weak BGG and also derive its corollary, Bott's Theorem (stated and used in the last section).

Recall that the Weak BGG theorem claimed the existence of a resolution of form

$$0 \longrightarrow B_{|R_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

such that

$$\operatorname{typ} B_k = \{w \circ \lambda : w \in W_k\}.$$

We will first exhibit this resolution for $\Pi_0 = \mathbb{C}$ the trivial (irreducible) representation, then use this exhibition to obtain this resolution for all other irreps.

[28] I think the most satisfying way to see this is to use that Tor and Ext, being derived functors, are universal delta functors; it then suffices to check these natural isomorphisms in degree 0, where it is very easy.

5.1. **General Lemmas.** First some generalities for arbitrary $\mathfrak{g}$. Recall the construction of a resolution of $\mathbb{C}$ by

$$\mathbb{C}_n \coloneqq U\mathfrak{g} \otimes \mathfrak{g}^{\wedge n}.$$

There is a variant of this which says

Theorem (9.1)**.** For any Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \subseteq \mathfrak{g}$ a subalgebra, there is a resolution of the trivial representation

$$\cdots \longrightarrow \mathbb{C}_2 \xrightarrow{\mathrm{d}_2} \mathbb{C}_1 \xrightarrow{\mathrm{d}_1} \mathbb{C}_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

whose terms are

$$\mathbb{C}_n \coloneqq U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n}$$

and whose differentials are

$$\mathrm{d}_n \colon U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n} \longrightarrow U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n-1}$$

$$\alpha \otimes \bigwedge \widetilde{\xi} \longmapsto \sum_{i=1}^{n} (-1)^{i+1} \alpha\xi_i \otimes \bigwedge_{\setminus i} \widetilde{\xi} + \sum_{1 \le i < j \le n} (-1)^{i+j} \alpha \otimes [\xi_i, \xi_j] \wedge \bigwedge_{\setminus i,j} \widetilde{\xi}.$$

Part of the theorem is that this is well-defined.

The well-defined-ness of the differential is an immediate check, and the proof of exactness is very similar to the one in Professor Gaitsgory's notes for M222, so we skip it here.

The idea is that we will use this resolution of $\mathbb{C}$ to build the weak BGG resolution of $\mathbb{C}$. First let us note

Fact (9.3)**.**

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} \square \colon \mathsf{Rep}\,\mathfrak{b} \xrightarrow{\text{exact}} \mathsf{Rep}\,\mathfrak{g}$$

is an exact functor. If $V \in \mathsf{Rep}\,\mathfrak{b}$ moreover has $\dim V = 1$, $\mathfrak{h}v = \lambda v$, and $\mathfrak{n}^+ v = 0$, then

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} V = M_\lambda.$$

This is pretty obvious since $U\mathfrak{g}$ is $U\mathfrak{b}$-free (recall PBW), and the second bit is definitional (we include it for completeness since BGG did). Next let us establish some facts about types in moving to construct weak BGG for $\mathbb{C}$.

Lemma (9.5)**.** For $N \in \mathsf{Rep}\,\mathfrak{b}$ such that $\dim N < \infty$ and N is $\mathfrak{h}$-semisimple, we have

$$\operatorname{typ}(U\mathfrak{g} \otimes_{U\mathfrak{b}} N) = \operatorname{wt} N.$$

Proof. Since $\mathfrak{b}$ is solvable, by Lie's theorem there exists a filtration of $\mathfrak{b}$-modules

$$\exists\, 0 = N_0 \subseteq \cdots \subseteq N_n = N$$

such that

$$\dim N_i/N_{i-1} = 1.$$

To exhibit $\operatorname{typ}(U\mathfrak{g} \otimes_{U\mathfrak{b}} N) = \operatorname{wt} N$, let us give a filtration

$$0 = U\mathfrak{g} \otimes_{U\mathfrak{b}} N_0 \subseteq \cdots \subseteq U\mathfrak{g} \otimes_{U\mathfrak{b}} N_n = U\mathfrak{g} \otimes_{U\mathfrak{b}} N,$$

where the quotients are

$$\Longrightarrow\ U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i \big/ U\mathfrak{g}\otimes_{U\mathfrak{b}} N_{i-1} \cong U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i \big/ N_{i-1}$$

since $U\mathfrak{g}\otimes_{U\mathfrak{b}}$ is exact and therefore preserves quotients:

$$0 \longrightarrow N_{i-1} \longrightarrow N_i \longrightarrow N_i \big/ N_{i-1} \longrightarrow 0$$

$$\Big\Downarrow U\mathfrak{g}\otimes_{U\mathfrak{b}}\square$$

$$0\longrightarrow U\mathfrak{g}\otimes_{U\mathfrak{b}} N_{i-1}\longrightarrow U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i\longrightarrow U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i\big/N_{i-1}\longrightarrow 0$$

which forces the last term of the second sequence to be $(U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i)/(U\mathfrak{g}\otimes_{U\mathfrak{b}} N_{i-1})$.

Recall the way Lie was proved was by induction[29] on $\dim V$ and using

$$\exists\, v : \mathfrak{b}v=\chi(\mathfrak{b})v$$

(where it is a formal consequence that $\chi|_{[\mathfrak{b},\mathfrak{b}]=\mathfrak{n}^+}=0$), so that N_i/N_{i-1} is a 1-dimensional space on which $\mathfrak{h}$ acts by a character and on which $\mathfrak{n}^+$ acts by zero. Therefore

$$\Longrightarrow\ U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i\big/N_{i-1}\cong M_\mu\qquad \mu\in\operatorname{wt}N,$$

where $\mu=\chi\in\operatorname{wt}N$ is a weight of N since there was a vector $\widehat{v}\in N_i\subseteq N$ on whom $\mathfrak{h}$ (through which the character must pass since it vanishes on $[\mathfrak{b},\mathfrak{b}]=\mathfrak{n}^+$) acts by a character μ. It is clear that, as we run across i, the weights will precisely run across and exhaust all of $\operatorname{wt}N$, exhibiting the desired claim. ■

Next, we will establish a fact about the type of M^ϑ (recall the bit about central characters earlier in Section 2).

Lemma (9.7)**.** For ϑ a central character of M,

$$\operatorname{typ}M^\vartheta=\{\psi\in\operatorname{typ}M:\vartheta_\psi=\vartheta\}\coloneqq\operatorname{typ}^\vartheta M,$$

where the second equality sign is a definition of the third object and by ϑ_ψ we mean the central character of the Verma M_ψ.

Proof. Let us take a filtration[30] exhibiting $\operatorname{typ}M$:

$$0=M_0\subseteq\cdots\subseteq M_n=M$$

such that

$$M_i/M_{i-1}\cong M_\psi,\qquad \psi\in\operatorname{typ}M.$$

Recall that the functor $\square^\vartheta$ is exact. Let us then apply it to

$$0\longrightarrow M_{i-1}\hookrightarrow M_i\twoheadrightarrow M_i/M_{i-1}\longrightarrow 0$$

$$\Big\Downarrow\square^\vartheta$$

$$0\longrightarrow {M_{i-1}}^\vartheta\hookrightarrow {M_i}^\vartheta\twoheadrightarrow \left(M_i\big/M_{i-1}\right)^\vartheta\longrightarrow 0,$$

whereupon we obtain a filtration

$$0={M_0}^\vartheta\subseteq\cdots\subseteq {M_n}^\vartheta=M^\vartheta$$

with quotients

$$\Longrightarrow\ {M_i}^\vartheta\big/{M_{i-1}}^\vartheta\cong\left(M_i\big/M_{i-1}\right)^\vartheta\cong {M_\psi}^\vartheta.$$

However, recall from Section 2 that

$$\Theta(M_\psi)=\{\vartheta_\psi\},$$

[29] I.e., we exhibited a desired filtration by using this fact.

[30] Unfortunately our index for the notation here will collide with the place where the highest weight of a Verma module would be; but hopefully it is clear upon context whether we refer to a filtration index or a highest weight. I guess this collision was more or less unavoidable, since highest weights are labelled below and central characters are placed on top.

so that ${M_\psi}^\vartheta$ is either all of M_ψ if $\vartheta = \vartheta_\psi$ or 0 if $\vartheta \neq \vartheta_\psi$:

$$\implies {M_i}^\vartheta \big/ {M_{i-1}}^\vartheta \cong {M_\psi}^\vartheta = \begin{cases} M_\psi & \vartheta = \vartheta_\psi \\ 0 & \vartheta \neq \vartheta_\psi \end{cases}.$$

Note that in the case of 0, we have ${M_i}^\vartheta = {M_{i-1}}^\vartheta$, so there is sort of a redundant term.

Then we can appropriately delete all such redundant terms in the filtration

$$0 = {M_0}^\vartheta \subseteq \cdots \subseteq {M_n}^\vartheta = M^\vartheta$$

to obtain a filtration exhibiting the claimed $\operatorname{typ} M^\vartheta = \operatorname{typ}^\vartheta M$, as desired. ■

Lastly, let us compute the type of the tensor product of a finite-dimensional representation and a Verma module. Note in particular by taking $V = \mathbb{C}$ the trivial representation we obtain the type of a Verma module (not that this is needed for us).

Lemma (9.10)**.** For $V \in \mathsf{Rep}_{\mathrm{fd}}\, \mathfrak{g}$, $\psi \in \mathfrak{h}^*$, we have

$$\operatorname{typ}(M_\psi \otimes_{\mathbb{C}} V) = \{\lambda + \psi\}_{\lambda \in \operatorname{wt} V} = \psi + \operatorname{wt} V.$$

Proof. Let us take a weight basis of V

$$V = \mathbb{C}\{v_1, \cdots, v_n\}$$

ordered so that their corresponding weights are ordered

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

Let v^ψ be the highest weight vector of M_ψ.

Consider then the set of vectors

$$\{v^\psi \otimes v_i\}_i.$$

The claim is that these guys are weight vectors. Indeed, compute

$$\begin{aligned} \mathfrak{h}(v^\psi \otimes v_i) &= \mathfrak{h} v^\psi \otimes v_i + v^\psi \otimes \mathfrak{h} v_i \\ &= \psi(\mathfrak{h}) v^\psi \otimes v_i + \lambda_i(\mathfrak{h}) v^\psi \otimes v_i \\ &= (\psi + \lambda_i)(\mathfrak{h})(v^\psi \otimes v_i), \end{aligned}$$

so that in fact

$$\implies v^\psi \otimes v_i \in (M_\psi \otimes V)^{\psi + \lambda_i}.$$

Moreover,

$$\begin{aligned} \mathfrak{n}^+(v^\psi \otimes v_i) &= \cancel{\mathfrak{n}^+ v^\psi} \otimes v_i + v^\psi \otimes \mathfrak{n}^+ v_i \\ &= v^\psi \otimes \mathfrak{n}^+ v_i, \end{aligned}$$

where $\mathfrak{n}^+ v_i \in \operatorname{Span}\{v_1, \cdots, v_{i-1}\}$ since $\mathfrak{n}^+$ raises the eigenvalue.

To exhibited the claimed type, consider now the filtration

$$0 = N_0 \subseteq \cdots \subseteq N_n = M_\psi \otimes V$$

where

$$N_k := \operatorname{Span}_{U\mathfrak{g}}\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\},$$

which is clearly a filtration. It is not yet obvious that

$$N_n = M_\psi \otimes V;$$

we will show this. We will also claim that

$$N_k / N_{k-1} = M_{\psi + \lambda_k}.$$

It is obvious that

$$N_k/N_{k-1} = \mathrm{Span}_{U\mathfrak{g}}\{v^\psi \otimes v_k\}$$

is generated by a single vector (here we really mean the equivalence class). Moreover, by the above computations of the action of $\mathfrak{h}$ and $\mathfrak{n}^+$, we have

$$v^\psi \otimes v_k \in \left(N_k \big/ N_{k-1}\right)^{\psi+\lambda_k}$$

which is sent under $\mathfrak{n}^+$ to[31]

$$\mathfrak{n}^+ \colon v^\psi \otimes v_k \longmapsto 0 \in N_k \big/ N_{k-1}.$$

Therefore

$$\implies N_k = \mathrm{Span}_{U\mathfrak{n}^-}\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\}.$$

We claim that this is moreover free over $U\mathfrak{n}^-$. To see this, let $\sum \prod \xi \in U\mathfrak{n}^-$ denote any arbitrary element in the universal enveloping algebra, and suppose there was a linear dependence relation:

$$\sum_{i=1}^{k} (\sum \prod \xi)(v^\psi \otimes v_i) = 0.$$

Then, exchanging the sums out and distributing according to Leibniz, we obtain

$$\begin{aligned} 0 = &\sum \sum_{i=1}^{k} ((\prod \xi) v^\psi) \otimes v_i \\ &+ \sum \sum_{i=1}^{k} (\prod_{\text{smaller}} \xi) v^\psi \otimes \underbrace{(\prod_{\text{smaller}} \xi) v_i}_{\text{absorb into Span}\{v_{i+1}, \cdots, v_n\}} + \cdots + \sum \sum_{i=1}^{k} v^\psi \otimes \underbrace{(\prod \xi) v_i}_{\text{absorb into Span}\{v_{i+1}, \cdots, v_n\}}. \end{aligned}$$

But now, after converting each instance of an element of $U\mathfrak{n}^-$ acting on v_i to a linear combination of v_i's, the second line in the above is a sum of pure tensors where the second factor is some v_i and the first factor is some element of $U\mathfrak{n}^-$ acting on v^ψ, where the element is of filtration degree strictly less than that of the element in the first line. Hence the first line cannot be cancelled out (recall that $v^\psi \in M_\psi$ belongs to a module which is $U\mathfrak{n}^-$-free), and we have contradiction. Hence N_k is moreover $U\mathfrak{n}^-$-free,

$$\implies N_k = U\mathfrak{n}^-\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\}.$$

But then this implies N_k/N_{k-1} is free also,

$$\implies N_k \big/ N_{k-1} = U\mathfrak{n}^-\{v^\psi \otimes v_k\} = M_{\psi+\lambda_k},$$

where we recall the weight of this vector is $\mathrm{wt}(v^\psi \otimes v_k) = \psi + \lambda_k$.

Moreover, at $k = n$, this gives $N_n = U\mathfrak{n}^-\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_n\}$, i.e. (since $U\mathfrak{n}^-$ acting on v^ψ generates all of M_ψ)

$$\implies N_n = M_\psi \otimes V.$$

This completes showing that the filtration we constructed exhibits the claimed type, so we are done. ■

[31]This is since $\mathfrak{n}^+(v^\psi \otimes v_k) \in v^\psi \otimes \mathrm{Span}\{v_1, \cdots, v_{k-1}\} \subseteq N_{k-1}$.

5.2. **Base Case of Weak BGG.** In particular, let us apply these facts (9.10 is not necessary) to the resolution of $\mathbb{C}$ given in 9.1.

Lemma. Weak BGG holds for $\Pi_0 = \mathbb{C}$, i.e. there is a resolution of $\mathbb{C}$ with terms in $\mathcal{O}$

$$B_k(\mathbb{C}) = \left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right)^{\vartheta_0}$$

of type

$$\operatorname{typ} B_k(\mathbb{C}) = \{w \circ 0\}_{w \in W_k}.$$

Proof. Recall the resolution of $\mathbb{C}$ given at the beginning of this section. In particular, we will take $\mathfrak{g}$ to be semisimple and $\mathfrak{a} = \mathfrak{b}$ to be the Borel subalgebra. This is a resolution of form

$$\cdots \longrightarrow \mathbb{C}_2 \xrightarrow{d_2} \mathbb{C}_1 \xrightarrow{d_1} \mathbb{C}_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

where

$$\mathbb{C}_n = U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge n}.$$

This is by construction finitely-generated over $U\mathfrak{g}$ (since $(\mathfrak{g}/\mathfrak{b})^{\wedge n}$ is finite-dimensional), $\mathfrak{h}$-semisimple by the Cartan root decomposition ($\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-$ has $\mathfrak{h}$ acts by roots), and is locally $U\mathfrak{n}^+$-finite since we can pass $U\mathfrak{n}^+$ across the tensor $\otimes_{U\mathfrak{b}}$ to act on $(\mathfrak{g}/\mathfrak{b})^{\wedge n}$, which is finite-dimensional. Hence $\mathbb{C}_n \in \mathcal{O}$, and applying the functor $\square^{\vartheta_0}$ keeps it in $\mathcal{O}$. This checks that the terms are indeed in $\mathcal{O}$.

The claim is that Weak BGG is realized by the resolution

$$\cdots \longrightarrow {\mathbb{C}_2}^{\vartheta_0} \longrightarrow {\mathbb{C}_1}^{\vartheta_0} \longrightarrow {\mathbb{C}_0}^{\vartheta_0} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let us compute the type of these terms:

$$\begin{aligned}\operatorname{typ} \mathbb{C}_k &= \operatorname{typ}\left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right) \\ &= \operatorname{wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k} \\ &= \left\{\sum_{\alpha \in S \subseteq R_-} \alpha\right\}_{\substack{S \subseteq R_-, \\ |S| = k}},\end{aligned}$$

where in the last line we have recalled that $\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-$, so that $\operatorname{wt}(\mathfrak{g}/\mathfrak{b}) = \operatorname{wt}(\mathfrak{n}^-) = R_-$; then it is a purely linear algebraic fact to see that, with the action of $\mathfrak{h}$ defined by Leibniz[32], the eigenvalues of the k-th wedge space are the set of sums of k distinct eigenvalues of the 1-st wedge space. In summary

$$\implies \operatorname{typ} \mathbb{C}_k = \left\{-\sum_{\alpha \in S} \alpha\right\}_{\substack{S \subseteq R_+, \\ |S| = k}}.$$

Since $\square^{\vartheta}$ is exact, applying this functor gives a resolution

$$\cdots \longrightarrow {\mathbb{C}_2}^{\vartheta} \longrightarrow {\mathbb{C}_1}^{\vartheta} \longrightarrow {\mathbb{C}_0}^{\vartheta} \longrightarrow \mathbb{C} \longrightarrow 0$$

where $\mathbb{C} = \mathbb{C}^{\vartheta}$ since it is a one-dimensional space. Then, by Lemma 9.7,

$$\implies \operatorname{typ} {\mathbb{C}_k}^{\vartheta} = \{\psi \in \operatorname{typ} \mathbb{C}_k : \vartheta_\psi = \vartheta\} = \{\psi \in \operatorname{wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k} : \vartheta_\psi = \vartheta\}.$$

In particular, let us take $\vartheta = \vartheta_0$ the central character of the Verma module of weight 0. Then $\operatorname{typ} {\mathbb{C}_k}^{\vartheta_0} = \{\psi \in \operatorname{wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k} : \vartheta_\psi = \vartheta_0\}$. By the Harish-Chandra theorem (see end of Section 2),

$$\vartheta_\psi = \vartheta_0 \iff \psi = w \circ 0 \text{ for some } w.$$

[32]I.e. $T(v \wedge w) = Tv \wedge w + v \wedge Tw$.

Now recall that

$$w \circ 0 = w\rho - \rho = -\sum_{\alpha \in R_+ : w^{-1}\alpha \in R_-} \alpha$$

and also that[33]

$$\ell(w) = \#\{\alpha \in R_+ : w(\alpha) \in R_-\} = \#\{\alpha \in R_+ : w^{-1}(\alpha) \in R_-\} = \ell(w^{-1}).$$

Combining all this with our description of $\operatorname{typ} \mathbb{C}_k{}^{\vartheta}$ and $\operatorname{typ} \mathbb{C}_k$, we have

$$\implies \operatorname{typ} \mathbb{C}_k{}^{\vartheta_0} = \left\{ -\sum_{\alpha \in S} \alpha : S \subseteq R_+,\ |S| = k,\ -\sum_{\alpha \in S} \alpha = -\sum_{\alpha \in R_+ : w^{-1}\alpha \in R_-} \alpha \right\}.$$

At this point, let us cite some combinatorial lemmas.

Fact (9.8)**.**

$$\sum_{\alpha \in R_+ : w^{-1}\alpha \in R_-} \alpha = \sum_{\alpha \in S} \alpha$$

$$\Updownarrow$$

$$S = \{\alpha \in R_+ : w^{-1}\alpha \in R_-\}.$$

In fact, Exercise 7.7 of Kirillov gives much more precise answers (though we won't need this here):

Fact. For $w^{-1} = s_{i_1} \cdots s_{i_k}$ a reduced expression,

$$\{\alpha \in R_+ : w^{-1}\alpha \in R_-\} = \{\alpha_{i_1},\ s_{i_1}\alpha_{i_2},\ s_{i_1}s_{i_2}\alpha_{i_3},\ \cdots\ , s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\}.$$

9.8 then tells us that

$$\operatorname{typ} \mathbb{C}_k{}^{\vartheta_0} = \left\{ -\sum_{\alpha \in R_+ : w^{-1}\alpha \in R_-} \alpha : w \in W_k \right\},$$

i.e.

$$\implies \operatorname{typ} \mathbb{C}_k{}^{\vartheta_0} = \{w \circ 0\}_{w \in W_k}.$$

This shows that the resolution

$$\cdots \longrightarrow \mathbb{C}_2{}^{\vartheta_0} \longrightarrow \mathbb{C}_1{}^{\vartheta_0} \longrightarrow \mathbb{C}_0{}^{\vartheta_0} \longrightarrow \mathbb{C} \longrightarrow 0$$

has terms of the type claimed, as desired. ■

5.3. **Proving Weak BGG.** Now we are finally in a position to prove Weak BGG. Let me reproduce the statement for convenience:

[33] From the reduced simple word description we can see $\ell(w) = \ell(w^{-1})$.

Theorem (Weak BGG, 9.9)**.** For $\lambda \in P_+$ and $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$, there is a resolution of $\mathfrak{g}$-modules

$$0 \longrightarrow B_{|R_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

with terms in $\mathcal{O}$

$$B_k = \left(\left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right)^{\vartheta_0} \otimes \Pi_\lambda\right)^{\vartheta_\lambda}$$

such that

$$\operatorname{typ} B_k = \{w \circ \lambda\}_{w \in W_k}.$$

Proof. We already have the theorem proved for the base case of $\Pi_0 = \mathbb{C}$. Now we will construct a new resolution of Π_λ as follows: take the weak BGG resolution for $\mathbb{C}$ and apply to it first $\square \otimes_{\mathbb{C}} \Pi_\lambda$ then $\square^{\vartheta_\lambda}$:

$$0 \longrightarrow B_{|R_+|}(\mathbb{C}) \longrightarrow \cdots \longrightarrow B_1(\mathbb{C}) \longrightarrow B_0(\mathbb{C}) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$\Big\Downarrow \square \otimes_{\mathbb{C}} \Pi_\lambda$$

$$0 \longrightarrow B_{|R_+|}(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow \cdots \longrightarrow B_1(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow B_0(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0$$

$$\Big\Downarrow \square^{\vartheta_\lambda}$$

$$0 \longrightarrow \left(B_{|R_+|}(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \cdots \longrightarrow \left(B_1(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \left(B_0(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \Pi_\lambda \longrightarrow 0,$$

where since Π_λ is a finite-dimensional vector space we know $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact[34], so the composition of two exact functors[35] $\square^{\vartheta_\lambda}$ and $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact. Note that by construction[36] these terms live in the category $\mathcal{O}$. We have also noted $\mathbb{C} \otimes_{\mathbb{C}} \Pi_\lambda = \Pi_\lambda$, as well as $\Pi_\lambda^{\vartheta_\lambda} = \Pi_\lambda$ since Π_λ is irreducible[37].

Having constructed a resolution, it remains to see each term has the desired type. To exhibit

$$\operatorname{typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{w \circ \lambda\}_{w \in W_k},$$

let us first compute $\operatorname{typ}(B_k(\mathbb{C}) \otimes \Pi_\lambda)$ by exhibiting a filtration. Recall

$$\operatorname{typ} B_k(\mathbb{C}) = \{w \circ 0\}_{w \in W_k} \implies \exists\, 0 = B_k(\mathbb{C})_0 \subsetneq \cdots \subsetneq B_k(\mathbb{C})_n = B_k(\mathbb{C}) \;:\; B_k(\mathbb{C})_i \big/ B_k(\mathbb{C})_{i-1} \cong M_{w_i \circ 0}$$

for $w_i \in W_k$; then the filtration of $B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda$ given by

$$0 = B_k(\mathbb{C})_0 \otimes_{\mathbb{C}} \Pi_\lambda \subsetneq \cdots \subsetneq B_k(\mathbb{C})_n \otimes_{\mathbb{C}} \Pi_\lambda = B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda$$

has each quotient

$$\implies B_k(\mathbb{C})_i \otimes_{\mathbb{C}} \Pi_\lambda \big/ B_k(\mathbb{C})_{i-1} \otimes_{\mathbb{C}} \Pi_\lambda \cong B_k(\mathbb{C})_i \big/ B_k(\mathbb{C})_{i-1} \otimes_{\mathbb{C}} \Pi_\lambda \cong M_{w_i \circ 0} \otimes_{\mathbb{C}} \Pi_\lambda,$$

where we have passed the tensor product outside the quotient since $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact (as Π_λ is a vector space).

For a similar reason[38] as in the JH case,

---

[34] Recall the huge block proposition in Section 2, where we noted that the tensor product with a finite-dimensional space is exact and in particular stays in $\mathcal{O}$.

[35] Since the tensor product stayed in $\mathcal{O}$, we are allowed to use that $\square^{\vartheta_\lambda}$ is an exact functor from $\mathcal{O}$ to $\mathcal{O}$.

[36] We saw that $B_k(\mathbb{C}) \in \mathcal{O}$, so applying these functors to it keeps things in $\mathcal{O}$.

[37] Recall M^{ϑ} is a subrep of M.

[38] Just refine the filtration; see the earlier footnote about JH (I think it's footnote 17 though of course this changes as I edit unfortunately).

Fact. If
$$0 = N_0 \subseteq \cdots \subseteq N_n = M,$$
then
$$\operatorname{typ} M = \bigsqcup_i \operatorname{typ}(N_i/N_{i-1}).$$

For this reason, the above filtration of $B_k(\mathbb{C}) \otimes \Pi_\lambda$ buys us
$$\operatorname{typ}(B_k(\mathbb{C})\otimes_{\mathbb{C}}\Pi_\lambda) = \bigsqcup_i \operatorname{typ} B_k(\mathbb{C})_i\otimes_{\mathbb{C}}\Pi_\lambda\big/B_k(\mathbb{C})_{i-1}\otimes_{\mathbb{C}}\Pi_\lambda = \bigsqcup_i \operatorname{typ}(M_{w_i\circ 0}\otimes_{\mathbb{C}}\Pi_\lambda) = \bigsqcup_i \{\mu+w_i\circ 0\}_{\mu\in\operatorname{wt}\Pi_\lambda},$$
where we have appealed to Lemma 9.10 in the last equality. Hence
$$\implies \operatorname{typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right) = \{\mu + w\circ 0\}_{\substack{\mu\in\operatorname{wt}\Pi_\lambda,\\ w\in W_k}}.$$

Now let us pass to $\square^{\vartheta_\lambda}$. By Lemma 9.7,
$$\operatorname{typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{\psi \in \operatorname{typ}(B_k(\mathbb{C}) \otimes \Pi_\lambda) : \vartheta_\psi = \vartheta_\lambda\} = \{\mu + w\circ 0 : \vartheta_{\mu+w\circ 0} = \vartheta_\lambda\}_{\substack{\mu\in\operatorname{wt}\Pi_\lambda,\\ w\in W_k}}.$$
By Harish-Chandra again, we have
$$\vartheta_{\mu+w\circ 0} = \vartheta_\lambda \iff \mu + w\circ 0 = u\circ\lambda \text{ for some } u.$$

Now we cite another combinatorial fact about Weyl groups:

Fact (K8.22b)**.** For any $\mu \in P$, the Weyl group orbit (not under the shifted action) of μ contains exactly one element of P_+.

Hence, for any $\mu + w\circ 0 \in \operatorname{typ}\left(B_k(\mathbb{C}) \otimes \Pi_\lambda\right)^{\vartheta_\lambda}$, since we know
$$\begin{aligned}\mu + w\circ 0 &= u\circ\lambda\\ \mu + w\rho - \rho &= u\lambda + u\rho - \rho\\ u^{-1}(\mu + w\rho - u\rho) &= \lambda,\end{aligned}$$
where $\mu + w\rho - u\rho \in P$ and $\lambda \in P_+$, we know that u^{-1} is the only element of W which turns $\mu + w\rho - u\rho$ into λ, so that there is a one-to-one correspondence between $\mu + w\circ 0$ and such u.

Moreover, since the set of weights of a finite-dimensional representation are closed under W[39], we have
$$u^{-1}\mu \in \operatorname{wt}\Pi_\lambda \implies u^{-1}\mu \le \lambda.$$
Since $\rho - u^{-1}w\rho = \sum_{\alpha\in R_+ : w^{-1}u\alpha\in R_-} \alpha$, we also have
$$u^{-1}w\rho \le \rho.$$
Together these two give
$$u^{-1}\mu + u^{-1}w\rho \le \lambda + \rho.$$
But as we saw in the last paragraph, this inequality is actually an equality, which forces
$$u^{-1}\mu = \lambda, \quad u^{-1}w\rho = \rho,$$

[39] In fact, the full statement is this: for any $V \in \mathsf{Rep}_{\mathrm{fd}}\,\mathfrak{g}$ and any $w \in W$,
$$w(\chi_V) = \chi_V,$$
where we define
$$w(e^\lambda) := e^{w(\lambda)}.$$
In particular the weights and the dimensions of their weight subspaces are invariant under W. This is Theorem K8.8. Its proof is very short and can be summarized by "reduce to $\mathfrak{sl}_2$", like many other proofs in Kirillov.

which means

$$u = w^{-1}.$$

In particular this means $\ell(u) = \ell(w) = k$, which gives

$$\implies \operatorname{typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{u \circ \lambda\}_{u \in W_k}.$$

This is precisely as claimed.

We have, at last, shown Weak BGG. ■

5.4. **A Corollary of Weak BGG.** As mentioned (and used!) earlier, Bott's Theorem[40] on Lie algebra cohomology can be derived as a corollary of Weak BGG;

Theorem (Bott)**.** For $\Pi \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$,

$$\dim H^k(\mathfrak{n}^- : \Pi) = |W_k|.$$

Proof. Recall

$$H^k(\mathfrak{n}^- : \Pi) = \operatorname{Ext}^k_{U\mathfrak{n}^-}(\mathbb{C}, \Pi) = \operatorname{Tor}_k^{U\mathfrak{n}^-}(\Pi^*, \mathbb{C})^* = \operatorname{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*})^*.$$

Let us compute the latter by resolving the second variable

$$0 \longrightarrow B_{|R_+|}(\Pi^{\dagger,*}) \longrightarrow \cdots \longrightarrow B_1(\Pi^{\dagger,*}) \longrightarrow B_0(\Pi^{\dagger,*}) \longrightarrow \Pi^{\dagger,*} \longrightarrow 0;$$

then Tor is the homology of (here we suppress the Π)

$$0 \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_{|R_+|} \longrightarrow \cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_1 \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_0 \longrightarrow 0,$$

which is the same thing as writing

$$0 \longrightarrow B_{|R_+|}\big/\mathfrak{n}^- B_{|R_+|} \longrightarrow \cdots \longrightarrow B_1\big/\mathfrak{n}^- B_1 \longrightarrow B_0\big/\mathfrak{n}^- B_0 \longrightarrow 0.$$

As remarked in a footnote earlier, $\mathfrak{h}$ acts naturally on this sequence.

By Weak BGG, since $B_k \in \mathcal{O}$, we have B_k is $U\mathfrak{n}^-$-finitely-generated, so that

$$\dim B_k\big/\mathfrak{n}^- B_k < \infty.$$

In particular, $B_k/\mathfrak{n}^- B_k$ admits a weight space decomposition, where the weight vectors are precisely[41] the quotient images of the highest weight vectors of B_k, which may be obtained by looking in a filtration

$$0 = (B_k)_0 \subseteq \cdots \subseteq (B_k)_n = B_k,$$

where $(B_k)_i/(B_k)_{i-1} = M_{w_i \circ \lambda}$, and taking the highest weight vector e.g. $v^{w_i \circ \lambda} \in (B_k)_i/(B_k)_{i-1}$. Lifting this to a $\widehat{v}^{w_i \circ \lambda} \in (B_k)_i \subseteq B_k$, and then projecting down to a $\widetilde{v}^{w_i \circ \lambda} \in B_k/\mathfrak{n}^- B_k$, we obtain a nonzero weight vector (nonzero since $v^{w_i \circ \lambda}$ was taken to be highest weight and so cannot be in the image of $\mathfrak{n}^-$). Hence the weights of $B_k/\mathfrak{n}^- B_k$ are

$$\implies \operatorname{wt} B_k\big/\mathfrak{n}^- B_k = \{w \circ \lambda\}_{w \in W_k},$$

and in particular

$$\implies \dim B_k\big/\mathfrak{n}^- B_k = |W_k|.$$

[40] Interestingly I can't quite find this online under this name. I thought at first BGG might be using out-of-date terminology, but Humphreys uses this name also. I'm probably just bad at using the Internet.

[41] It is clear that the process we describe below gives weight vectors, and moreover all weight vectors must arise this way since the following: for $B_k \in \mathcal{O}$, recall $\operatorname{wt} B_k$ lies in a finite union of cones. Then, since $\mathfrak{n}^-$ drops the weights, $\mathfrak{n}^- B_k$ has all the weights of B_k except for the highest ones in each cone, so that $B_k/\mathfrak{n}^- B_k$ has weights precisely the highest ones.

But the maps in the resolution $B_\bullet/\mathfrak{n}^- B_\bullet$, which commute with $\mathfrak{h}$ (see a previous footnote), must preserve weight spaces; since $w \circ \lambda \neq u \circ \lambda$ for $w \neq u$, we have these maps must actually all be zero, so that its homology is simply

$$\implies \operatorname{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*}) = B_k(\Pi^{\dagger,*}) \big/ \mathfrak{n}^- B_k(\Pi^{\dagger,*}),$$

and in particular

$$\dim H^k(\mathfrak{n}^- : \Pi) = \dim \operatorname{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*}) = \dim B_k \big/ \mathfrak{n}^- B_k = |W_k|,$$

i.e.

$$\implies \dim H^k(\mathfrak{n}^- : \Pi) = |W_k|,$$

as is stated. ∎

With this we have finally cleared all of our debts; at last we have done what we set out to do. What a journey!

6. References

D.J. Benson. *Representations and Cohomology I.*

I.N. Bernstein, I.M. Gelfand, S.I. Gelfand. "Differential operators on the base affine space and a study of $\mathfrak{g}$-modules".

J.E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$.*

A. Kirillov. *An Introduction to Lie Groups and Lie Algebras.*

J.J. Rotman. *An Introduction to Homological Algebra.*